Affine processes under parameter uncertainty

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Why non-linear affine processes?

- Affine processes have been considered in many variants (in particular in interest rate and credit risk markets)
- In applications, the parameters of those processes have to be estimated, thus leading to a certain amount of model risk
- What can be done to incorporate this model risk into our affine models

Philosophical aspects

Uncertainty can be motivated in a number of situations:

1. We have no idea about the distribution of the future evolution of X, except some rough guesses about intervals of parameters
2. We have some past data and are able to estimate the parameters, but this has uncertainty $\rightarrow$ we could use confidence intervals for the parameters
3. We believe that the future evolution is close to the observed evolution, but not exactly like it.

Where do we place ourselves? Certainly, this depends on the task we want to achieve!

Introduction

- Consider the state space $\mathscr{X} = \mathbb{R}$ or $\mathscr{X} = \mathbb{R}^+$ (canonical state space)
- A (time-homogeneous) Markov processes X is called **affine**, if

$$\mathbb{E}[e^{iuX_T} \mid \mathscr{F}_t] = e^{\phi(T-t,u)+\psi(T-t,u)X_t}$$

for all $u \in i\mathbb{R}$, $0 \leq t \leq T$ with appropriate functions ϕ and ψ.

- Then, $X = X^x$ is the strong solution of

$$dX_t = (b^0 + b^1 X_t)dt + \sqrt{a^0 + a^1 X_t}\, dW_t, \qquad X_0 = x. \tag{1}$$

where the parameter vector $\theta := (b^0, b^1, a^0, a^1)^\top$ satisfies certain admissibility conditions. Here, W is a standard Brownian motion.

Framework

- Let $\Omega = C([0,T];\mathbb{R})$ be the canonical space of continuous paths.
- We endow Ω with the topology of locally uniform convergence and denote by $\mathscr{F}$ its Borel σ-field.

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- We endow Ω with the topology of locally uniform convergence and denote by $\mathscr{F}$ its Borel σ-field.
- Let X be the canonical process $X_t(\omega) = \omega_t$, and let $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ with $\mathscr{F}_t = \sigma(X_s, 0 \leq s \leq t)$ be the (raw) filtration generated by X.
- Denote by $\mathfrak{P}(\Omega)$ the Polish space of all probability measures on Ω equipped with the topology of weak convergence.

- X is a continuous P-$\mathbb{F}$-semimartingale, if there exist $B = B^P$ and $M = M^P$ satisfying $B_0 = M_0 = 0$ such that

$$X = X_0 + B + M.$$

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- We focus on semimartingales where there exist predictable processes β^P and $\alpha \geq 0$, such that

$$B^P = \int_0^\cdot \beta_s^P ds, \quad C = \int_0^\cdot \alpha_s ds.$$

- We denote

$$\mathfrak{P}^{ac}_{sem} = \{P \in \mathfrak{P}(\Omega) \,|\, X \text{ is a } (P, \mathbb{F})\text{-semimartingale with a.c. characteristics}\}.$$

- We will consider **model risk** in the sense that there is uncertainty on the parameter vector $\theta = (b^0, b^1, a^0, a^1)$ of the affine process.
- Assume there is additional information on bounds on the parameter vector θ, denoted by

$$\underline{b}_i, \ \bar{b}_i, \ \underline{a}_i, \ \bar{a}_i$$

leading to

$$\Theta = [\underline{b}^0, \bar{b}^0] \times [\underline{b}^1, \bar{b}^1] \times [\underline{a}^0, \bar{a}^0] \times [\underline{a}^1, \bar{a}^1]. \quad (2)$$

- We are interested in the intervals generated by the associated affine functions: let

$$b^*(x) := \{b^0 + b^1 x : (b^0, b^1) \in [\underline{b}^0, \bar{b}^0] \times [\underline{b}^1, \bar{b}^1]\},$$
$$a^*(x) := \{a^0 + a^1 x^+ : (a^0, a^1) \in [\underline{a}^0, \bar{a}^0] \times [\underline{a}^1, \bar{a}^1]\} \tag{3}$$

for $x \in \mathbb{R}$.

- Due to the nice structure of Θ the sets are always intervals: indeed,

$$b^*(x) = [\underline{b}^0 + (\underline{b}^1 \mathbb{1}_{\{x \geq 0\}} + \bar{b}^1 \mathbb{1}_{\{x<0\}})x, \bar{b}^0 + (\bar{b}^1 \mathbb{1}_{\{x \geq 0\}} + \underline{b}^1 \mathbb{1}_{\{x<0\}})x],$$
$$a^*(x) = [\underline{a}^0 + \underline{a}^1 x^+, \bar{a}^0 + \bar{a}^1 x^+]. \tag{4}$$

Definition

We call P **affine-dominated** by Θ, if (β^P, α) satisfy

$$\beta_s^P \in b^*(X_s), \quad \text{and} \quad \alpha_s \in a^*(X_s), \tag{5}$$

for dt-almost all $s \in [0, T]$ for P-almost all $\omega \in \Omega$.

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Definition

A **non-linear affine process** is a family of semimartingale laws $P \in \mathfrak{P}_{\text{sem}}^{ac}$ such that

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Denote by $\mathscr{A}(x, \Theta)$ those semimartingale laws $P \in \mathfrak{P}_{\text{sem}}^{\text{ac}}$, satisfying $P(X_0 = x) = 1$ and being dominated by Θ.

- The state space $\mathscr{O}$ will be either $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{>0}$.
- $\mathscr{O}$ depends on the choice of Θ.
- A family of non-linear affine processes $(\mathscr{A}(x,\Theta))_{x\in\mathscr{O}}$ with state space $\mathscr{O}$ is called **proper**, if either $a^0 > 0$ holds, or $\underline{a}^0 = \bar{a}^0 = 0$ and $\underline{b}^0 \geq \bar{a}^1/2 > 0$.

Proposition

Consider $x > 0$ and assume that $\underline{a}^0 = \bar{a}^0 = 0$ and that $\underline{b}^0 \geq \bar{a}^1/2 > 0$. Then for any $P \in \mathscr{A}(x,\Theta)$ it holds that

$$P(X_t > 0,\ 0 \leq t \leq T) = 1.$$

Dynamic programming

- We utilize the general results on dynamic programming obtained in Nutz & van Handel (2013) and El Karoui & Tan (2013).

Definition (Conditional non-linear affine processes)

Denote by $\mathscr{A}(t,x,\Theta)$ those semimartingale laws $P \in \mathfrak{P}^{ac}_{sem}$ such that

(i) $P(X_s = x, 0 \leq s \leq t) = 1$,

(ii) P is affine-dominated on $(t,T]$ by Θ.

Proposition

Consider a proper family of non-linear affine processes with state space $\mathscr{O}$. For any $(t,x) \in [0,T] \times \mathscr{O}$ and any stopping time τ taking values in $[t,T]$, we obtain

$$v(t,x) = \sup_{P \in \mathscr{A}(t,x,\Theta)} E^P\left[v(\tau, X_\tau)\right].$$

The Kolmogorov equation

- Consider the state space $\mathscr{O}$ which will be either $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{>0}$. Fix $\psi : \mathscr{O} \to \mathbb{R}$.
- The affine process X given in Equation (1) is uniquely characterized by its infinitesimal generator,

$$\mathscr{L}^{\theta} = (b^0 + b^1 x)\partial_x + \frac{1}{2}(a^0 + a^1 x)\partial_{xx}.$$

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- We study the non-linear PDE

$$\begin{cases} -\partial_t v(t,x) - \sup_{\theta \in \Theta} \mathscr{L}^{\theta} v(t,x) = 0 & \text{on } [0,T) \times \mathscr{O} \\ v(T,x) = \psi(x) & x \in \mathscr{O}. \end{cases}$$

Theorem

Consider a proper family of non-linear affine processes with state space $\mathscr{O}$ and let $\psi : \mathscr{O} \to \mathbb{R}$ be Lipschitz continuous. Then

$$v(t,x) := \sup_{P \in \mathscr{A}(t,x,\Theta)} E^P\big[\psi(X_T)\big], \quad x \in \mathscr{O}$$

is a viscosity solution of the non-linear PDE above.

Existence and Uniqueness

- Existence follows from the standard argument in stochastic control.

Proposition (Uniqueness)

Assume that ψ is Lipschitz-continuous. Then $v(t,x)$ is a viscosity solution of (6) If in addition,

(i) $\underline{a}^0 > 0$ and $\mathscr{O} = \mathbb{R}$, then $v(t,x)$ is the unique solution of (6), or

(ii) if $\underline{a}^0 = \bar{a}^0 = 0$, $\underline{b}^0 \geq \bar{a}^1/2 > 0$ and $\mathscr{O} = \mathbb{R}_{>0}$, then $v(t,x)$ is the only viscosity solution, such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}_{>0}} \frac{|v(t,x)|}{1+x} < \infty.$$

Motivating example

- Non-linear Vasiček-CIR model with state space $\mathbb{R}$ and $a^1 \in [0, \overline{a}^1]$, and $a^0 \in [0, \overline{a}^0]$

An Itô formula for non-linear affine processes

Proposition

Let $\mathscr{A}(x,\Theta)$ be a non-linear affine process and $F \in C^2$. Then, for every $P \in \mathscr{A}(x,\Theta)$, $Y = F(X)$ is a P-semimartingale with differential characteristics $\tilde{\alpha}$ and $\tilde{\beta}$ satisfying

$$\tilde{\beta}_s^P \in b^F(X_s), \qquad \tilde{\alpha}_s \in a^F(X_s).$$

We define the two interval-valued functions a^F and b^F by

$$a^F(x) := [(F'(x))^2(\underline{a}^0 + \underline{a}^1 x^+), (F'(x))^2(\bar{a}^0 + \bar{a}^1 x^+)] \tag{6}$$

and

$$b^F(x) := \Big[\inf_{(\beta,\alpha)\in b^*(x)\times a^*(x)} \Big(F'(x)\beta + \frac{1}{2}F''(x)\alpha\Big), \sup_{(\beta,\alpha)\in b^*(x)\times a^*(x)} \Big(F'(x)\beta + \frac{1}{2}F''(x)\alpha\Big)\Big] \tag{7}$$

- Assume there exist set-valued functions $\tilde{a}$ and $\tilde{b}$, such that for all $x \in \mathscr{O}$,

$$a^F(x) = \tilde{a}(F(x)), \qquad \text{and} \qquad b^F(x) = \tilde{b}(F(x)). \tag{8}$$

  For example, if F is invertible on $\mathscr{O}$, then condition (8) holds which will be the case in the below example.

Definition

Consider $F \in C^2(\mathbb{R})$ and a non-linear affine process $\mathscr{A}(x, \Theta)$ and assume that condition (8) holds. The non-linear process $F(\mathscr{A}(x, \Theta))$ is a family of semimartingale laws $P \in \mathfrak{P}^{ac}_{sem}$ with differential semimartingale characteristics (β^P, α) such that

(i) $P(X_0 = F(x)) = 1$,

(ii) (β^P, α) satisfy

$$\beta^P_s \in \tilde{b}(X_s), \quad \text{and} \quad \alpha_s \in \tilde{a}(X_s), \tag{9}$$

for $dP \otimes dt$-almost all $(\omega, s) \in \Omega \times (0, T]$.

Proposition

Let $F \in C^2$ such that $F'(x) \neq 0$ for all x in $\mathscr{O}$. Then

$$F(\mathscr{A}(x,\Theta)) = \left\{P \circ (F(X))^{-1} : P \in \mathscr{A}(x,\Theta)\right\}.$$

Example

Let $\mathscr{A}(x,\Theta)$ be a non-linear Vasiček model satisfying $\bar{b}^0 = \underline{b}^0 = 0$, and $Y = F(X) = X^2$. We apply the result (Itô) above and calculate the non-linear process $F(\mathscr{A}(x,\Theta))$. First, note that since $F'' = 2 > 0$,

$$b^F(x) = [2x^2\underline{b}^1 + \underline{a}^0, 2x^2\bar{b}^1 + \bar{a}^0]$$

and $a^F(x) = [4x^2\underline{a}^0, 4x^2\bar{a}^0]$. Thus, $F(X)$ is a non-linear CIR process.

Affine term structure model

- Uncertainty on the default intensity, recovery rate - (Fadina & Schmidt 2018).

Proposition

Assume that f is Lipschitz-continuous. Then $F(t,x)$ is a viscosity solution of

$$\partial_t F(t,x) - \sup_{\theta \in \theta} \mathscr{L}^{\theta} F(t,x) + xF(t,x) = 0, \tag{10}$$

with $F(T,x) = f(x)$.

The dynamic programming yields for any stopping time τ taking values in $[t, T]$ that

$$F(T-t,x) = \sup_{P\in\mathscr{A}(t,x,\Theta)} E^P\left[e^{-\int_t^\tau X_s ds}F(T-\tau,X_\tau)\right].$$

The term-structure equation now allows to obtain the bond prices by considering the pay-off $f(X_T) = 1$. Here, upper bond prices under the non-linear affine term structure model, $x \in \mathscr{O}$, are given by

$$\bar{p}(t,T,x) = \sup_{P \in \mathscr{A}(t,x,\Theta)} E^P\left[e^{-\int_t^T X_s ds} | X_t = x\right], \quad 0 \leq t \leq T.$$

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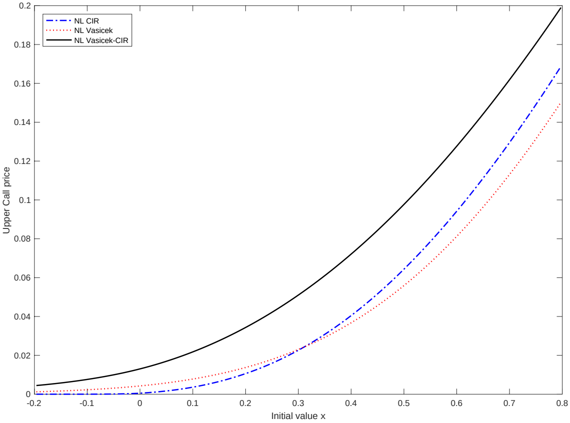
Tolulope Fadina, Ariel Neufeld, Thorsten Schmidt, (2018)
Preprint is available on arxiv.org/abs/1806.02912

Example 2 - The non-linear affine model



Thank you for your attention.

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