On pathwise quadratic variation for càdlàg functions

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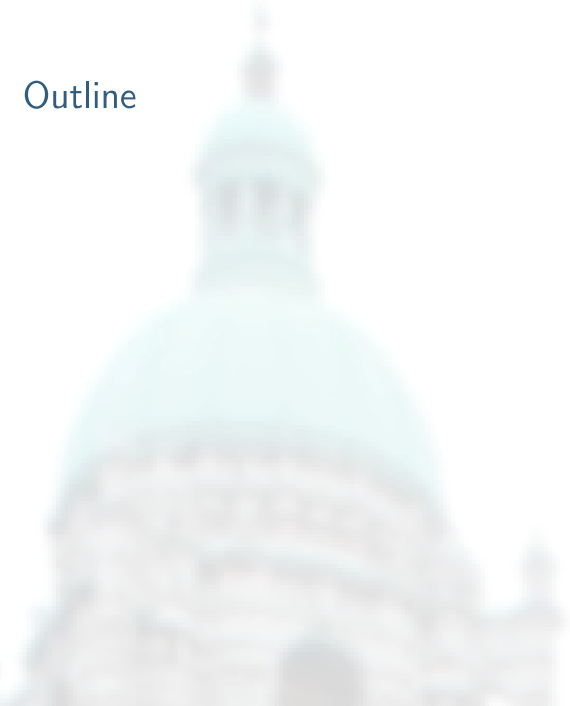
Föllmer(1981), Calcul d'Ito sans probabilités.

Outline



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- Revisit quadratic variation
- Reformulate quadratic variation
- Demo

Notations

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- $D^m := D([0,\infty),\mathbb{R}^m)$, $D^{m\times m} := D([0,\infty),\mathbb{R}^{m\times m})$: Skorokhod spaces each equipped with a complete metric which induces the standard Skorokhod (a.k.a. J_1) topology.

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- $\mathcal{F}$ canonical filtration generated by the coordinate maps $x \longmapsto x(t), t \geq 0$ on $D := D^1$.
- $Q \subset D$: subset of càdlàg functions with finite quadratic variation along π.

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Remark

The **existence** *of quadratic variation depends on the* **terms of the sequence** *of distribution functions of (μ_n) as well as the* **limit** *$[x]$.*

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Proposition

There exists continuous x such that $[x]$ is finite but discontinuous.

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- Unless one could give meaning to (3), one may not remove the $\Delta[x] = (\Delta x)^2$ **literally**.
- Can we reformulate quadratic variation such that we no longer need to worry about $\Delta[x] = (\Delta x)^2$?

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We say that $\mathbf{x} := (x^1, \ldots, x^m)^T \in D^m$ *has finite quadratic variation along* π *if all* x^i, $x^i + x^j$ $(1 \leq i, j \leq m)$ *have finite quadratic variation. In this case, denotes*

$$[x^i, x^j](t) := \frac{1}{2}\left([x^i + x^j](t) - [x^i](t) - [x^j](t)\right),$$

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- m=1?

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$\mathbf{x} \in D^m$ *has finite quadratic variation along* π *if*

$$\mathbf{q}_n(t) := \sum_{\pi_n \ni t_i \leq t} (\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i))(\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i))^T$$

converges in $(D^{m\times m}, d)$. *The limit* $[\mathbf{x}] := ([x^i, x^j])_{1\leq i\leq j\leq m}$ *is called the quadratic variation of* $\mathbf{x}$.

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- The definitions are equivalent. (Chiu & Cont 2018)

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Recap

We have **uncovered the connection** between **quadratic variation** and the **Skorokhod topology**.

Thank you for your attention