L6: Flows of Measures / B-B / Displacement interpolation

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Intro & links to previous problems

1) Monge Transport for $\mathcal{X}=\mathcal{Y}$.

If $\pi = (\mathrm{Id}, T)_{\#}\mu \in \Pi(\mu,\nu)$ is a Monge coupling

then we can naturally consider "intermediate" measures

$$\mu_t := ((1-t)\mathrm{Id} + tT)_{\#}\mu \qquad t\in[0,1]$$

so that $\mu_0=\mu$ & $\mu_1=\nu$.

2) An easy way to construct a martingale $(M_t)_{t\in[0,1]}$ with $M_0=0$, $M_1 \sim \nu \in \mathcal{P}_1(\mathbb{R})$ centred is

$$M_t = \mathbb{E}\left[F_\nu^{-1}(\Phi(B_1)) \mid \mathcal{F}_t\right] \qquad t\in[0,1].$$

Note that $\Phi(B_1)\sim \mathrm{Unif}[0,1] \Rightarrow F_\nu^{-1}(\Phi(B_1))\sim\nu$

$\Rightarrow M_0=0$, $M_1 = F_\nu^{-1}(\Phi(B_1))\sim\nu$

& $M_t = F(t,B_t) \sim \mu_t$

$$M_t = \mathbb{E}\left[F_\nu^{-1}(\Phi(B_t+B_1-B_t)) \mid \mathcal{F}_t\right]$$

$$= F(t,B_t) \qquad F_\nu^{-1}\circ\Phi =: g$$

$$F(t,x) = \mathbb{E}\left[F_\nu^{-1}(\Phi(x+\sqrt{1-t}\,N))\right] = \int p_{1-t}(y)\,g(x+y)\,dy$$

implies a flow of measures

3) Given τ which solves $(SEP)(\nu)$ I can define

$$L_t := \mathcal{L}(B_\tau \mid \mathcal{F}_{t\wedge\tau})$$ so that now $L_0 \sim \nu$

and $\tau = \inf\{t \geq 0 : L_t \text{ is a dirac}\} = \inf\{t \geq 0 : \mathrm{Var}(L_t) = 0\}$

L_t is a measure-valued martingale (i.e., $\forall f$ any $\int f(x) L_t(dx) =: M_t^f$ is since $M_t^f = \mathbb{E}[f(B_\tau) \mid \mathcal{F}_{t\wedge\tau}]$)

Note that $B_{t\wedge\tau} = \int x \, dL_t$

__Eldan '17__ $\rightarrow$ solve (SEP) via a Markov flow L_t looking at

$$\rho(t) = \frac{dL_t}{d\mu}$$ which solves an SDE. See the paper for comparison with Bass.

__Lagrangian vs Eulerian__ points of view [technicalities aside]

__on $X = \mathbb{R}^n$__

Consider particles moving according to a velocity field v:

(Lag.) $$\frac{dX(t)}{dt} = v_t(X(t))$$ describes the movement/position of a particle X.

We can also ask how does the density of particles evolve? If $\rho(t,x)$ is the density at time t, then (Lag) is equivalent to

(Eul) $$\frac{d\rho_t}{dt} + \nabla_x \cdot (\rho_t V_t) = 0$$ (transport equation / continuity — " — / conservation of mass eq.)

$\nabla \cdot$ via $$\int \varphi \, d(\nabla \cdot m) = -\int \nabla\varphi \cdot dm$$

for a vector-valued measure m.

For a smooth ρ, v in a Euclidean setting

$$\nabla \cdot (\rho v) = \sum_{i=1}^n \frac{\partial(\rho v_i)}{\partial x_i}$$

Kinetic energy of particles is given by $E(t) = \int_{\mathbb{R}^n} \rho_t(x)\, |v_t(x)|^2 dx$

$\Rightarrow$ action A given by $A[\rho, v] = \int_0^1 E(t)\, dt$

"total effort to move particles using the velocity field v"

<u>B-B formulation</u>: $\inf_{(\rho, v) \in V(\mu, \nu)} A[\rho, v]$, where

$$V(\mu, \nu) = \left\{ (\rho, v) : \begin{array}{l} \rho(0,x)\,dx = \mu(dx) \\ \rho(1,x)\,dx = \nu(dx) \end{array} \quad \frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t v_t) = 0 \right.$$
$\quad$ + regularity

Thm [B-B] For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^n)$, $\mu, \nu \ll Leb$,

$$W_2^2(\mu, \nu) = \inf \{ A[\rho, v] : (\rho, v) \in V(\mu, \nu) \}$$

<u>Some ideas for the proof:</u>

$$W_2^2(\mu, \nu) = \inf \left\{ \int \rho_0(x) |T(x) - x|^2 dx \; : \; T_\# \rho_0 = \rho_1 \right\}$$

Given $(\rho, v) \in V(\mu, \nu)$, define T_t via $\begin{cases} \frac{d}{dt} T_t(x) = v_t(T_t(x)) \\ T_0(x) = x. \end{cases}$

$\Rightarrow \rho_t = T_t \# \rho_0$

& $E(t) = \int \rho_0(x) |v_t(T_t(x))|^2 dx = \int \rho_0(x) \left| \frac{d}{dt} T_t(x) \right|^2 dx$

$$\rightarrow A[\rho, v] = \int_0^1 \int \rho_0(x) \left| \frac{d}{dt} T_t(x) \right|^2 dx\, dt$$

$$= \int_{\mathbb{R}^n} \rho_0(x) \int_0^1 \left| \frac{d}{dt} T_t(x) \right|^2 dt\, dx \overset{\text{Jensen}}{\geq} \int \rho_0(x) |T_1(x) - T_0(x)|^2 dx$$

$$= \int \rho_0(x) |T_1(x) - x|^2 dx \geq W_2^2(\mu, \nu)$$

with equality iff $\frac{d}{dt} T_t(x) \equiv \text{const} \Rightarrow v_t(T_t(x))$ const.

i.e. optimal flow has particles moving at a constant speed.

This is achieved taking $T = D\varphi$ the optimal Monge transport &

letting $T_t := (1-t)\,Id + t\,T(x) = D\varphi_t(x)$

& $v_t = \left(\frac{d}{dt} T_t\right) \circ T_t^{-1}$ /Recall $D\varphi^{-1} = D\varphi_t^*$/

$= (T - Id) \circ T_t^{-1}$ /clearly $v_t(T_t) = \frac{d}{dt} T_t$

& $\frac{d}{dt} T_t = (T - Id)$ indep of t./

& $\int \rho_t f(v_t)\,dx = \int \rho_0 f(v_t \circ T_t)\,dx$

$= \int \rho_0 f(T - Id)\,dx$

So $E(t) = \int \rho_0 |T(x) - x|^2 dx = W_2^2(\mu, \nu)$ indep. of t. ∎

Rk Above we optimised over (ρ, v) but among all v compatible with the flow ρ via the cont. eq. we want to select the one with minimal energy $E(t)$ $\Rightarrow$ it should be $\perp$ to divergence-free vector fields $\Rightarrow$ should be a gradient in some sense

$\Rightarrow$ this leads to seeing the [B-B] pb as a gradient flow on $\mathcal{P}_2(\mathbb{R}^n)$ $\rightsquigarrow$ $\mathcal{P}_2(X)$.

$\Rightarrow$ the flows trace geodesics (compare with geodesics formula in Riemannian geometry)

Rk The dual pb also has a dynamic reformulation.

Prop If c is convex on $\mathbb{R}^n$ then c-concave functions are equivalent to all viscosity solutions of the H-J eq.

$$\frac{\partial u}{\partial t} + c^*(Du) = 0 \quad \text{at } t = 1 \quad (\text{or } t \geq 1)$$

$\Rightarrow$ Thm Let $c : \mathbb{R}^n \to \mathbb{R}_+$ be convex & superlinear. Then

$$D(\mu, \nu) = \sup \left\{ \int \varphi(1, \cdot)\,d\nu - \int \varphi(0, \cdot)\,d\mu : \varphi : [0,1] \times \mathbb{R}^n \to \mathbb{R}\right.$$

solves $\begin{cases} \frac{\partial \varphi}{\partial t} + c^*(D_x \varphi) = 0 \\ \varphi(0,\cdot) \in C_b(\mathbb{R}^n) \end{cases}$

This leads to a dual proof of B-B (use F-R in the proof)

Displacement Interpolation

Go back to the idea above that a tr. plan T defines a flow via T_t.

Consider instead of $P(\mu,\nu)$, the problem of

$$P^{[0,1]}(\mu,\nu) = \inf \left\{ \int_X C\left((T_t x)_{0\le t\le 1}\right) \mu(dx) : \begin{array}{l} T_0 = Id, \; T_1 \# \mu = \nu, \\ t \mapsto T_t x \quad C^1 \text{ (say)} \end{array} \right\}.$$

When does $P^{[0,1]}$ & P yield the same value & transport $T = T_1$?

A sufficient condition is that $c(x,y) = \inf\{ C((z_t)_{t\le 1}) : z_0 = x, z_1 = y\}$

If we have a nice diff structure then consider $C((z_t)) = \int_0^1 \tilde{c}(\dot{z}_t)\,dt$

ex $\quad C((z_t)) = \int_0^1 |\dot{z}_t|^p dt$ on $\mathbb{R}^n$ $\Rightarrow$ $c(x,y) = |x-y|^p$, $p \ge 1$ via Jensen

c convex $\qquad c(\dot{z}_t) \Rightarrow c(x-y)$

$C((z_t)) = \int_0^1 \|\dot{z}_t\|^p dt$ on a smooth complete R. manifold M $\Rightarrow$ $c(x,y) = d(x,y)^p$, $p>1$

$\Leftrightarrow (z_t)$ minimizing geodesics with arc length parametrization.

$\Rightarrow$ optimal trajectory a straight line $z_t = x + t(y-x)$, $t \le 1$.

Thm Intermediate optimality: $\mu,\nu \ll Leb$, $c(x,y) = |x-y|^p$, $p>1$

$$T(x) = x - Dc^*(D\varphi(x)) \quad \& \quad T_t = Id\,(1-t) + t\cdot T$$
$$= x - t\,Dc^*(D\varphi)$$

$(Id, T)\#\mu$ is optimal in $\Pi(\mu,\nu)$ and

In fact: $W_p(\mu_s,\mu_t) = |t-s| W_p(\mu_0,\mu_1)$

$(\mathrm{Id}, T_t)_\# \mu$ is — " — $\Pi(\mu, T_{t\#}\mu)$ and

$(T_t, T)_\# \mu$ — " — $\Pi(T_t \mu, \nu)$.

so we are tracing constant speed geodesics in $\mathcal{P}_p(X)$.

$Q \subseteq \mathcal{P}(X)$ is <u>convex</u> iff $\forall\, \mu,\nu \in Q$ $\quad t\mu + (1-t)\nu \in Q$

<u>Rk</u> $t \mapsto t\mu_0 + (1-t)\mu_1$ is not a <u>geodesic</u>.

is <u>displacement convex</u> (McCann) iff $\forall\, \mu,\nu \in Q$ $\quad \rho_t \in Q$

$$\rho_t := T_t \# \mu$$

As it turns out a lot of important functionals $F : \mathcal{P}(X) \mapsto \mathbb{R}$ are displacement convex (i.e., $t \mapsto F(\rho_t)$ is convex)

Thm

If $U(0)=0$ & $r \mapsto r\,U(r^{-n})$ is convex non-increasing on $(0,+\infty)$

then $\mathcal{U}(\rho) = \int U(\rho(x))\,dx$ is displacement convex

(internal energy) e.g. $U(r) = r\log r$ (on p-tiles with density ok.)

If $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex then $\mathcal{V}(\rho) = \int V\, d\rho$ is displacement convex

(potential energy)