

Indefinite Theta Series

V is a \mathbb{Q} -Vector space, with $\dim V = m$ and a quadratic form

$$Q(x) = \frac{1}{2}(x, x)$$

$L \subset V$ even Lattice, $L \subset L^\#$, $h \in L^\#$

a) (V, Q) positive definite: Let $p \in \mathcal{H}^L(V_C)$ harmonic polynomials

$$\Theta(\tau, p, \zeta, h) = \sum_{x \in L+h} p(x) q^{(x, \tau)} \in M_{\frac{m}{2}+l}(\Gamma)$$

Proof idea: $p(x) e^{-2\pi Q(x)}$ is an eigenfunction under Fourier transform

- arithmetic considerations to determine Γ'

b) What if V is indefinite? Consider signature $(1,1)$

- $V = k = \mathbb{Q}(-\sqrt{d})$ real quadratic
- $\alpha = x + y\sqrt{d} \Rightarrow Q(\alpha) = N(\alpha) = x^2 - dy^2$
- $L \subseteq \mathcal{O}_K$, $\Gamma \leq \mathcal{O}_K^*$

Thm (Hecke 1927)

$$\sum_{\substack{\alpha \in L+h \\ N(\alpha) > 0 \text{ mod } \Gamma}} \operatorname{sgn}(\alpha) q^{N(\alpha)} \in S_1(\Gamma')$$

Another example: $V = \mathbb{Q} \oplus \mathbb{Q}$, $Q(x, y) = xy$, $L = \mathbb{Z}^2$, k even

$$E_k(\tau) = -\frac{B_k}{k} + 2 \sum_{x, y > 0} x^{k-1} q^{Q(x)} \in M_k(SL_2(\mathbb{Z}))$$

The classical Eisenstein series, but viewed as an indefinite Θ -series restriction to positive norm

KM - Theory : Complete θ -series

Let V have signature (p, q) , and the associated symmetric space

$$D = \{ z \in V_{\mathbb{R}} \mid \dim z = q, (z, z) < 0 \} = D_+ \sqcup D_-$$

"negative Grassmannian"

$$\Gamma \subseteq \text{Stab}(L+h) \subseteq SO_0(V_{\mathbb{R}}) = G$$

$$M = \Gamma \backslash D \text{ locally symmetric space, } \dim_{\mathbb{R}} = p \cdot q$$

Consider $x \in V$, $Q(x) > 0$, define the cycles as

$$D_x = \{ z \in D \mid z \perp x \} \cong D_{p-1, q}$$

$$C_x = \bigwedge_{\Gamma_x} D_x \rightarrow \frac{D}{\Gamma} = M$$

$$C_N = \sum_{\substack{x \in L+h \\ Q(x) = N \text{ mod } \Gamma}} C_x \in H_{(p-1)q} (M, \partial M)$$

KM defined Schwarz-functions ↓
 closed differential forms

$$\Psi_{KM} = \Psi_q \in [S(V_{\mathbb{R}}) \otimes A^q(D)]^G$$

↑
 eigenfunctions under Fourier transform

So, with the same method as before, we can define a θ series

$$\Theta_q(\tau, z, L, h) = \sum_{x \in L+h} \Psi_q(x, \tau, z) \quad \Phi \in N\text{hol} M_{m_2}(\Gamma') \otimes A^q(D)$$

↑
 non-holomorphic modular forms

Theorem (Kudla - Millson, '79 - '90)

intersection number

Let $C \in H_q(M)$. Then

$$I(\tau, C) := \int_C \Theta_q(\tau, z) = \sum_{N \geq 0} I(C, C_N) q^N \in M_{m_2}(\mathbb{P}^1)$$

(the integral over z kills all negative coefficients in the q -expansion \Rightarrow holomorphic)

One aspect of the proof.

$$\exists \Theta(\varphi, \tau, z) \in N\text{Hol}M_{m_2-2}(\mathbb{P}^1) \otimes \mathcal{A}^{q^{-1}}(M) \text{ s.t.}$$

$$\bar{\partial}_\tau \Theta(\varphi, \tau, z) = d_z \Theta(\varphi, \tau, z)$$

$$\Rightarrow \bar{\partial}_\tau \int_C \Theta(\varphi, \tau, z) = \int_C d_z \Theta(\varphi, \tau, z) = \int_{\partial C} \dots = 0$$

\Rightarrow Holomorphicity \mathbb{W}_n (another argument needed for showing absence of negative q -powers)

Incomplete Θ -series

Let S be any compact q -cell in D . Consider

$$I(S, \tau) = \int_S \Theta_q(\tau, z) = \sum_{x \in L + h} \left(\int_S \varphi_q(\tau x) \right) q^{Q(x)}$$

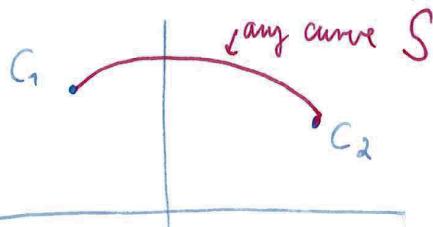
What are interesting, non-homological S ?

Example ($p, 1$) (Zwegers)

D hyperbolic space; $p=2$, $D = \mathbb{H}_+ \sqcup \mathbb{H}_-$, Take $C_1, C_2 \in V_R$ with $Q(C_i) = -1 < 0$

\Rightarrow We can think of C_1, C_2 as points in the negative Grommann = hyperbolic space

Furthermore, assume $(C_1, C_2) < 0$



Theorem (F.-Kud(a))

Set $E_1(t) := 2 \int_0^t e^{-\pi u^2} du$ (the error function).

$$I(S, \tau) = \frac{1}{2} \sum_{x \in L+h} [E_1(\sqrt{\nu}(x, c_1)) - E_1(\sqrt{\nu}(x, c_2))] q^{Q(x)}$$

(depth 1 Mass forms)

$$= \frac{1}{2} \sum_{x \in L+h} [\operatorname{sgn}(x, c_1) - \operatorname{sgn}(x, c_2)] q^{Q(x)}$$

$$+ \frac{1}{2\pi} \sum_{x \in L+h} [\operatorname{sgn}(x, c_2) \Gamma(\gamma_2, 2\pi\nu(x, c_2)^2) - \dots (c_1) - \dots - c_1] q^{Q(x)}$$

Holomorphic, what does it mean?

For positive x_1 , the geodesic D_x intersects S precisely if $\operatorname{sgn}(x, c_1) = -\operatorname{sgn}(x, c_2)$

\Rightarrow The term is the intersection number $I(S, D_x)$

For negative $Q(x) < 0$, it gives always zero

Γ if c_1, c_2 lie on \mathbb{R} , if they are also Γ -equivalent, S is a relative homology cycle in the symmetric space \Rightarrow we know that

$\sum I(S, D_x) q^{Q(x)}$ is modular \Rightarrow and form vanishes !!!

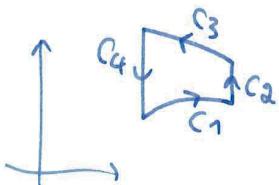
Example: Signature $(p, 2)$ T A B M P (p=1)

Considerations in II-B
string theory

$\bullet Q(c) = -1 < 0$ defines a geodesic in $D = H_+ \sqcup H_-$

\bullet Points in D are defined by oriented negative 2-frames $[c_1, c_2]$

\bullet Assume c_1, \dots, c_N fulfil a) $Q(c_i) = -1$ b) $[c_j, c_{j+1}] =: z_j$ lie in the same component of D



Theorem: let $E_2(C_1, C_2, x) = \int e^{2\pi i Q(Y - xz)} \operatorname{sgn}(C_1, Y) dy$

$[C_1, C_2] = z$

and let S denote the
2-cell with $\partial S = C_1 + C_2 + \dots + C_N$. Then

$$I(S, \tau) = \frac{1}{4} \sum_{x \in L+h} \left(\sum_{j=1}^N E_2(C_j, C_{j+1}, \sqrt{\nu} x) + 4w - N \right) q^{Q(\tau)}$$

"polygonal density"
= largest winding number in