# Conical sums and multiple polylogarithms

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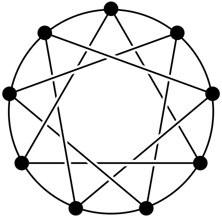

Multiple polylogarithms (MPL) are multivariate special functions,

$$\mathrm{Li}_{n_1,\dots,n_d}(z_1,\dots,z_d) = \sum_{1\le k_1<\dots<k_d} \frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}} \quad \text{indexed by} \quad \vec{n}\in\mathbb{N}^d.$$

### Example

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \int_{0<t_1<t_2<z} \frac{\mathrm{d}t_1\mathrm{d}t_2}{(1-t_1)t_2}$$

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This integral representation generalizes (hyperlogarithms) and the duality

nested sums $\longleftrightarrow$ iterated integrals

makes MPL amenable to powerful methods in symbolic summation and integration at the same time (many dedicated programs are available).

## Problem

In practice, MPL are often "hidden" in more complicated expressions.

MPL at $z_1 = \cdots = z_d = 1$ are called multiple zeta values (MZV)

$$\zeta(n_1, \ldots, n_d) = \sum_{1 \le k_1 < \cdots < k_d} \frac{1}{k_1^{n_1} \cdots k_d^{n_d}} \qquad \left( \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad \text{[Euler]} \right)$$

$$= \sum_{\vec{k} \in \mathbb{N}^d} \frac{1}{k_1^{n_1}(k_1 + k_2)^{n_2} \cdots (k_1 + \ldots + k_d)^{n_d}}$$

**Definition (Mordell-Tornheim '58)**

$$T(n_1, \ldots, n_d; s) := \sum_{\vec{k} \in \mathbb{N}^d} \frac{1}{k_1^{n_1} \cdots k_d^{n_d} (k_1 + \ldots + k_d)^s}$$

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**Theorem (Bradley & Zhou '12)**

*Every convergent $T(\vec{n};s)$ is a $\mathbb{Q}$-linear combination of MZV.*

## Definition (Matsumoto & Tsumura '06)

$$\zeta_{\mathfrak{sl}(d+1)}(\vec{n}) = \sum_{\vec{k}\in\mathbb{N}^d} \prod_{i\le j} \frac{1}{(k_i + \cdots + k_j)^{n_{ij}}}$$

## Example for $\zeta_{\mathfrak{sl}(4)}$

$$\sum_{k_1,k_2,k_3=1}^{\infty} \frac{1}{k_1 k_2 k_3 (k_1+k_2)^2 (k_2+k_3)(k_1+k_2+k_3)^2} = \frac{2}{875}\zeta^4(2) + \frac{1}{5}\zeta(3,5)$$

Many papers study (Witten-) zeta functions of other root systems, like

$$\zeta_{\mathfrak{so}(7)}(\vec{n}) = \sum_{a,b,c=1}^{\infty} \frac{1}{a^{n_1} b^{n_2} c^{n_3} (a+b)^{n_4} (b+c)^{n_5} (2b+c)^{n_6}} \times \frac{1}{(a+b+c)^{n_7} (a+2b+c)^{n_8} (2a+2b+c)^{n_9}}$$

$$\zeta_{\mathfrak{g}_2}(\vec{n}) = \sum_{a,b=1}^{\infty} \frac{1}{a^{n_1} b^{n_2} (a+b)^{n_3} (a+2b)^{n_4} (a+3b)^{n_5} (2a+3b)^{n_6}}$$

**Theorem (Zhao '09)**

*All convergent $\zeta_{\mathfrak{g}_2}(\vec{n})$ are $\mathbb{Q}$-linear combinations of MPL at twelfth roots of unity.*

## Definition (Dupont & Zerbini)

For an integer matrix $A = (a_{ij}) \in \mathbb{N}_0^{w \times d}$ consider

$$\zeta(A) := \sum_{\vec{k} \in \mathbb{N}^d} \frac{1}{\ell_1(\vec{k}) \cdots \ell_w(\vec{k})} \quad \text{where} \quad \ell_i(\vec{k}) = \sum_{j=1}^{d} a_{ij} k_j.$$

$$\zeta \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \sum_{a,b,c,d=1}^{\infty} \frac{1}{(a+b)(a+b+c)(a+c+d)(a+b+d)^2}$$

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$$= \frac{15}{32}\zeta(5) - \frac{9}{4}\zeta(2)\zeta(3) + \frac{9}{4}\ln(2)\zeta^2(2)$$

## Conjecture (Dupont)

*Let $N$ denote the least common multiple of all minors of $A$. Then $\zeta(A)$ is linear combination of MPL at $N$th roots of unity.*

# Some tools for symbolic summation

Specialized:

- Weinzierl, Symbolic expansion of transcendental functions, Comput. Phys. Commun., 2002, 145, 357-370
- Moch & Uwer, `-XSummer-` Transcendental functions and symbolic summation in Form, Comput. Phys. Commun., 2006, 174, 759-770

Much more general (including all $\zeta(A)$, in principle):

- Anzai & Sumino, Algorithms to evaluate multiple sums for loop computations, J. Math. Phys., 2013, 54, 033514
- Ablinger, Blümlein, De Freitas, Raab, Round, Schneider
  $\Rightarrow$ `Sigma`, `HarmonicSums`, `EvaluateMultiSums`, `SumProduction`

## Integral representation

$$\zeta(A) = \int_{[0,1]^w} \frac{y_1^{l_1-1}\cdots y_w^{l_w-1}\mathrm{d}^w\vec{y}}{\prod_{j=1}^d (1 - y_1^{a_{1j}}\cdots y_w^{a_{wj}})} \quad \text{with} \quad l_i := \sum_{j=1}^d a_{ij}$$

Proof: introduce a variable $y_i$ for each linear form $\ell_i$ to write

$$\frac{1}{\ell_i(\vec{k})} = \frac{1}{k_1 a_{i1} + \cdots + k_d a_{id}} = \int_0^1 \frac{\mathrm{d}y_i}{y_i}(y_i^{a_{i1}})^{k_1}\cdots(y_i^{a_{id}})^{k_d}$$

Then each summation $k_j \in \mathbb{N}$ becomes a geometric series.

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## Lemma (Zerbini)

*If the rows of $A \in \{0,1\}^{w\times d}$ can be permuted such that in each column, all 1's are consecutive, then $\zeta(A)$ is a $\mathbb{Q}$-linear combination of MZV.*

Proof: Integrand has denominators $1 - y_i y_{i+1}\cdots y_j$ (moduli space [Brown])

# HyperInt

The integrals over $y_i$ can be expressed as hyperlogarithms. Such integrals are called linearly reducible and can be computed in Maple with HyperInt (open source): https://bitbucket.org/PanzerErik/hyperint/

```
> read "HyperInt.mpl":
> hyperInt(log((1+z)/x)/((1+x)^2+y)/(1+x+z)/(y+z^2)/(1+z)
  ,{x,y,z}):
> fibrationBasis(%);
```

$$\frac{11}{4}\zeta(3) - 4\zeta(2) + \frac{3}{8}\zeta^2(2) + 3\ln(2)\zeta(2)$$

This computes the integral

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{\log((1+z)/x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{[(1+x)^2+y)(1+x+z)(y+z^2)(1+z)}$$

The integral representation for $\zeta(A)$ is suitable for `HyperInt`. Example:

$$\sum_{a,b,c,d=1}^{\infty} \frac{1}{a(a+b)(a+b+c)(a+b+c+d)^5(b+c+d)^3}$$

```
> ConicalSum([[1,0,0,0],[1,1,0,0],[1,1,1,0],[1,1,1,1]$5
  ,[0,1,1,1]$3]);
```

$$-\frac{1301}{210}\zeta_2^3\zeta_5 + \frac{2951}{200}\zeta_2^2\zeta_7 + \frac{23167}{72}\zeta_2\zeta_9 - \frac{521}{375}\zeta_2^4\zeta_3$$
$$-\frac{17}{5}\zeta_3\zeta_{3,5} + \frac{13}{2}\zeta_3^2\zeta_5 + \frac{32}{5}\zeta_{3,3,5} - \frac{21469}{40}\zeta_{11} - \frac{5}{6}\zeta_2\zeta_3^3$$

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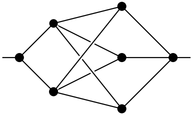
**Theorem**

*All $\zeta(A)$ are linear combinations of MPL at $N^{th}$ roots of unity. Their integral representation is linearly reducible and an upper bound on $N$ can be obtained from polynomial reduction [Brown].*

# More general sums from applications

- Mellin-Barnes representation [talk by De Freitas]
- Expansion in Gegenbauer polynomials [Broadhurst '85]



$$= 16 \sum_{h>a,b,c\geq 1} \frac{\mu(a,b,c;h)}{a^2 b^2 c^2 h^3} \left\{ 2 + 9\frac{a}{h} + 18\frac{ab}{h^2} + 15\frac{abc}{h^3} \right\}$$

$$= \frac{288}{5}\Big(58\zeta(8) - 45\zeta(3)\zeta(5) - 24\zeta(3,5)\Big) \quad \text{where}$$

$$\mu(a,b,c;h) = |2h-a-b-c| + |h-a-b-c| - |h-a-b| - |h-b-c| - |h-c-a|$$

- Superstring amplitudes [Zerbini]
- Runtime bound for the simplex algorithm on the Klee-Minty cube [Pemantle & Schneider]

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)} = 2\zeta(5) + 4\zeta(2)\zeta(3) - 2\zeta(3) - 4\zeta(2)$$

# Conical sums

We want to allow for linear constraints $a_i, b_i$ in the summations:

$$\sum_{k_1=a_1}^{b_1} \sum_{k_2=a_2(k_1)}^{b_2(k_2)} \cdots \sum_{k_d=a_d(k_1,\ldots,k_{d-1})}^{b_d(k_1,\ldots,k_{d-1})} \frac{1}{\ell_1(\vec{k}) \cdots \ell_w(\vec{k})}$$

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This is a special case of

### Definition (Terasoma '04)

The conical zeta value for a rational cone $C$ and linear forms $\ell_i$ is

$$\zeta(C; \ell_1, \ldots, \ell_w) := \sum_{\vec{k} \in \mathbb{Z}^d \cap \mathrm{int}(C)} \frac{1}{\ell_1(\vec{k}) \cdots \ell_w(\vec{k})}$$

$$C := \bigcap_{i=1}^{d} \{a_i \leq k_i \leq b_i\}$$

1. Rational cone: $C = \left\{\vec{x} = \lambda_1 v_1 + \ldots \lambda_m v_m : \ \vec{\lambda} \in \mathbb{R}_+^m\right\} \subset \mathbb{R}^d$, $v_i \in \mathbb{Z}^d$
   $\Leftrightarrow$ intersection of half-spaces

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## Sum over a unimodular cone

$$\sum_{\vec{x} \in \mathbb{Z}^d \cap \operatorname{int}(C)} \frac{1}{\ell_1(\vec{x}) \cdots \ell_w(\vec{x})} = \sum_{\vec{k} \in \mathbb{N}^d} \prod_{i=1}^{w} \frac{1}{\ell_i(k_1 v_1 + \ldots + k_d v_d)}$$

## Theorem (Terasoma '04)

*Let $C \subset \mathbb{R}^d$ be a rational cone, $\ell_i = b_i + \sum_j a_{ij} k_j$ affine forms and $\vec{z} \in \mathbb{C}^d$ variables such that*

$$\sum_{\vec{k} \in \mathbb{Z}^d \cap \mathrm{int}(C)} \frac{z_1^{k_1} \cdots z_d^{k_d}}{\ell_1(\vec{k}) \cdots \ell_w(\vec{k})}$$

*converges. Then it is a $\mathbb{Q}[e^{2\pi i/N}]$-linear combination of MPL with arguments of the form $z_1^{r_1} \cdots z_d^{r_d} \cdot e^{2\pi i m/N}$ for rational $r_i$ and some $N \in \mathbb{N}$.*

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## Open question

What is the minimal $N$? Is there a geometric interpretation?

# Summary

- conical sums are always MPL at $N^{\text{th}}$ roots of unity
- for unimodular cones, they are simple to evaluate
- complexity introduced by summation constraints can be separated into convex geometry (unimodular subdivision)
- optimal $N$ unclear

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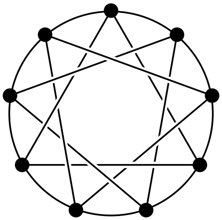
## Thanks

Thank you for your attention!



## Relations for special values

- datamine for MZV and alternating sums [Blümlein, Broadhurst & Vermaseren '10]
- datamine for arbitrary sixth roots up to weight 6 [Henn, Smirnov$^2$ '15]
- multiple Deligne values (MDV) [Broadhurst '15] (datamine for weight $\leq 11$ MDV)
- generalized parity theorem [Panzer '15]
- motivic decomposition algorithm [Brown '11]