Feynman integral relations from parametric annihilators

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Loops & Legs, St. Goar

joint work with Thomas Bitoun, Christian Bogner, René Pascal Klausen
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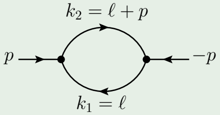
An integral family is defined by a set of denominators $\mathsf{D}_1, \ldots, \mathsf{D}_N$ that are quadratic (or linear) forms in loop momenta $\ell_1, \ldots, \ell_L$:

$$\mathcal{I}(\nu_1, \ldots, \nu_N; d) = \left(\prod_{k=1}^{L} \int \frac{\mathrm{d}^d \ell_k}{\mathrm{i}\pi^{d/2}} \right) \prod_{a=1}^{N} \mathsf{D}_a^{-\nu_a}$$

Example

$$\mathcal{I}(\nu_1, \nu_2; d) = \int \frac{\mathrm{d}^d \ell}{\mathrm{i}\pi^{d/2}} \frac{1}{(\ell^2)^{\nu_1}((\ell + p)^2)^{\nu_2}}$$



A family is also described by a matrix Λ, vectors Q_i and a scalar J such that

$$\sum_{a=1}^{N} x_a \mathsf{D}_a = -\sum_{i,j=1}^{L} \Lambda_{ij}(\ell_i \cdot \ell_j) + \sum_{i=1}^{L} 2(Q_i \cdot \ell_i) + J$$

Associated polynomials: $\mathcal{U} := \det \Lambda$, $\mathcal{F} := \mathcal{U} \left(Q^\intercal \Lambda^{-1} Q + J \right)$

In terms of $\omega := \nu_1 + \ldots + \nu_N - L\frac{d}{2}$ and $\mathcal{G} := \mathcal{U} + \mathcal{F}$ (Lee-Pomeransky),

$$\mathcal{I}(\nu) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)}\right) \mathcal{G}^{-d/2}$$

Example

$$\mathcal{I}(\nu_1, \nu_2) = \frac{\Gamma(\frac{d}{2})}{\Gamma(d - \nu_1 - \nu_2)} \int_0^\infty \frac{x_1^{\nu_1 - 1} \mathrm{d}x_1}{\Gamma(\nu_1)} \int_0^\infty \frac{x_2^{\nu_2 - 1} \mathrm{d}x_2}{\Gamma(\nu_2)} \Big(\underbrace{x_1 + x_2}_{\mathcal{U}} \underbrace{- p^2 x_1 x_2}_{\mathcal{F}}\Big)^{-\frac{d}{2}}$$

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The Mellin transform of a function $f : \mathbb{R}_+^N \longrightarrow \mathbb{C}$ is

$$\mathcal{M}\{f\}(\nu) := \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) f(x_1, \ldots, x_N),$$

whenever this integral exists. Special case:

$$\mathcal{I}(\nu) = \frac{\Gamma(d/2)}{\Gamma(d/2 - \omega)} \widetilde{\mathcal{I}}(\nu) \quad \text{for} \quad \widetilde{\mathcal{I}}(\nu) = \mathcal{M}\left\{ \mathcal{G}^{-d/2} \right\}(\nu).$$

Speer

- Such integrals converge in a non-empty, open domain wrt (d, ν).
- They have a unique, meromorphic extension to $\mathbb{C}^{1+N}$.
- The poles are simple and located on rational hyperplanes.

$\Rightarrow$ To prove relations between regularized Feynman integrals, we may assume convergent values of the parameters.

Example

$$\mathcal{I}(\nu_1, \nu_2) = (-p^2)^{d/2-\nu_1-\nu_2} \frac{\Gamma(d/2-\nu_1)\Gamma(d/2-\nu_2)\Gamma(\nu_1+\nu_2-d/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(d-\nu_1-\nu_2)}$$

Poles: $\{d/2-\nu_1=k\} \cup \{d/2-\nu_2=k\} \cup \{\nu_1+\nu_2-d/2=k\}$; $k \in \mathbb{Z}_{\leq 0}$

(If d is the only regulator ($\nu \in \mathbb{Z}^N$), poles coalesce and cease to be simple.)

Properties of the Mellin transform

1. $\mathcal{M}\{\alpha f + \beta g\}(\nu) = \alpha \mathcal{M}\{f\}(\nu) + \beta \mathcal{M}\{g\}(\nu) \qquad (\alpha, \beta \in \mathbb{C})$
2. $\mathcal{M}\{x_i f\}(\nu) = \nu_i \mathcal{M}\{f\}(\nu + \mathrm{e}_i)$

$$\int_0^\infty \frac{x_i^{\nu_i - 1}\mathrm{d}x_i}{\Gamma(\nu_i)}(x_i f) = \int_0^\infty \frac{\nu_i x_i^{\nu_i}\mathrm{d}x_i}{\nu_i \Gamma(\nu_i)} f = \int_0^\infty \frac{\nu_i x_i^{(\nu_i+1)-1}\mathrm{d}x_i}{\Gamma(\nu_i + 1)} f$$

3. $\mathcal{M}\{-\partial_i f\}(\nu) = \mathcal{M}\{f\}(\nu - \mathrm{e}_i)$

$$\int_0^\infty \frac{x_i^{\nu_i - 1}\mathrm{d}x_i}{\Gamma(\nu_i)}(-\partial_i f) = -\left[\frac{x_i^{\nu_i - 1}}{\Gamma(\nu_i)} f\right]_{x_i=0}^\infty + \int_0^\infty \frac{x_i^{\nu_i - 2}\mathrm{d}x_i}{\Gamma(\nu_i - 1)} f$$

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Shift operators:

$$(\mathbf{i}^- F)(\nu) := F(\nu - \mathrm{e}_i) \qquad (\mathbf{n}_i F)(\nu) = \nu_i F(\nu) \quad \text{for}$$

$$(\hat{\mathbf{i}}^+ F)(\nu) := \nu_i F(\nu + \mathrm{e}_i) \qquad \mathbf{n}_i := \hat{\mathbf{i}}^+ \mathbf{i}^-$$

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Given any differential operator $P \in A^N[d]$ in the Weyl algebra

$$A^N[d] := \mathbb{C}[d] \langle x_1, \dots, x_N, \partial_1, \dots, \partial_N \mid [\partial_i, x_j] = \delta_{i,j} \rangle$$

such that $P \bullet \mathcal{G}^{-d/2} = 0$ (annihilator), the substitutions

$$x_i \mapsto \hat{\mathbf{i}}^+, \quad \partial_i \mapsto -\mathbf{i}^-, \quad x_i\partial_i \mapsto -\mathbf{n}_i$$

define a shift operator $\mathcal{M}\{P\} \in S^N[d]$ in the shift algebra

$$S^N[d] := \mathbb{C}[d] \left\langle \hat{\mathbf{1}}^+, \dots, \hat{\mathbf{N}}^+, \mathbf{1}^-, \dots, \mathbf{N}^- \;\middle|\; [-\mathbf{j}^-, \hat{\mathbf{i}}^+] = \delta_{i,j} \right\rangle$$

such that $\mathcal{M}\{P\} \bullet \mathcal{M}\left\{\mathcal{G}^{-d/2}\right\} = \mathcal{M}\left\{P \bullet \mathcal{G}^{-d/2}\right\} = 0$ (relation).

Example ($\mathcal{G} = x_1 + x_2 - p^2 x_1 x_2$)

1. $\left[(-p^2)(-d/2 - x_1\partial_1 + 1)x_1 + (-d/2 - x_1\partial_1 - x_2\partial_2)\right] \bullet \mathcal{G}^{-d/2} = 0$
2. $(-p^2)(-d/2 + \mathbf{n}_1 + 1)\hat{\mathbf{1}}^+ \widetilde{\mathcal{I}} = -(-d/2 + \mathbf{n}_1 + \mathbf{n}_2)\widetilde{\mathcal{I}}$
3. $(-p^2)\nu_1 \widetilde{\mathcal{I}}(\nu_1 + 1, \nu_2) = -\dfrac{-d/2 + \nu_1 + \nu_2}{-d/2 + \nu_1 + 1} \widetilde{\mathcal{I}}(\nu_1, \nu_2)$

$$\mathcal{M}\{\cdot\}$$

$$\left\{P \in A^N[d]\colon \; P \bullet \mathcal{G}^{-d/2} = 0\right\} \qquad \left\{\tilde{P} \in S^N[d]\colon \; \tilde{P} \bullet \widetilde{\mathcal{I}} = 0\right\}$$

(annihilator) (shift relation)

$$\mathcal{M}^{-1}\{\cdot\}$$

The inverse Mellin transform of $f^{\star}(\nu) := \mathcal{M}\{f\}(\nu)$ is

$$f(x) = \mathcal{M}^{-1}\{f^{\star}\}(x) = \left(\prod_{i=1}^{N} \int_{\sigma_i + \mathrm{i}\mathbb{R}} \frac{\Gamma(\nu_i)\,\mathrm{d}\nu_i}{(2\pi\mathrm{i})\, x_i^{\nu_i}} \right) f^{\star}(\nu).$$

Therefore, **every** shift relation comes from an annihilator.

Open problems

A finite list of generators for all annihilators (i.e. IBP relations) can sometimes be computed with SINGULAR.

Question 1

Is the annihilator of $\mathcal{G}^{-d/2}$ linearly generated?

For a full set of ISPs, the action of $\dfrac{\partial}{\partial q_i} \cdot q_j$ on the *momentum space* integrand leads to IBP relations that map to linear annihilators $\widetilde{\mathbf{O}}^i_j$.

Question 2

Do the momentum space IBP's generate all annihilators (relations)?

No counterexamples found, but only few cases tested.

We define the number of master integrals of $\widetilde{\mathcal{I}}(\nu) = \mathcal{M}\left\{\mathcal{G}^{-d/2}\right\}(\nu)$ as

$$\mathfrak{C}\left(\mathcal{G}\right) := \dim_{\mathbb{C}(d,\nu)}\left(\sum_{n\in\mathbb{Z}^N} \mathbb{C}(d,\nu)\cdot\widetilde{\mathcal{I}}(\nu+n)\right)$$

We define the number of master integrals of $\widetilde{\mathcal{I}}(\nu) = \mathcal{M}\left\{\mathcal{G}^{-d/2}\right\}(\nu)$ as

$$\mathfrak{C}(\mathcal{G}) := \dim_{\mathbb{C}(d,\nu)} \left(\sum_{n \in \mathbb{Z}^N} \mathbb{C}(d,\nu) \cdot \widetilde{\mathcal{I}}(\nu + n) \right)$$

- $\mathfrak{C}(\mathcal{G}) \geq \dim_{\mathbb{C}(d)} \left(\sum_{n \in \mathbb{Z}^N} \mathbb{C}(d) \cdot \widetilde{\mathcal{I}}(n) \right)$
- no symmetries
- not "modulo subtopologies"
- **exactly computable!**

Using the Mellin transform, $\theta_i := x_i \partial_i = \mathcal{M}^{-1}\{\mathbf{n}_i\}$,

$$\mathfrak{C}(\mathcal{G}) = \dim_{\mathbb{C}(d,\theta)} \left(\underbrace{\mathbb{C}(d,\theta) \otimes_{\mathbb{C}[d,\theta]} A^N[d] \mathcal{G}^{-d/2}}_{\mathfrak{M}} \right)$$

Here, $A^N(d)\mathcal{G}^{-d/2}$ is a holonomic D-module, and $\mathfrak{M}$ is a holonomic system of finite difference equations [Loeser & Sabbah '91].

Theorem

$$(-1)^N \mathfrak{C}(\mathcal{G}) = \chi\left(\mathbb{C}^N \setminus \{x_1 \cdots x_N \mathcal{G} = 0\}\right) = \chi\left((\mathbb{C}^*)^N \setminus \{\mathcal{G} = 0\}\right)$$

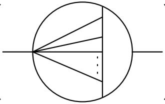
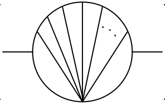
$\Rightarrow$ implies finiteness [Smirnov & Petukhov]

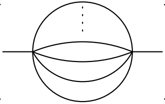
The Euler characteristic $\chi(X) = \sum_i (-1)^i \dim H^i(X)$ is a fundamental invariant and can be computed with many different tools, for example:

- $\chi(X) = \chi(X \setminus Z) + \chi(Z)$
- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$
- $\chi(E) = \chi(B) \cdot \chi(F)$ for fibrations $F \to E \to B$
- D-modules and Groebner bases (e.g. SINGULAR) [Oaku & Takayama]
- algorithms by M. Helmer (`CharacteristicClasses` in *Macaulay2*)
- Kouchnirenko/Khovanskii's theorem: For *non-degenerate* $\mathcal{G}$,

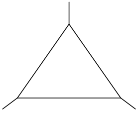
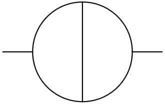
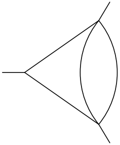
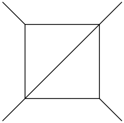
$$\mathfrak{C}(\mathcal{G}) = N! \cdot \operatorname{Vol} \operatorname{NP}(\mathcal{G})$$

For some infinite families one can prove explicit formulas:

$\mathfrak{C}\Big($  $\Big) = \mathfrak{C}\Big($  $\Big) = \dfrac{L(L+1)}{2}$

$\mathfrak{C}\Big($  $\Big) = 2^{L+1} - 1$ [Kalmykov & Kniehl]

Plenty of further computations agreed with predictions by AZURITE, e.g.

Graph G				
$\mathfrak{C}(G)$ massless	4	3	4	20
$\mathfrak{C}(G)$ massive	7	30	19	55

Massive one-loop sunrise

$$\mathcal{U} = x_1 + x_2 \qquad \mathcal{F} = (x_1 + x_2)^2 + x_1 x_2$$

In *Macaulay2*, the Euler characteristic $\mathfrak{C}(\mathcal{G}) = 3$ can be computed with

```
load "CharacteristicClasses.m2"
R=QQ[x0,x1,x2]
I=ideal(x0*x1*x2*((x1+x2)*x0+(x1+x2)^2+x1*x2))
Euler(I)
```

The individual cohomology groups can also be obtained with

```
load "Dmodules.m2"
R=QQ[x1,x2]
f=x1*x2*(x1+x2+(x1+x2)^2+x1*x2)
deRham f
```

$\Rightarrow H^0(X) \cong \mathbb{Q}, \quad H^1(X) \cong \mathbb{Q}^3, \quad H^2(X) \cong \mathbb{Q}^5 \quad \Rightarrow \chi(X) = 5 - 3 + 1 = 3$

The same can be done in SINGULAR.

Thanks

Thank you for your attention!

- The Mellin transform translates IBP relations to annihilators [Tkachov, Baikov, Lee, Pomeransky].
- Algorithms for computations with D-modules are available.
- Application: The number of master integrals, for free ν's, is

$$\mathfrak{C}(\mathcal{G}) = (-1)^N \chi((\mathbb{C}^*)^N \setminus \{\mathcal{G} = 0\}) < \infty$$

- Goal: Extend IBP reduction from $\nu \in \mathbb{Z}^N$ to free ν.

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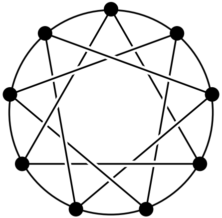


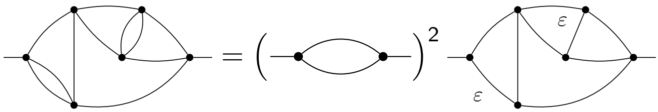
Parametric representations

$$\omega := \nu_1 + \cdots + \nu_N - L\frac{d}{2}$$

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}},$$

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \Gamma(\omega) \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) \frac{\delta\left(1 - \sum_{j=1}^{N} x_j\right)}{\mathcal{U}^{d/2-\omega} \mathcal{F}^{\omega}}$$

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) \mathcal{G}^{-d/2}.$$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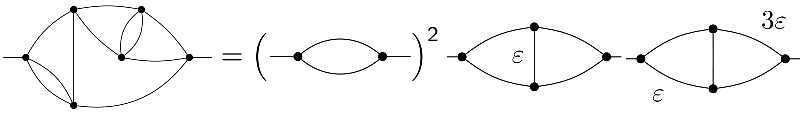
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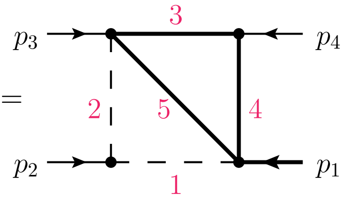
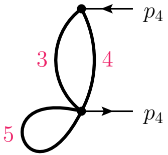
Miscounting in Azurite

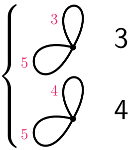
$G =$



$G/\{1,2\} =$



$\mapsto$



$G' =$