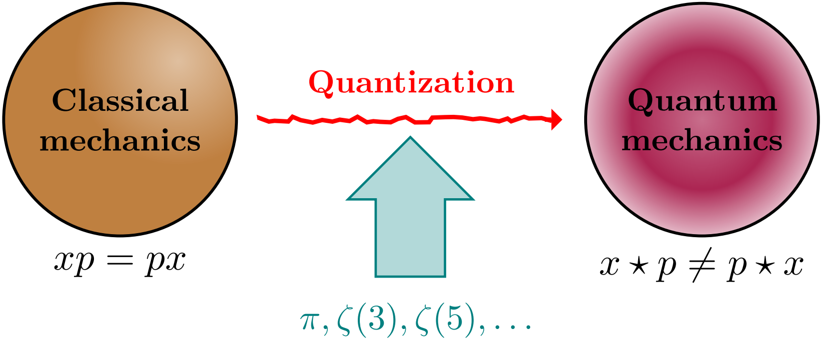
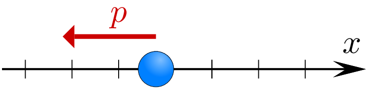


joint work with **Peter Banks** and **Brent Pym**

Erik Panzer 10 August 2021 Royal Society UK

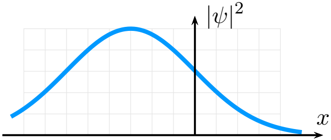
Classical mechanics

- functions $f \in \mathcal{C}^\infty(\mathbb{R}^2)$
- $\dfrac{\partial}{\partial t} f = \{f, H\}$
- $\{x, p\} = 1$



Quantum mechanics

- operators $\widehat{f}$ on $L^2(\mathbb{R}^2)$
- $\mathrm{i}\hbar \dfrac{\partial}{\partial t} \langle \widehat{f} \rangle = \langle [\widehat{f}, \widehat{H}] \rangle$
- $[\widehat{x}, \widehat{p}] = \mathrm{i}\hbar$



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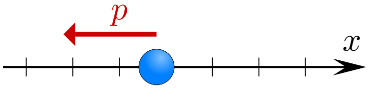

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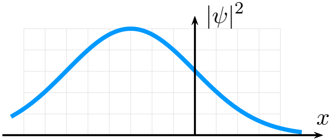

Canonical quantization (Dirac, Groenewold & Moyal)

$$[\widehat{f}, \widehat{g}] = i\hbar \widehat{\{f, g\}} + \mathcal{O}(\hbar^2)$$

- commutative $fg = gf$
- **star product**

$$f \star g$$

- commutative limit $\hbar \to 0$
- non-commutative

$$\widehat{f}\,\widehat{g} = \widehat{f \star g}$$

Given: **any** Poisson bracket on $\mathcal{C}^\infty(\mathbb{R}^d)$:

1. Odd: $\{f, g\} = -\{g, f\}$
2. Jacobi: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
3. Leibniz: $\{fg, h\} = f\{g, h\} + g\{f, h\}$

$\downarrow$

Deformation quantization

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Deformation quantization

Problem: Find product $\star$ on $\mathcal{C}^\infty(\mathbb{R}^d)[[\hbar]]$ with

1. $f\star(g\star h) = (f\star g)\star h$
2. $f\star g = fg + \sum_{n=1}^{\infty} (\mathrm{i}\hbar)^n B_n(f,g)$
3. $f\star g - g\star f = \mathrm{i}\hbar\{f,g\} + \mathcal{O}(\hbar^2)$

[Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer]

Every Possion bracket admits a $\star$ product!

Kontsevich's universal formula

$$B_n(f,g) = \frac{1}{n!}\sum_G w_G \cdot B_G(f,g)$$

real number   differential operator

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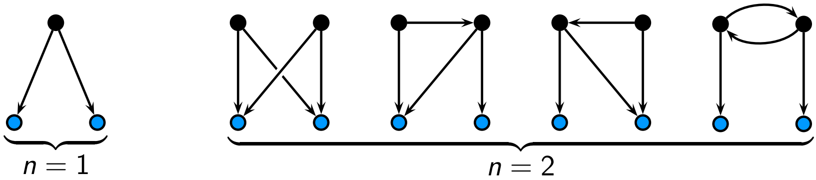
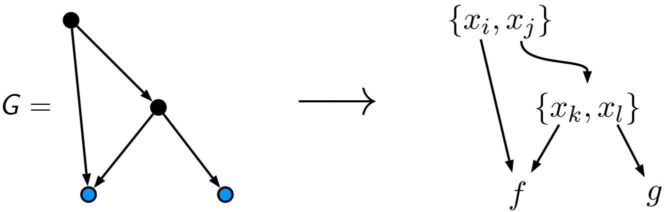
Graphs G:

- $\{1,\ldots,n\}$: 2 edges going out
- $\{L,R\}$: nothing going out



Operator $B_G(f, g)$



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- derivatives $\partial_j = \dfrac{\partial}{\partial x_j}$

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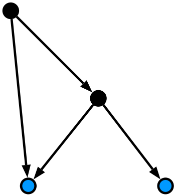
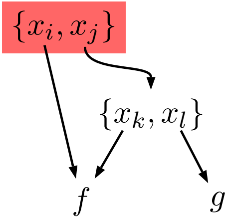
$G =$  $\longrightarrow$ 

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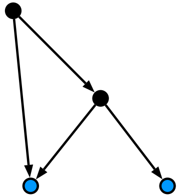
$G =$ 

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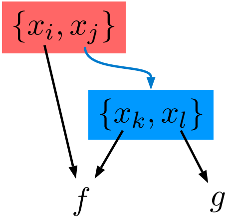

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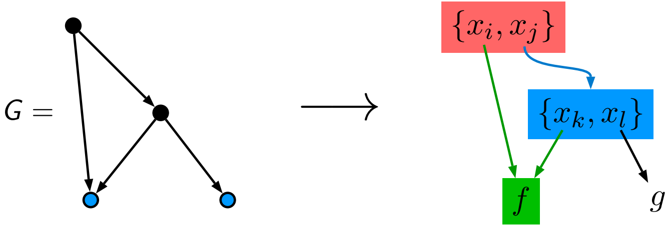

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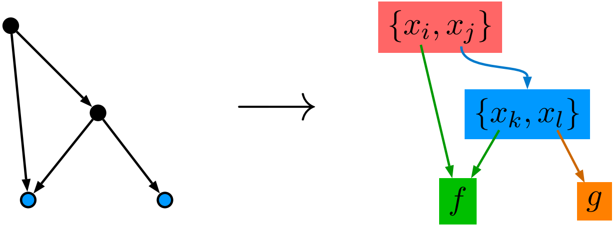

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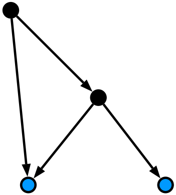

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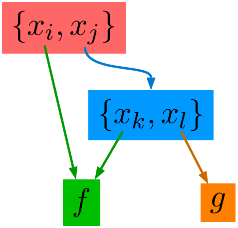

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Weight w_G

- coordinates at each vertex:

$$z_L = 0 \qquad z_R = 1 \qquad z_1, \dots, z_n \in \mathbb{H} = \{x + \mathrm{i}y : y > 0\}$$

- differentials for each edge $i \to j$: (harmonic propagator)

$$\mathrm{d}\phi_{i \to j} = \frac{1}{2\pi} \mathrm{d}\arg \frac{z_i - z_j}{\bar{z}_i - z_j}$$

- n vertices $\Rightarrow$ edges $e_1, \dots, e_{2n}$

$$w_G = \int_{\mathbb{H}^n} \mathrm{d}\phi_{e_1} \wedge \ldots \wedge \mathrm{d}\phi_{e_{2n}} \in \mathbb{R}$$

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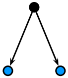
- or log propagator: [Alekseev–Rossi–Torossian–Willwacher]

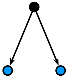
$$\mathrm{d}\phi_{i\to j} = \frac{1}{2\mathrm{i}\pi}\mathrm{d}\log \frac{z_i - z_j}{\bar{z}_i - z_j}$$

$G =$ 

$$\alpha = \phi_{1\to L}\, \mathrm{d}\phi_{1\to R} = \frac{1}{(2\mathrm{i}\pi)^2} \log\frac{z}{\bar{z}}\ \mathrm{d}\log\frac{z-1}{\bar{z}-1}$$

Integral:

$$w_G = \int_{\mathbb{H}} \mathrm{d}\alpha$$

$G =$ 

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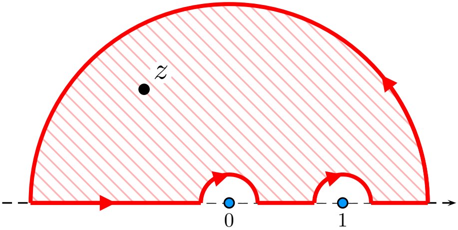
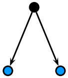
$= \lim \int$ $\mathrm{d}\alpha = \lim \int$  α



$G =$ 

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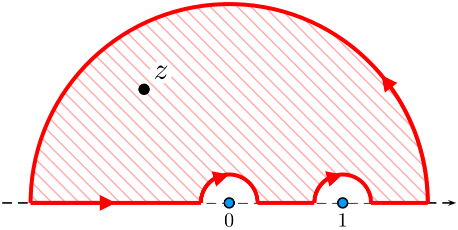
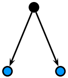


$$G = $$ 

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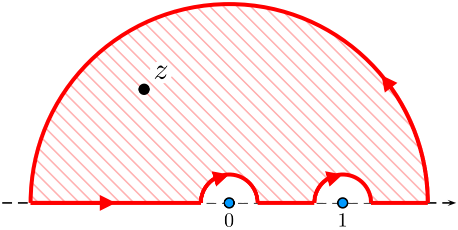

First term in Kontsevich's formula:

$$B_1(f,g) = w_G B_G(f,g) = \frac{1}{2}\sum_{i,j}\{x_i,x_j\}(\partial_i f)(\partial_j g) = \frac{1}{2}\{f,g\}$$

$$f \star g = fg + \tfrac{1}{2}\mathrm{i}\hbar\{f,g\} + \dots$$

Generalizations:

- Graphs with $n + m$ vertices $\bullet^n \sqcup \circ^m$ (previously: $m = 2$)
- Configuration space of marked discs:

$$C_{n,m} = \left(\mathbb{H}^n \setminus \bigcup_{i<j} \{z_i = z_j\}\right) \times \{q_1 < \cdots < q_m\} \Big/ \mathbb{R} \rtimes \mathbb{R}^\times$$

- $\dim(C_{n,m}) = 2n + m - 2$
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- Integrands:

$$\omega_G = \bigwedge_{\text{edges } e} \mathrm{d}\phi_e \quad \in \quad \Omega^\bullet(C_{n,m})$$

- Forgetful maps: $C_{n+r,m+s} \longrightarrow C_{n,m}$
- Integration over fibres:

$$\int_{C_{n+r,m+s} \longrightarrow C_{n,m}} \omega_G \quad \in \quad \Omega^\bullet(C_{n,m})$$

1. Star product:

$$f \star g = fg + \sum_{n,G} \frac{\hbar^n}{n!} \left(\int_{C_{n,2}} \omega_G \right) B_G(f, g)$$

2. Formality L_∞ morphism $\mathcal{U}\colon\ T_{\mathrm{poly}}(\mathbb{R}^d) \longrightarrow D_{\mathrm{poly}}(\mathbb{R}^d)$ [Kontsevich]

$$\mathcal{U}_n(\gamma_1, \ldots, \gamma_n) = \sum_{m,G} \left(\int_{C_{n,m}} \omega_G \right) B_G(\gamma_1, \ldots, \gamma_n)$$

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3. Lift to homotopy Gerstenhaber formality [Willwacher]

$$\mathrm{Graphs}_{n,m} \longrightarrow \Omega^\bullet(\overline{C}_{n,m}), \qquad G \mapsto \int_{C_{n+r,m+s} \longrightarrow C_{n,m}} \omega_G$$

4. Formality of little discs D_2 [Kontsevich, Lambrechts & Volić]

$$\mathrm{Graphs}_n \longrightarrow \Omega^\bullet(\overline{C}_n), \qquad G \mapsto \int_{C_{n+r} \longrightarrow C_n} \omega_G$$

What are these integrals?

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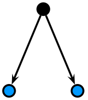
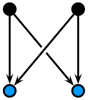
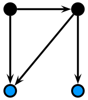
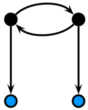
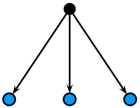
[Special cases: Kontsevich, Merkulov, Kathotia, Ben Amar, Van den Bergh, Buring, Kiselev, . . .]

Graph G					
weight w_G	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{6}$

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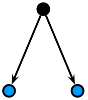
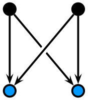
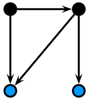
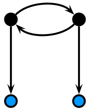
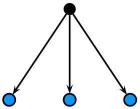
Graph G					
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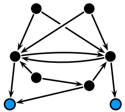
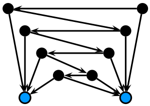
Not always rational: [Felder & Willwacher]

Graph G		
weight w_G	$\frac{1}{1920} - \frac{17}{64}\frac{\zeta(3)^2}{\pi^6}$	$\frac{22427}{1990656000} + \frac{65\zeta(3)\zeta(5)}{1024\pi^8} + \frac{417\zeta(3,5)}{10240\pi^8}$

Zeta values:

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} \qquad \zeta(5) = \sum_{k=1}^{\infty} \frac{1}{k^5} \qquad \zeta(3,5) = \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \frac{1}{k^3 l^5}$$

Transcendence conjecture

$\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent. $\Rightarrow$ $\zeta(3)^2/\pi^6 \notin \mathbb{Q}$

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Multiple zeta values (MZV): [Écalle, Zagier, ...] ("weight")

- $$\zeta(s_1, \ldots, s_d) = \sum_{1 \leq k_1 < \cdots < k_d} \frac{1}{k_1^{s_1} \cdots k_d^{s_d}} \qquad |s| = s_1 + \ldots + s_d$$
- many relations: $\zeta(2) = \pi^2/6$, $\zeta(1,2) = \zeta(3)$, [Euler,...]

$$\zeta(1,1,2) = (4/3)\zeta(2,2) = 4\zeta(1,3) = \zeta(4) = \pi^4/90$$

Theorem & Algorithm (Banks–Panzer–Pym)

$$w_G \in \mathcal{Z}_{n+m-2} \quad \text{where} \quad \mathcal{Z}_w := \sum_{|s| \leq w} \frac{\zeta(s)}{(\mathrm{i}\pi)^{|s|}} \mathbb{Q}$$

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- $\zeta(s_1, \ldots, s_d) = \sum_{1 \le k_1 < \cdots < k_d} \frac{1}{k_1^{s_1} \cdots k_d^{s_d}}$ $\qquad |s| = s_1 + \ldots + s_d$
- many relations: $\zeta(2) = \pi^2/6$, $\zeta(1,2) = \zeta(3)$, [Euler,…]

$$\zeta(1,1,2) = (4/3)\zeta(2,2) = 4\zeta(1,3) = \zeta(4) = \pi^4/90$$

Theorem & Algorithm (Banks–Panzer–Pym)

$$w_G \in \mathcal{Z}_{n+m-2} \quad \text{where} \quad \mathcal{Z}_w := \sum_{|s| \le w} \frac{\zeta(s)}{(\mathrm{i}\pi)^{|s|}} \mathbb{Q}$$

n	0	3	5	6	7	8
$\Re \mathcal{Z}_n$	1			$\frac{\zeta(3)^2}{\pi^6}$		$\frac{\zeta(3)\zeta(5)}{\pi^8}, \frac{\zeta(3,5)}{\pi^8}$
$\Im \mathcal{Z}_n$		$\frac{\zeta(3)}{\mathrm{i}\pi^3}$	$\frac{\zeta(5)}{\mathrm{i}\pi^5}$		$\frac{\zeta(7)}{\mathrm{i}\pi^7}$	

$\Rightarrow$ $w_G \in \mathbb{Q}$ for $\hbar^{\le 5}$ in $f \star g$

Software (on BitBucket):

- computation of weights w_G
- expansion of $\star$-products

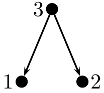
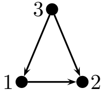
Example: log-canonical bracket

- $\mathbb{R}^2 = \{(x, y)\}$ with bracket $\{x, y\} = xy$
- Kontsevich $\star$-product:

$$x \star y = xy\left(1 + \frac{\hbar}{2} + \frac{\hbar^2}{24} - \frac{\hbar^3}{48} - \frac{\hbar^4}{1440} + \frac{\hbar^5}{480} \right.$$
$$\left. + \left[\frac{251\zeta(3)^2}{2048\pi^6} - \frac{17}{184320}\right]\hbar^6 + \cdots \right)$$

- Transcendentals do not cancel.
- Nevertheless, $x \star y = e^{\hbar} y \star x$.

Some non-constant fibre integrals:

G	$\displaystyle\int_{C_{n,m}\longrightarrow C_{2,0}} \omega_G$
	$\dfrac{1}{2\mathrm{i}\pi} \log \dfrac{z_1 - \bar{z}_2}{z_2 - \bar{z}_1}$
	$\dfrac{1}{2\mathrm{i}\pi} \left(\log \dfrac{z_1 - \bar{z}_2}{z_2 - \bar{z}_1} \right) \dfrac{1}{4\mathrm{i}\pi} \mathrm{d} \log \dfrac{(z_1 - z_2)(z_1 - \bar{z}_2)}{(\bar{z}_1 - z_2)(\bar{z}_1 - \bar{z}_2)}$

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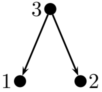
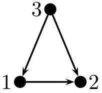
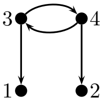
	$\frac{1}{2\mathrm{i}\pi} \log \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_1}$
	$\frac{1}{2\mathrm{i}\pi} \left(\log \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_1} \right) \frac{1}{4\mathrm{i}\pi} \mathrm{d} \log \frac{(z_1 - z_2)(z_1 - \bar{z}_2)}{(\bar{z}_1 - z_2)(\bar{z}_1 - \bar{z}_2)}$
	$\frac{3}{4} - \left(\frac{1}{4\mathrm{i}\pi} \log \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_1} \right)^2 - \frac{1}{8\pi^2} \mathrm{Li}_2 \left(1 - \frac{(z_1 - \bar{z}_1)(z_2 - \bar{z}_2)}{(z_1 - \bar{z}_2)(\bar{z}_1 - z_2)} \right)$

Polylogarithms:

$$\mathrm{Li}_1(z) = -\log(1-z) = \int_0^z \frac{\mathrm{d}t}{1-t}$$

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{s-1}(t)$$

- Rational forms on $\mathrm{Conf}_{2n+m}(\mathbb{C})$: [Arnol'd]

$$\mathcal{A}^\bullet(C_{n,m}) = \mathbb{Q}\begin{bmatrix} \mathrm{d}\log(z_i - z_j), \mathrm{d}\log(z_i - \bar{z}_j), \\ \mathrm{d}\log(z_i - q_j), \mathrm{d}\log(\bar{z}_i - q_j) \end{bmatrix} \subset \Omega^\bullet(C_{n,m})$$

- double interior points to punctured curve

$$\mathbb{C} \setminus \Sigma \quad \text{where} \quad \Sigma = \{z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n, q_1, \ldots, q_m\}$$

- holomorphic hyperlogarithms [Lappo-Danilevskyy, Goncharov, Brown]

$$\mathcal{V}_{n,m} = \mathbb{Q}\langle L_w(\sigma)\colon \sigma w \in \Sigma^*\rangle$$

- single-valued real analytic functions:

$$\iota^*(\mathcal{V}_{n,m}) \subset \Omega^0(C_{n,m})$$

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Single-valued integration [O. Schnetz]

1. Find a primitive:

$$\int_{\mathbb{C}} \frac{\mathrm{d}\bar{z}}{\bar{z}} \wedge \left(\frac{\mathrm{d}z}{z-1} - \frac{\mathrm{d}z}{z-p} \right) = \int_{\mathbb{C}} \mathrm{d} \left[\log \bar{z} \left(\frac{\mathrm{d}z}{z-1} - \frac{\mathrm{d}z}{z-p} \right) \right]$$

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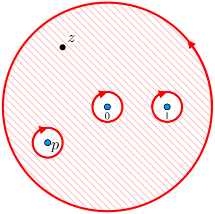
2. Make it single-valued: [F. Brown]

$$= \int_{\mathbb{C}} \mathrm{d} \left[(\log \bar{z} + \log z) \left(\frac{\mathrm{d}z}{z-1} - \frac{\mathrm{d}z}{z-p} \right) \right]$$



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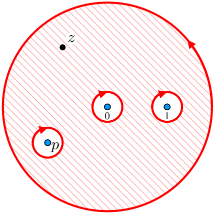

3. Truncate $\mathbb{C} \setminus \{0, 1, p\} = \bigcup_{\epsilon > 0}$, apply Stokes:

$$= \sum_{w \in \{0,1,p,\infty\}} \lim_{\epsilon \to 0} \pm \oint_{|z-w|=\epsilon} \log |z|^2 \left(\frac{\mathrm{d}z}{z-1} - \frac{\mathrm{d}z}{z-p}\right)$$

$$= 2\mathrm{i}\pi \log |p|^2$$

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$$L_{\sigma w}(z) = \int_0^z \frac{\mathrm{d}t}{t-\sigma} L_w(t) \quad \Rightarrow \quad \mathrm{d}L_{\sigma w}(z) = \frac{\mathrm{d}z}{z-\sigma} L_w(z)$$

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$$\mathscr{L}_w(z) = \sum_{u,v} c_{u,v} L_u(z) \overline{L_v(z)} \quad \text{s.t.} \quad \partial \mathscr{L}_{\sigma w}(z) = \frac{\mathrm{d}z}{z-\sigma} \mathscr{L}_w(z)$$

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Single-valued integration on $\mathbb{C}^n$

$$f \in \mathbb{Q}\left[z_i, \bar{z}_i, \frac{1}{z_i - z_j}, \frac{1}{\bar{z}_i - \bar{z}_j}\right] \quad \Rightarrow \quad \int_{\mathbb{C}^n} f \bigwedge_i \mathrm{d}^2 z_i \in (2\mathrm{i}\pi)^n \mathcal{Z}^{\mathrm{sv}}$$

$$\int_{\mathbb{C}} \mathrm{d}z \wedge \mathrm{d}\bar{z} \quad \Rightarrow \quad 2\mathrm{i}\pi \cdot \int_0^1 \mathrm{d}z$$

Generalization to integrals over $\mathbb{H}^n$:

- The integrand $\omega = \bigwedge_e \mathrm{d}\phi_e$ is a rational form on

$$\mathfrak{M}_{0,2n+3} \cong (\mathbb{C} \setminus \{0,1\})^{2n} \setminus \bigcup_{i,j} (\{z_i = z_j\} \cup \{\bar{z}_i = \bar{z}_j\} \cup \{z_i = \bar{z}_j\})$$

- Integrate one variable at a time:

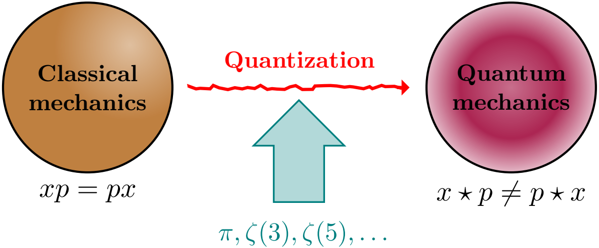
$$\mathfrak{M}_{0,2n+3} \longrightarrow \mathfrak{M}_{0,2n+1} \longrightarrow \cdots \longrightarrow \mathfrak{M}_{0,3}$$

- Find primitive α of $\omega = \mathrm{d}\alpha$ with multiple polylogarithms on $\mathfrak{M}_{0,2n+3}$
- Make α single-valued (cancel unipotent monodromy)
- Apply regularized Stokes' theorem (new boundary $\int_{\mathbb{R}}$ & corners)
- Repeat until $n = 0$, leftover polylogs on $\mathfrak{M}_{0,3} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$:

$$L_{0^{[s_d-1]}1 \ldots 0^{[s_1-1]}1}(1) = (-1)^d \zeta(s_1, \ldots, s_d)$$

Summary

1. Quantization:



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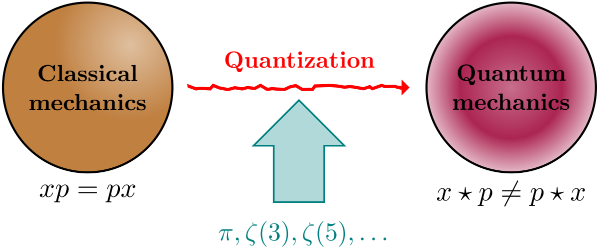

2. Every Poisson bracket on $\mathbb{R}^d$ can be quantized: (Kontsevich)

$$f \star g = fg + \sum_{n=1}^{\infty} \frac{(i\hbar)^n}{n!} \sum_G w_G B_G(f, g)$$

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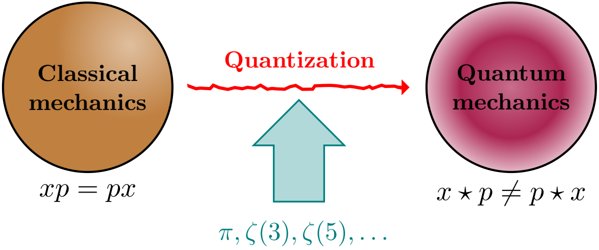

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3. weights $w_G \in \sum_s \frac{\zeta(s)}{(\mathrm{i}\pi)^{|s|}} \mathbb{Q}$ multiple zeta values

Next steps

1. Replace Axelrod–Singer/Fulton–MacPherson operad $\left\{\overline{C}_{n,m}\right\}$ by algebraic varieties with mixed Tate cohomology
2. Define motivic lifts $w_G^{\mathfrak{m}} \in \mathcal{O}(\mathsf{Isom}^{\otimes}(\omega_{\mathrm{B}}, \omega_{\mathrm{dR}}))$
3. ⇒ action of the motivic Galois (Tannaka) group of $\mathcal{MT}(\mathbb{Z})$
4. Compare with GRT action on stable formality morphisms
5. Generalize disc $\{z \in \mathbb{C} : \ |z| \leq 1\}$ to higher genus surface

Poisson σ-model: [Cattaneo & Felder]

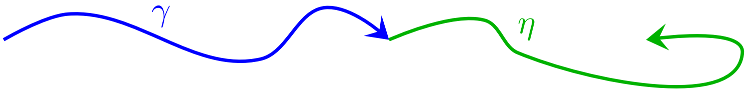
$$(f \star g)(x) = \int_{X(\infty)=x} f(X(1))g(X(0)) \exp\left(\frac{\mathrm{i}}{\hbar} S[X,\eta]\right) [\mathscr{D}X][\mathscr{D}\eta]$$

- Poisson manifold (M, π)
- $X \colon \mathbb{H} \longrightarrow M$
- η is a section of $T^*\mathbb{H} \otimes X^{-1}(T^*M)$

$$S[X,\eta] = \int_{\mathbb{H}} \left[\eta_i(u) \wedge \mathrm{d}X^i(u) + \frac{1}{2}\pi^{ij}(X(u))\eta_i(u) \wedge \eta_j(u)\right]$$

Path concatenation

Let $\gamma \star \eta$ denote the concatenation of γ and η at $\gamma(1) = \eta(0) = (\gamma \star \eta)(\frac{1}{2})$:



To decompose

$$\int_{\gamma\star\eta} \omega_2\omega_1 = \int_{0\le t_1\le t_2\le 1} (\gamma \star \eta)^*(w_2)(t_2)(\gamma \star \eta)^*(w_1)(t_1),$$

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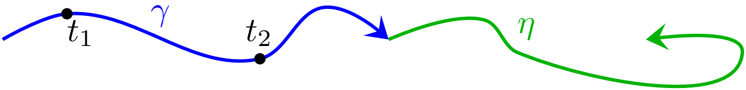

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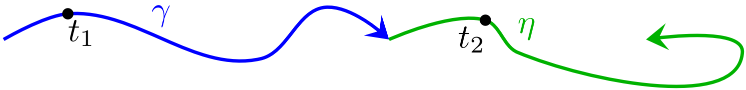

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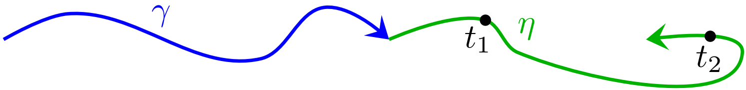

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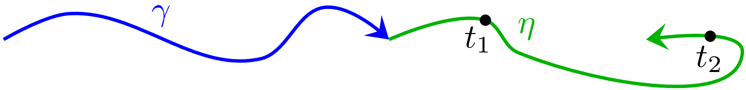

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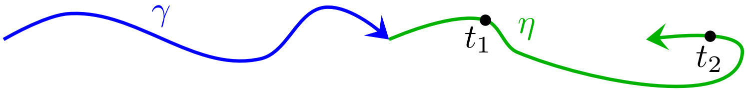

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More generally, the path concatenation formula reads

$$\int_{\gamma\star\eta} \omega_r \cdots \omega_1 = \sum_{k=0}^{r} \int_\eta \omega_r \cdots \omega_{k+1} \int_\gamma \omega_k \cdots \omega_1.$$

Shuffle product

The shuffle product of two words

$$w_{n+m} \cdots w_{n+1} \sqcup\!\sqcup\, w_n \cdots w_1 = \sum_{\sigma} w_{\sigma(n+m)} \cdots w_{\sigma(1)}$$

is the sum of all their shuffles σ, i.e. permutations which preserve the relative order of letters in both factors:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(n) \quad \text{and} \quad \sigma^{-1}(n+1) < \cdots < \sigma^{-1}(n+m).$$

For arbitrary words u and v, we find that ($\int_\gamma$ is linearly extended)

$$\left(\int_\gamma u\right) \cdot \left(\int_\gamma v\right) = \int_\gamma (u \sqcup\!\sqcup\, v).$$

Example

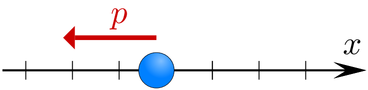
$$\int_\gamma \omega_3 \cdot \int_\gamma \omega_2\omega_1 = \int_\gamma (\omega_3\omega_2\omega_1 + \omega_2\omega_3\omega_1 + \omega_2\omega_1\omega_3)$$

$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

Classical mechanics

States: phase space $\{(x,p)\} = \mathbb{R}^2$

- position x
- momentum p



Observables: functions $f \in \mathcal{C}^\infty(\mathbb{R}^2)$

- x, p
- $H(x,p) = \frac{p^2}{2m} + V(x)$

Poisson bracket:

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}$$

Equations of motion: $x(t)$, $p(t)$ **(Hamilton 1833)**

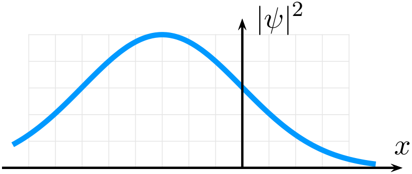
$$\left.\begin{aligned} \frac{\partial}{\partial t}x &= \frac{p}{m} = \frac{\partial H}{\partial p} \\ \frac{\partial}{\partial t}p &= -V'(x) = -\frac{\partial H}{\partial x} \end{aligned}\right\} \quad \frac{\partial}{\partial t}f = \{f,H\}$$

Canonical brackets: $\{x,x\} = \{p,p\} = 0$, $\{x,p\} = 1$

Quantum mechanics

States: Hilbert space $L^2(\mathbb{R})$ of wave functions

$$\psi\colon \mathbb{R} \longrightarrow \mathbb{C}$$



Observables: operators

- $\widehat{x}\psi(x) = x\psi(x)$
- $\widehat{p}\psi(x) = -\mathrm{i}\hbar\frac{\partial}{\partial x}\psi(x)$

Expectation value:

$$\langle \widehat{x} \rangle = \int_{\mathbb{R}} \overline{\psi}(x) \cdot x \cdot \psi(x)\, \mathrm{d}x$$

Equations of motion: $\psi(x, t)$ **(Schrödinger 1925)**

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = \widehat{H}\psi \quad \Rightarrow \quad \mathrm{i}\hbar\frac{\partial}{\partial t}\langle \widehat{f} \rangle = \langle \widehat{f}\widehat{H} - \widehat{H}\widehat{f} \rangle$$

Canonical commutators: $[\widehat{x}, \widehat{x}] = [\widehat{p}, \widehat{p}] = 0$, $[\widehat{x}, \widehat{p}] = \mathrm{i}\hbar$