HUMBOLDT-UNIVERSITÄT ZU BERLIN

Analytic regularization of divergent integrals and hyperlogarithmic integration

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Outline

1. The hyperlogarithm method of integration
2. Examples of linearly reducible Feynman integrals
3. Divergences and analytic regularization

Motivation: Feynman integrals in Schwinger parameters

Scalar propagators $(p_e^2 + m_e^2)^{-a_e}$, $\mathrm{sdd} = \sum_e a_e - D/2 \cdot \mathrm{loops}(G)$:

$$\Phi(G) = \frac{\Gamma(\mathrm{sdd})}{\prod_e \Gamma(a_e)} \int_0^\infty \psi^{\mathrm{sdd} - D/2} \cdot \varphi^{-\mathrm{sdd}} \cdot \prod_{e \in E} \alpha_e^{a_e - 1} \mathrm{d}\alpha_e \cdot \delta(1 - \alpha_N)$$

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Graph polynomials:

$$\psi = \mathcal{U} = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \mathcal{F} = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e$$

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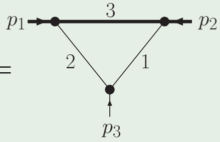
Example

$$\Phi(G) = \int_0^\infty \frac{\Gamma(1+\varepsilon)\ \delta(1-\alpha_N)\ \mathrm{d}\alpha_1\ \mathrm{d}\alpha_2\ \mathrm{d}\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)^{1-2\varepsilon}\left[m^2(\alpha_1+\alpha_2+\alpha_3) + p_1^2\alpha_2 + p_2^2\alpha_1\right]^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$$

$G =$ 

$D = 4 - 2\varepsilon \qquad a_e = 1 \qquad \text{sdd} = 1 + \varepsilon$

Hyperlogarithms

Definition (Poincaré, Lappo-Danilevsky)

To words $w = \omega_{\sigma_1} \ldots \omega_{\sigma_n}$ with $\sigma_i \in \mathbb{C}$ associate *hyperlogarithms*

$$L_{\omega_0^n}(z) := \frac{\log^n z}{n!} \quad \text{and} \quad L_{\omega_\sigma w}(z) := \int_0^z \frac{\mathrm{d}z'}{z' - \sigma} L_w(z').$$

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Example (alternative names)

Goncharov polylogarithms, generalized harmonic polylogarithms:

$$L_{\omega_{\sigma_1} \ldots \omega_{\sigma_n}}(z) = G(\sigma_1, \ldots, \sigma_n; z).$$

Polylogarithms $L_{\omega_0^{n-1}\omega_\sigma}(z) = -\operatorname{Li}_n\left(\frac{z}{\sigma}\right)$ and multiple polylogarithms:

$$L_{\omega_0^{n_r-1}\omega_{\sigma_r} \cdots \omega_0^{n_2-1}\omega_{\sigma_2}\omega_0^{n_1-1}\omega_{\sigma_1}}(z) = (-1)^r \operatorname{Li}_{n_1,\ldots,n_r}\left(\frac{\sigma_2}{\sigma_1}, \ldots, \frac{\sigma_r}{\sigma_{r-1}}, \frac{z}{\sigma_r}\right).$$

Integration with hyperlogarithms following Brown [7]

Applications by Chavez & Duhr [8], Wißbrock [1], Anastasiou et al. [3, 2]

To compute $\int_0^\infty f \, \mathrm{d}\alpha_e$ where f is a rational linear combination of polylogarithms that depend rationally on α_e:

1. Rewrite f using hyperlogarithms:

$$f = \sum_{w,\sigma,n} \frac{L_w(\alpha_e)}{(\alpha_e - \sigma)^n} \lambda_{w,\sigma,n} \quad \text{with constants } \lambda_{w,\sigma,n} \text{ w.r.t. } \alpha_e.$$

Implemented in the `Maple` code `HyperInt` [13], completely algebraic.

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2. Construct an antiderivative $\partial_{\alpha_e} F = f$.
3. Evaluate the limits

$$\int_0^\infty f \, \mathrm{d}\alpha_e = \lim_{\alpha_e \to \infty} F(\alpha_e) - \lim_{\alpha_e \to 0} F(\alpha_e).$$

Implemented in the `Maple` code `HyperInt` [13], completely algebraic.

Linear reducibility

Precondition: For all $n < N$,

$$f_n := \left[\prod_{e=1}^{n-1} \int_0^\infty \mathrm{d}\alpha_e\right] \psi^{\mathrm{sdd} - D/2} \varphi^{-\mathrm{sdd}} \prod_{e \in E} \alpha_e^{a_e - 1}$$

can be written as hyperlogarithms of α_e over denominators that factor linearly in α_e (a very strong constraint).

Definition

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Polynomial reduction

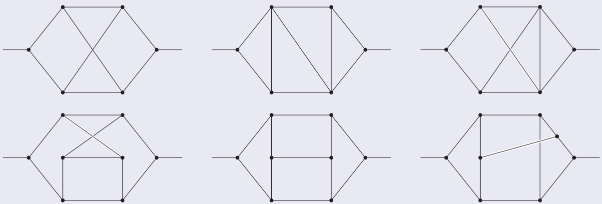
Simple algorithms [7, 6] are available to check sufficient criteria for linear reducibility.

Are there linearly reducible Feynman graphs?

Massless propagators (single-scale integrals)

Theorem ([12])

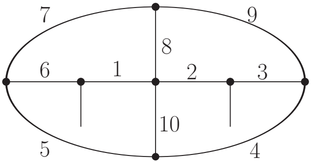
All massless propagators up to four loops are linearly reducible. Their ε-expansions only contain alternating Euler sums $\mathrm{Li}_{n_1,\ldots,n_r}(\pm 1,\ldots,\pm 1)$.



Hence all these graphs can be computed

- to any order in ε expanded near arbitrary even dimension $D|_{\varepsilon=0} \in 2\mathbb{N}$,
- with any tensor structures and
- for arbitrary powers $a_e = n_e + \varepsilon\nu_e$ of propagators ($n_e \in \mathbb{Z}, \nu_e \in \mathbb{C}$).

Massless propagators: 4-loop example

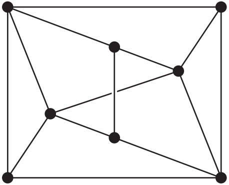
$M_{5,1} =$ 

$$\frac{\Phi\left(M_{5,1}\right)\cdot(1+\varepsilon[3+\nu_{345678910}])(4+\nu_{345678910})}{G_0^4(1-2\varepsilon)^3} = -20\zeta_5\varepsilon^{-1} - \tfrac{80}{7}\zeta_2^3$$
$$-\zeta_3^2\left(68+6\,p_1\right) - \varepsilon\left\{\tfrac{1}{5}\zeta_2^2\zeta_3\left(408+36\,p_1\right)+\zeta_7\left(170-7\,p_2\right)\right\} + \mathcal{O}\left(\varepsilon^2\right),$$

where the polynomials $p_1, p_2 \in \mathbb{Q}[\nu_1,\ldots,\nu_{10}]$ are given by

$$p_1 = 2\,\nu_{36810} + 3\,\nu_{4579} = 2(\nu_3+\nu_6+\nu_8+\nu_{10}) + 3(\nu_4+\nu_5+\nu_7+\nu_9)$$
$$p_2 = 8\nu_{12} - \tfrac{55}{4}\nu_{4579} - \tfrac{5}{2}\nu_{36810} - \tfrac{1}{8}p_1^2$$
$$+2\left(\nu_8-\nu_{10}\right)\left(\nu_{4510}-\nu_{789}\right) + 2\left(\nu_3-\nu_6\right)\left(\nu_{567}-\nu_{349}\right)$$
$$+2(\nu_{12}\nu_{345678910}+\nu_{36}\nu_{810}-\nu_{47}\nu_{59}) - 4\left(\nu_4^2+\nu_5^2+\nu_7^2+\nu_9^2\right).$$

Massless φ^4 theory: No alternating sums, but $e^{i\pi/3}$

$P_{7,11} =$ 

$P_{7,11}$ is not linearly reducible: After integrating ten variables, denominator

$$\begin{aligned} d_{10} = {} & \alpha_2\alpha_4^2\alpha_1 + \alpha_2\alpha_4^2\alpha_3 - \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_2^2\alpha_4\alpha_1 + \alpha_2^2\alpha_4\alpha_3 \\ & - 2\alpha_2\alpha_3^2\alpha_4 - \alpha_2^2\alpha_3^2 - 2\alpha_2^2\alpha_3\alpha_1 - 2\alpha_2\alpha_3^2\alpha_1 - \alpha_3^2\alpha_4^2 \\ & - 2\alpha_3^2\alpha_4\alpha_1 - \alpha_2^2\alpha_1^1 - 2\alpha_2\alpha_3\alpha_1^2 - \alpha_3^2\alpha_1^2. \end{aligned}$$

Changing variables $\alpha_3 = \frac{\alpha_3'\alpha_1}{\alpha_1+\alpha_2+\alpha_4}$, $\alpha_4 = \alpha_4'(\alpha_2 + \alpha_3')$ and $\alpha_1 = \alpha_1'\alpha_4'$,

$$d_{10}' = (\alpha_2 + \alpha_3')(\alpha_2 + \alpha_2\alpha_4' - \alpha_1')(\alpha_1'\alpha_4' + \alpha_2 + \alpha_2\alpha_4' + \alpha_3'\alpha_4')$$

factors linearly and $\alpha_1', \alpha_3', \alpha_4'$ can be integrated ($\alpha_2 = 1$).

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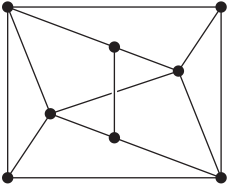
$P_{7,11} =$ 

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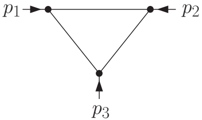
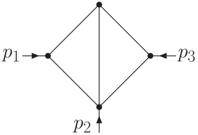
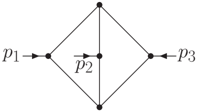
factors linearly and $\alpha_1', \alpha_3', \alpha_4'$ can be integrated ($\alpha_2 = 1$).
The final integrand is $\mathrm{HPL}(\alpha_1)/(1-\alpha_1+\alpha_1^2)$ and gives *not a multiple zeta value*, but a polylogarithm at sixth roots of unity.

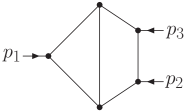
Off-shell massless three-point integrals

Internal $m_e = 0$, external $p_1^2, p_2^2 = |z|^2 \cdot p_1^2, p_3^2 = |1-z|^2 \cdot p_1^2 \neq 0$









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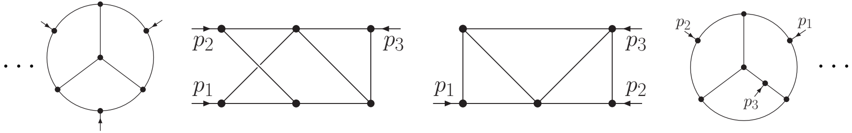

Theorem ([14])

All three-loop off-shell massless three-point functions are linearly reducible. They generalize single-valued multiple polylogarithms [8, 16] to the alphabet $\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}, z\bar{z}-1, z+\bar{z}-1, z\bar{z}-z-\bar{z}\}$.

$$p_2^2 = p_1^2 \cdot z\bar{z} \qquad \text{and} \qquad p_3^2 = p_1^2 \cdot (1-z)(1-\bar{z})$$

$$z - \bar{z} = \sqrt{p_1^2 + p_2^2 + p_3^2 - 2p_1p_2 - 2p_1p_3 - 2p_2p_3}$$

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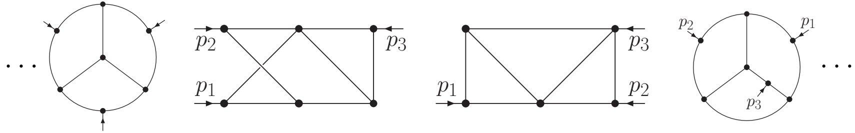

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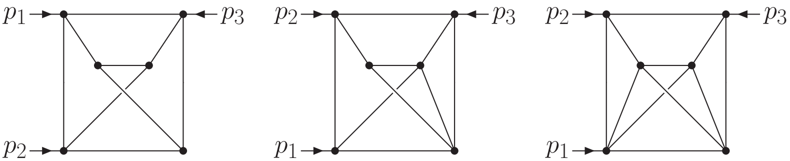
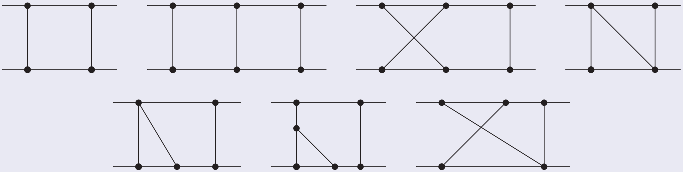
Some linearly reducible three-point functions with more loops:



Massless on-shell four-point graphs

Theorem ([4])

All massless four-point on-shell graphs ($p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$) with at most two loops are linearly reducible. In particular these include



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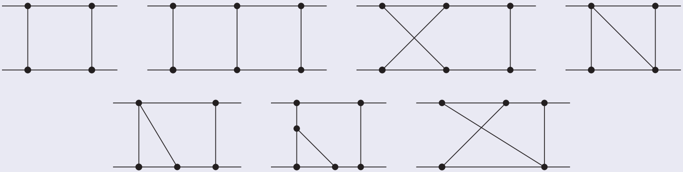
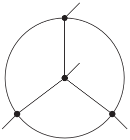


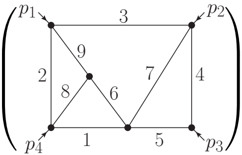
$K_4 =$



At three loops there are non-reducible graphs like K_4.
Still, K_4 evaluates to polylogarithms [10].
It becomes linearly reducible after a change of variables.

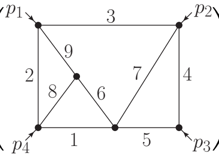
Massless on-shell four-point graphs

Linearly reducible four-loop example

Φ 

$$= \frac{\Gamma(1+4\varepsilon)}{s^{1+4\varepsilon}} \sum_{n=-1}^{\infty} f_n\left(\frac{s}{u}\right) \cdot \varepsilon^n$$

is linearly reducible along the sequence $1, 2, 8, 6, 9, 7, 5, 4$ of edges ($\alpha_3 = 1$). All f_n are harmonic polylogarithms of $\frac{s}{u}$:



$$\Phi\left(\text{graph}\right) = \frac{\Gamma(1+4\varepsilon)}{s^{1+4\varepsilon}} \sum_{n=-1}^{\infty} f_n\left(\frac{s}{u}\right)\cdot\varepsilon^n$$

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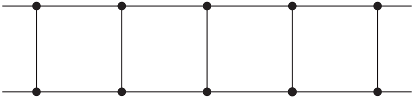
$$\begin{aligned}
f_{-1} = &-\frac{79}{70}\zeta_2^3 H_{-1} - \zeta_3\big(15\zeta_2 H_{-1,-1} - 9\zeta_2 H_{-1,0} - H_{-1,-2,-1} + H_{-1,-1,-2} + 6H_{-1,-1,0,0}\big) \\
&-6\zeta_3^2 H_{-1} - \frac{3}{2}\zeta_5(11H_{-1,-1} - 5H_{-1,0}) - \frac{3}{10}\zeta_2^2(H_{-1,-2} - 17H_{-1,-1,0} - 10H_{-1,-1,-1}) \\
&-\zeta_2\Big(H_{-1,-2,0,0} - 2H_{-1,-1,-2,0} + 3H_{-1,-1,-2,-1} - H_{-1,-1,-1,0,0} + 6H_{-1,-1,-3} \\
&\qquad -3H_{-1,-2,-1,-1} - 2H_{-1,-1,0,0,0}\Big) + H_{-1,-2,-1,0,0,0} - H_{-1,-1,-2,-1,0,0} \\
&+H_{-1,-1,-2,0,0,0} - 2H_{-1,-1,-3,0,0} + H_{-1,-2,-1,-1,0,0}
\end{aligned}$$

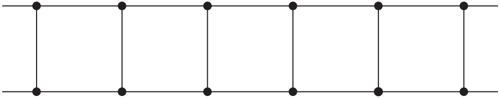
Massless ladder boxes with two off-shell legs

All ladder boxes are linearly reducible. For on-shell kinematics these are HPL of $x = \frac{u}{s}$ which we computed in $D = 6$ for $n \leq 6$.

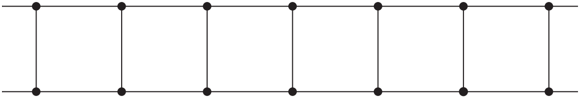
$B_4 =$



$B_5 =$



$B_6 =$



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The $u \to 0$ limit $B_n = c_n/s + \mathcal{O}(x)$ is

$$c_4 = -56\zeta_7 - 32\zeta_2\zeta_5 + 32\zeta_3^2 + \frac{8}{5}\zeta_3\left(4\zeta_2^2 - 15\right) + \frac{992}{35}\zeta_2^3 - 8\zeta_2^2 - 18\zeta_2$$

$$\begin{aligned} c_5 = {} & 56\zeta_7(-5+\zeta_3) + 26\zeta_5^2 + 4\zeta_5(-40\zeta_2 - 49 + 8\zeta_2\zeta_3 + 35\zeta_3) \\ & - \frac{4}{5}\zeta_3^2\left(-140 + 25\zeta_2 + 4\zeta_2^2\right) + 8\zeta_3\left(7\zeta_2 + 4\zeta_2^2 - 14\right) \\ & - \frac{1168}{385}\zeta_2^5 - \frac{24}{7}\zeta_2^4 + \frac{496}{5}\zeta_2^3 + 4\zeta_2\left(-21 + 2\zeta_{3,5}\right) + 20\zeta_{3,5} + 4\zeta_{3,7} \end{aligned}$$

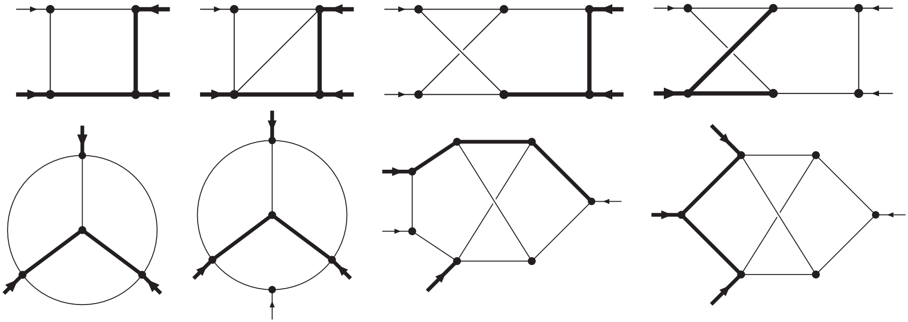
$$\begin{aligned} c_6 = {} & -1320\zeta_{11} - 12\zeta_9(20\zeta_2 + 161) + \frac{8}{5}\zeta_7\left(104\zeta_2^2 + 35\zeta_2 + 840\zeta_3 - 1120\right) \\ & + 624\zeta_5^2 + \frac{16}{35}\zeta_5\left(1680\zeta_2\zeta_3 - 3675 - 12\zeta_2^3 - 2240\zeta_2 + 490\zeta_2^2 + 5145\zeta_3\right) \\ & - \frac{48}{5}\zeta_3^2\left(35\zeta_2 + 8\zeta_2^2 - 60\right) - \frac{32}{5}\zeta_3\left(105 - 32\zeta_2^2 + 3\zeta_2^3 - 75\zeta_2\right) + 96\zeta_2^2 \\ & + 24\zeta_2\left(-21 + 8\zeta_{3,5}\right) - \frac{28032}{385}\zeta_2^5 - \frac{288}{5}\zeta_2^4 + \frac{18864}{35}\zeta_2^3 + 336\zeta_{3,5} + 96\zeta_{3,7} \end{aligned}$$

Massive integrals with up to 7 scales

Notation:

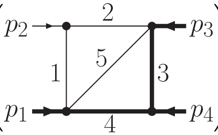
- thin lines: light-like momenta $p_e^2 = 0$, massless propagators $m_e = 0$
- thick lines: arbitrary (different) masses m_e and external momenta p_e^2

Linearly reducible examples:



Massive integrals with up to 7 scales: Example

Φ

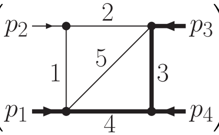


$$= \frac{\Gamma(1+2\varepsilon) p_3^{-2-4\varepsilon}}{1+p-s-u} \sum_{n=-1}^{\infty} f_n(s,u,p,m,M)\varepsilon^n$$

Kinematics:

$$p_4^2 = 0 \quad p = \frac{p_1^2}{p_3^2} \quad s = \frac{(p_1+p_2)^2}{p_3^2} \quad u = \frac{(p_1+p_4)^2}{p_3^2} \quad M = \frac{m_3^2}{p_3^2} \quad m = \frac{m_4^2}{p_3^2}$$

Massive integrals with up to 7 scales: Example



$$\Phi\left(\cdots\right) = \frac{\Gamma(1+2\varepsilon)p_3^{-2-4\varepsilon}}{1+p-s-u}\sum_{n=-1}^{\infty} f_n(s,u,p,m,M)\varepsilon^n$$

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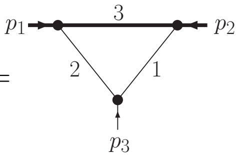
Leading order:

$$\begin{aligned}
f_{-1} =& \operatorname{Li}_{1,1,1}\left(\tfrac{1-u+p-s}{(1-s)(1-u)}, \tfrac{M(1-s)}{M-m}, \tfrac{M-m}{M}\right) - \operatorname{Li}_{1,1,1}\left(\tfrac{M(1-u+p-s)}{(M-m)(1-u)}, \tfrac{(u+M)(M-m)}{(-p-m+u+M)M}, \tfrac{-p-m+u+M}{u+M}\right) \\
&+ \operatorname{Li}_{1,1,1}\left(-\tfrac{u(1-u+p-s)}{(-u+p)(1-u)}, -\tfrac{(u+M)(-u+p)}{(-p-m+u+M)u}, \tfrac{-p-m+u+M}{u+M}\right) - \ln\left(\tfrac{m}{p+m}\right)\operatorname{Li}_{1,1}\left(-\tfrac{u(1-u+p-s)}{(-u+p)(1-u)}, -\tfrac{-u+p}{u}\right) \\
&- \operatorname{Li}_{1,1,1}\left(-\tfrac{u(1-u+p-s)}{(-u+p)(1-u)}, -\tfrac{M(-u+p)}{(M-m)u}, \tfrac{M-m}{M}\right) - \operatorname{Li}_{1,1,1}\left(\tfrac{1-u+p-s}{(1-s)(1-u)}, \tfrac{(1+M)(1-s)}{-m-s+1+M}, \tfrac{-m-s+1+M}{1+M}\right) \\
&+ \operatorname{Li}_{1,1,1}\left(\tfrac{M(1-u+p-s)}{(M-m)(1-u)}, \tfrac{(1+M)(M-m)}{(-m-s+1+M)M}, \tfrac{-m-s+1+M}{1+M}\right) - \ln\left(-\tfrac{p-s}{1-u}\right)\operatorname{Li}_2\left(\tfrac{p}{p+m}\right) + \ln\left(-\tfrac{p-s}{1-u}\right)\operatorname{Li}_2\left(\tfrac{s}{m+s}\right) \\
&+ \ln\left(\tfrac{m}{m+s}\right)\operatorname{Li}_{1,1}\left(\tfrac{1-u+p-s}{(1-s)(1-u)}, 1-s\right) - \ln\left(\tfrac{m}{m+s}\right)\operatorname{Li}_{1,1}\left(\tfrac{M(1-u+p-s)}{(M-m)(1-u)}, \tfrac{M-m}{M}\right) + \tfrac{1}{2}\ln\left(-\tfrac{p-s}{1-u}\right)\ln^2\left(\tfrac{m}{m+s}\right) \\
&- \tfrac{1}{2}\ln\left(-\tfrac{p-s}{1-u}\right)\ln^2\left(\tfrac{m}{p+m}\right) + \ln\left(\tfrac{m}{p+m}\right)\operatorname{Li}_{1,1}\left(\tfrac{M(1-u+p-s)}{(M-m)(1-u)}, \tfrac{M-m}{M}\right)
\end{aligned}$$

Analytic regularization of parametric integrals

Divergences in Schwinger parameters: Example

$G =$ 

$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$
$$\varphi = \alpha_3 \left(m^2\psi + p_1^2\alpha_2 + p_2^2\alpha_1 \right)$$

Parametric representation for $D = 4 - 2\varepsilon$, $a_1 = a_2 = a_3 = 1$, $\mathrm{sdd} = 1 + \varepsilon$:

$$\Gamma(1+\varepsilon) \int \frac{\Omega}{(\alpha_1 + \alpha_2 + \alpha_3)^{1-2\varepsilon} \left[m^2(\alpha_1 + \alpha_2 + \alpha_3) + p_1^2\alpha_2 + p_2^2\alpha_1 \right]^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$$

The projective volume form

$$\Omega = \delta(H) \cdot \mathrm{d}\alpha_1 \wedge \mathrm{d}\alpha_2 \wedge \mathrm{d}\alpha_3$$

restricts integration to an arbitrary hyperplane $H = 1 - \sum \lambda_e \alpha_e$ with $\lambda_e \geq 0$ not all zero.

Divergences at $\varepsilon = 0$ are manifest in the regions $\alpha_3 \to 0$ and $\alpha_1, \alpha_2 \to \infty$.

Divergences in Schwinger parameters: Power counting

$$F = \prod_{e\in E} \alpha_e^{a_e - 1} \cdot \psi^{\mathrm{sdd} - D/2} \cdot \varphi^{-\mathrm{sdd}}$$

Definition

For an integrand F and disjoint sets $J, K \subset E$ of edges, the rescaling

$$F_J^K := F \circ \left(\alpha_e \mapsto \begin{cases} \lambda\alpha_e, & e \in J \\ \lambda^{-1}\alpha_e, & e \in K \\ \alpha_e, & e \in E \setminus (J \cup K) \end{cases} \right)$$

defines a vanishing degree $\deg_J^K$ such that $F_J^K = \lambda^{\deg_J^K} \cdot \widetilde{F_J^K}$ where $\lim_{\lambda\to 0} \widetilde{F_J^K} \neq 0$ exists. Set $\omega_J^K := |J| - |K| + \deg_J^K$.

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Finiteness in simple cases

When all coefficients of φ are positive and $\omega_J^K > 0$ for all disjoint $J, K \subsetneq E$ with $0 \neq J \cup K \subsetneq E$, then $\int F \; \Omega$ converges absolutely.

Analytic regularization in Schwinger parameters

A partial integration yields the relation

$$\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda} \lambda^{\omega_J^K} \cdot \widetilde{F_J^K}(\lambda) = \left. \frac{\lambda^{\omega_J^K}}{\omega_J^K} \widetilde{F_J^K}(\lambda) \right|_{\lambda=0}^{\infty} - \frac{1}{\omega_J^K} \int_0^\infty \mathrm{d}\lambda \cdot \lambda^{\omega_J^K} \frac{\partial}{\partial \lambda} \widetilde{F_J^K}(\lambda)$$

with vanishing boundary contribution in the convergent regime.
The analytically regularized functions associated to both integrals thus coincide (they are meromorphic in the regulators).

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The analytically regularized functions associated to both integrals thus coincide (they are meromorphic in the regulators).

Example (Vertex with one mass and $p_3^2 = 0$ in $D = 4 - 2\varepsilon$)

We have $\omega_{\{3\}}^{\emptyset} = -\varepsilon$ with $\widetilde{F_{\{3\}}^{\emptyset}} = \psi^{2\varepsilon-1} \cdot \left[m^2\psi + p_2^2\alpha_1 + p_1^2\alpha_2\right]^{-1-\varepsilon}$:

$$\int \frac{\Omega}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}} = \left.\frac{\alpha_3^{-\varepsilon}\widetilde{F_{\{3\}}^{\emptyset}}}{-\varepsilon}\right|_{\alpha_3=0}^{\infty} + \frac{1}{\varepsilon} \cdot \int \frac{\Omega}{\alpha_3^{\varepsilon}}\frac{\partial}{\partial\alpha_3}\widetilde{F_{\{3\}}^{\emptyset}}$$

$$= \frac{1}{\varepsilon} \cdot \int \frac{\Omega\alpha_3}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}}\left[\frac{2\varepsilon-1}{\psi} - \frac{(1+\varepsilon)\alpha_3 m^2}{\varphi}\right] \text{ when } \varepsilon < 0.$$

Regularization in Schwinger parameters

General construction

Proposition

For disjoint subsets $J, K \subsetneq E$, acting with the differential operator

$$\mathcal{D}_J^K := 1 - \frac{1}{\omega_J^K}\Big[\sum_{e\in J}\frac{\partial}{\partial\alpha_e}\alpha_e - \sum_{e\in K}\frac{\partial}{\partial\alpha_e}\alpha_e\Big]$$

on the integrand F yields a new integrand $\widetilde{F} := \mathcal{D}_J^K(F)$ such that

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Thus finitely many partial integrations suffice to make all ω_J^K positive, thereby yielding a convergent integral representation (suitable for hyperlogarithmic integration).

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Thus finitely many partial integrations suffice to make all ω_J^K positive, thereby yielding a convergent integral representation (suitable for hyperlogarithmic integration).

This procedure does not introduce new singularities, i.e. the polynomial reduction and linear reducibility is not affected.

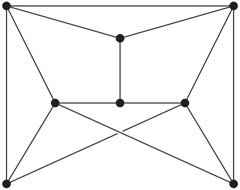
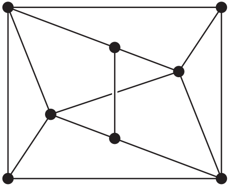
Summary: Hyperlogarithmic integration

1. many linearly reducible graphs with highly non-trivial kinematics
2. lin. reducible $\Rightarrow$ arbitrary ε-order, $D|_{\varepsilon=0} \in 2\mathbb{N}$, tensors, $a_e = n_e + \varepsilon\nu_e$
3. polynomial reduction determines nature of periods (constants) and letters (alphabet) of final polylogarithms
4. considers individual graphs, no boundary terms needed
5. applicable to other parametric integrals: hypergeometric functions, phase-space [3, 2], ...
6. divergent integrals computable in dimReg [14] (integration by parts); corresponds to a reduction to finite (master) integrals
7. `Maple`™ implementation: `HyperInt` [13]
8. problems with huge expressions for tensors and multiple divergences (IBP-reduction might help)
9. changes of variables: so far case-by-case
10. current implementation restricted to positive φ/Euclidean region

Thank you.

Massless φ^4 theory: No alternating sums

All periods of primitive φ^4-graphs up to seven loops are now known exactly. Among the most complicated ones in Schnetz' census [15] are

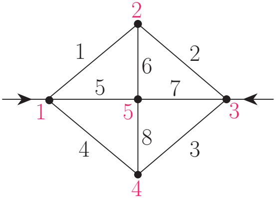
$P_{7,9} =$  $P_{7,11} =$ 

$P_{7,9}$ is linearly reducible and we verified David Broadhurst's PSLQ-fit [5]

$$P_{7,9} = \frac{92943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_{3,5}\zeta_3\right) - \frac{1155}{4}\zeta_3^2\zeta_5 + 896\zeta_3\left(\frac{27}{80}\zeta_{3,5} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8\right)$$

analytically. Interestingly, it appears as an alternating Euler sum: The last integrand has denominator $(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2)$.

HyperInt




```
> E := [[1,2],[2,3],[3,4],[4,1],[5,1],[5,2],[5,3],[5,4]]:
> psi := graphPolynomial(E):
> phi := secondPolynomial(E, [[1,1], [3,1]]):
> sdd := nops(E)-(1/2)*4*(4-2*epsilon):
> f := series(psi^(-2+epsilon+sdd)*phi^(-sdd), epsilon=0):
> f := add(coeff(f,epsilon,n)*epsilon^n,n=0..2):
> z := [x[1],x[2],x[6],x[5],x[3],x[4],x[7],x[8]]:
> hyperInt(eval(f,z[-1]=1), z[1..-2]):
> collect(fibrationBasis(%), epsilon);
```

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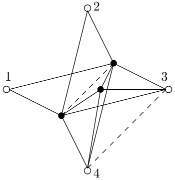
```

$$\left(254\zeta_7 + 780\zeta_5 - 200\zeta_2\zeta_5 - 196\zeta_3^2 + 80\zeta_2^3 - \frac{168}{5}\zeta_2^2\zeta_3\right)\varepsilon^2 + \left(-28\zeta_3^2 + 140\zeta_5 + \frac{80}{7}\zeta_2^3\right)\varepsilon + 20\zeta_5.$$

Conformal four-point integrals

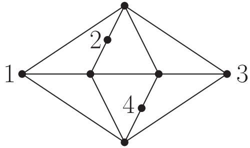
The cross ratios $z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $(1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ parametrize conformal four-point integrals [9]. Many are linearly reducible, e.g.

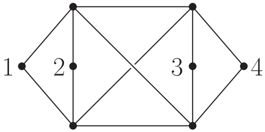


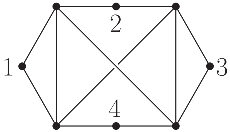
$$= H_{12;34} = \frac{x_{34}^2}{\pi^6} \int_{\mathbb{R}^{12}} \frac{\mathrm{d}^4 x_5\, \mathrm{d}^4 x_6\, \mathrm{d}^4 x_7 \cdot x_{57}^2}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2) x_{56}^2 (x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2)}$$

can be integrated parametrically to a rational function in x_{ij}^2 times a polylogarithm dependent on z and $\bar{z}$.

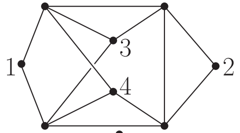
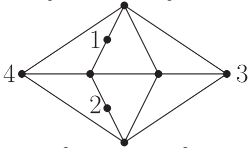
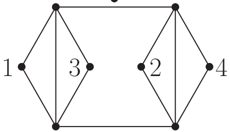
Without inverse (numerator) propagators, there are only three four-point integrals with 4 internal vertices ("loops") that are not linearly reducible:

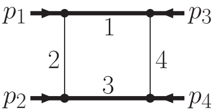






Conformal four-point integrals

graph	denominator	weight	additional symbol letters	terms
	$x_{34}^2 x_{13}^4 x_{24}^4 \cdot (z-\bar{z}) \cdot (z+\bar{z}-2)$	8	$z-\bar{z}$, $z\bar{z}-1$, $z+\bar{z}-1$	4235
	$x_{34}^2 x_{13}^4 x_{24}^4 \cdot (z-\bar{z})^2$	8	$\emptyset$	107
	$x_{13}^4 x_{24}^4 \cdot z\bar{z}(z-\bar{z})$	7	$z-\bar{z}$	146



The on-shell massive box $p_1^2 = p_2^2 = p_3^2 = p_4^2 = -m^2$ with $m_2 = m_4 = 0$ and $m_1 = m_3 = m \neq 0$ is not linearly reducible. The graph polynomials are

$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad \text{and} \quad \varphi = s\alpha_1\alpha_3 + t\alpha_2\alpha_4 + m^2(\alpha_1+\alpha_3)^2$$

and φ is only linear in α_2 and α_4. Reducing (integrating) α_4 we obtain

$$S_{\{4\}} = \left\{\alpha_1 + \alpha_2 + \alpha_3, s\alpha_1\alpha_3 + m^2(\alpha_1+\alpha_3)^2, R\right\} \quad \text{where the resultant}$$

$$R := [\psi, \varphi]_{\alpha_4} = s\alpha_1\alpha_3 + m^2(\alpha_1+\alpha_3)^2 - t\alpha_2(\alpha_1+\alpha_2+\alpha_3)$$

is irreducible and quadratic in all remaining Schwinger parameters, therefore prohibiting any further integration.

Extending linear reducibility

Box with two masses vis-à-vis

To proceed we change variables according to

$$\frac{s}{m^2} = \frac{(1-x)^2}{x} \quad \frac{t}{m^2} = \frac{(1-y)^2}{y} \quad \alpha_2 = \widetilde{\alpha_2}\,(\alpha_1 + x\alpha_3) \quad \alpha_4 = \widetilde{\alpha_4}\,(x\alpha_1 + \alpha_3)\,.$$

On one hand we reparametrized the kinematics via x and y to rationalize roots that would otherwise appear in the result, while afterwards

$$\varphi = \frac{m^2}{x}(\alpha_1 + x\alpha_3)(x\alpha_1 + \alpha_3) + \frac{m^2}{y}(1-y)^2\alpha_2\alpha_4$$

suggests to introduce the variables $\widetilde{\alpha_2}$ and $\widetilde{\alpha_4}$ with the effect that

$$\varphi = \frac{m^2}{xy}(\alpha_1 + x\alpha_3)(x\alpha_1 + \alpha_3)\left[y + x(1-y)^2\widetilde{\alpha}_2\widetilde{\alpha}_4\right]$$

factors linearly in these new parameters. Now $\left[\psi, y + x(1-y)^2\widetilde{\alpha}_2\widetilde{\alpha}_4\right]_{\widetilde{\alpha}_4}$ is linear in α_1 and α_3 allowing for a further integration.

Extending linear reducibility

Box with two masses vis-à-vis

Indeed we now observe linear reducibility in the new variables and obtain

$$S_{\{\widetilde{\alpha}_2,\alpha_3,\widetilde{\alpha}_4\}} = \{x+1, x-1, y+1, y-1, xy+1, x+y\}$$

as the final set in the polynomial reduction. Together with $\{x, y\}$ it defines the alphabet of the symbol of the resulting function of x and y in agreement with [11].

Soft phase space integrals

Using Mellin-Barnes representations in an intermediate step, a technology developed in [2, 3] allowed to rewrite soft phase space integrals in terms of parametric integrals like

$$\mathcal{I}_{9,1}(\varepsilon) = -\int_0^\infty \mathrm{d}t_1\ \mathrm{d}t_2 \int_0^1 \mathrm{d}x_1\ \mathrm{d}x_2\ \mathrm{d}x_3\ t_1^{2-4\varepsilon}(1+t_1)^{\varepsilon-1} t_2^{1-2\varepsilon}$$
$$\times x_1^{-\varepsilon}(1-x_1)^{2-4\varepsilon} x_2^{1-3\varepsilon}(1-x_2)^{-\varepsilon} x_3^{-\varepsilon}(1+t_2x_3)^{1-3\varepsilon}(1+t_2x_2x_3)^{\varepsilon}$$
$$\times \left(t_1 t_2^2 x_1x_2x_3 + t_2^2 x_2x_3 + t_1 t_2 x_1 x_2 + t_1 t_2 x_3 + t_2 x_2 x_3 + t_2 + t_1 + 1\right)^{3\varepsilon-3}.$$

These are linearly reducible and could be integrated along the sequence t_1, x_1, x_2, x_3, t_2 of variables and evaluate to MZV.

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