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Master integrals without subdivergences

Joint work with Andreas von Manteuffel and Robert Schabinger

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Sector decomposition

Subdivergences (IR and UV) of Feynman integrals result in (higher order) poles in ϵ (dimensional regularization) and obstruct both analytical and numerical evaluation. Standard solution: Sector decomposition [4, 5].

1. Split original integral into several sectors:

$$\int f(x)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N = \sum_i \int_{S_i} f(x)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N$$

2. Change of variables in each sector (monomialise):

$$\int_{S_i} f(x)\, \mathrm{d}x_1 \cdots \mathrm{d}x_N = \left(\prod_{k=1}^{N} \int_0^1 x_k^{\beta_k - 1} \mathrm{d}x_k \right)\; f_i'(x)$$

   such that $f_i'(x)$ is bounded on $[0,1]^N$

3. Regulate all (now normal crossing) divergences at zero:

$$\int_0^1 x_k^{\beta-1} \mathrm{d}x_k\; f_i(x) = \frac{1}{\beta} f(x)\big|_{x_k=1} - \frac{1}{\beta} \int_0^1 x_k^{\beta} \mathrm{d}x_k\; \partial_{x_k} f_i(x)$$

Undesirable features of sector decomposition

- Many different terms (sectors and subtraction terms) to consider
- Spurious structures (cancellation between sectors): An individual sector is more complicated than the total sum.

$$\int_0^1 \frac{\ln(1+x)}{x(1+x)} \mathrm{d}x = \frac{1}{2}\zeta(2) - \frac{1}{2}\ln^2(2)$$

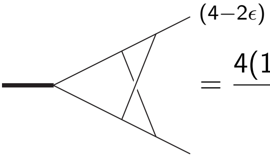
$$\int_1^\infty \frac{\ln(1+x)}{x(1+x)} \mathrm{d}x = \frac{1}{2}\zeta(2) + \frac{1}{2}\ln^2(2)$$

- Changes of variables are different in each sector and can destroy linear reducibility, a property allowing for analytical evaluation

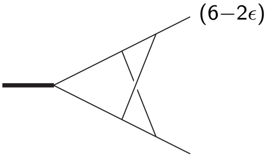
Idea

Avoid sector decomposition and write the integral as a linear combination of subdivergence-free ("primitive" or "quasi-finite") Feynman integrals.

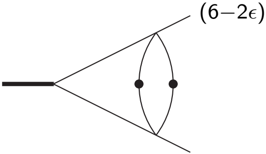
Example: Two-loop non-planar form factor



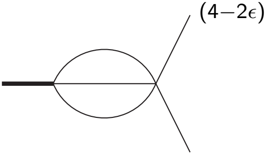
$$= \frac{4(1-\epsilon)(3-4\epsilon)(1-4\epsilon)}{\epsilon s^2}$$



$$- \frac{10 - 65\epsilon + 131\epsilon^2 - 74\epsilon^3}{\epsilon^3 s^2}$$



$$- \frac{14 - 119\epsilon + 355\epsilon^2 - 420\epsilon^3 + 172\epsilon^4}{(1-2\epsilon)\epsilon^3 s^3}$$



improving convergence via partial integration

Feynman integrals in Schwinger parameters

Scalar propagators $(p_e^2 + m_e^2)^{-a_e}$, $\mathsf{sdd} = \sum_e a_e - D/2 \cdot \mathsf{loops}(G)$:

$$\Phi(G) = \frac{\Gamma(\mathsf{sdd})}{\prod_e \Gamma(a_e)} \int_0^\infty \psi^{\mathsf{sdd}-D/2} \cdot \varphi^{-\mathsf{sdd}} \cdot \prod_{e \in E} \alpha_e^{a_e - 1} \mathrm{d}\alpha_e \cdot \delta(1 - \alpha_N)$$

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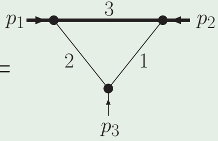
Graph polynomials:

$$\psi = \mathcal{U} = \sum_T \prod_{e \notin T} \alpha_e \qquad \varphi = \mathcal{F} = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e$$

Example (arbitrary ε)

$$\Phi(G) = \int_0^\infty \frac{\Gamma(1+\varepsilon)\ \delta(1-\alpha_N)\ \mathrm{d}\alpha_1\ \mathrm{d}\alpha_2\ \mathrm{d}\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)^{1-2\varepsilon} \left[m^2(\alpha_1+\alpha_2+\alpha_3) + p_1^2\alpha_2 + p_2^2\alpha_1\right]^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$$

$G =$ 

$D = 4 - 2\varepsilon \qquad a_e = 1 \qquad \mathrm{sdd} = 1 + \varepsilon$

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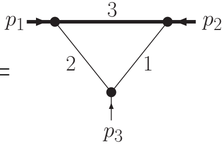
Example (expanded in $\varepsilon \to 0$)

$$\Phi(G) = \int_0^\infty \frac{\Gamma(1+\varepsilon)\, \delta(1-\alpha_N)\, \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)^{1-2\varepsilon} \left[m^2(\alpha_1+\alpha_2+\alpha_3) + p_1^2\alpha_2 + p_2^2\alpha_1\right]^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$$

$$= \Gamma(1+\varepsilon) \sum_{n=0}^\infty \frac{\varepsilon^n}{n!} \int_0^\infty \frac{\delta(1-\alpha_N)\, \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3}{\psi\varphi} \log^n \frac{\psi^2}{\varphi}$$

Divergences in Schwinger parameters

$G =$ 

$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$
$$\varphi = \alpha_3 \left(m^2 \psi + p_1^2 \alpha_2 + p_2^2 \alpha_1 \right)$$

In $D = 4 - 2\varepsilon$, sdd $= 1 + \varepsilon$ such that $\int_0^\infty \mathrm{d}\alpha_3$ diverges at the lower boundary when $\varepsilon \to 0$:

$$\int \frac{\Omega}{(\alpha_1 + \alpha_2 + \alpha_3)^{1-2\varepsilon} \left[m^2(\alpha_1 + \alpha_2 + \alpha_3) + p_1^2 \alpha_2 + p_2^2 \alpha_1 \right]^{1+\varepsilon} \alpha_3^{1+\varepsilon}}$$

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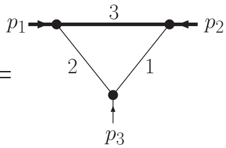
$G =$ 

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$$\int \frac{\Omega}{(\alpha_1+\alpha_2+\alpha_3)^{1-2\varepsilon}\left[m^2(\alpha_1+\alpha_2+\alpha_3)+p_1^2\alpha_2+p_2^2\alpha_1\right]^{1+\varepsilon}\alpha_3^{1+\varepsilon}}$$

Example (Regularization)

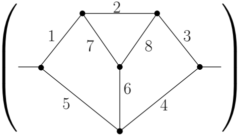
Let $\tilde{\phi} := \varphi/\alpha_3 = m^2\psi + p_2^2\alpha_1 + p_1^2\alpha_2$ and integrate by parts:

$$\int \frac{\Omega}{\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}\alpha_3^{\varepsilon}} = \frac{\alpha_3^{-\varepsilon}}{-\varepsilon\psi^{1-2\varepsilon}\tilde{\varphi}^{1+\varepsilon}}\Bigg|_{\alpha_3=0}^{\infty} + \frac{1}{\varepsilon}\cdot\int\frac{\Omega}{\alpha_3^{\varepsilon}}\frac{\partial}{\partial\alpha_3}\psi^{-1+2\varepsilon}\tilde{\varphi}^{-1-\varepsilon}$$

$$= \frac{1}{\varepsilon}\cdot\int\frac{\Omega\alpha_3}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}}\left[\frac{2\varepsilon-1}{\psi} - \frac{(1+\varepsilon)\alpha_3 m^2}{\varphi}\right] \quad \text{when } \varepsilon < 0.$$

Divergences in Schwinger parameters

In $D = 4 - 2\varepsilon$ dimensions,

Φ  $= \Gamma(2+3\varepsilon) \int \frac{\Omega}{\varphi^{2+3\varepsilon}\psi^{-4\varepsilon}}$

where $\varphi = \alpha_4(\cdots) + \alpha_5(\cdots) + \alpha_4\alpha_5(\cdots)$ vanishes at $\alpha_4 = \alpha_5 = 0$.
Rescale $\alpha_4 \to \lambda\alpha_4$ and $\alpha_5 \to \lambda\alpha_5$, so $\varphi \to \lambda\widetilde{\varphi}$ and $\psi \to \widetilde{\psi}$:

$$\int \frac{\Omega\psi^{4\varepsilon}}{\varphi^{2+3\varepsilon}} = \int \frac{\Omega\psi^{4\varepsilon}}{\varphi^{2+3\varepsilon}} \int_0^\infty \delta(\alpha_4 + \alpha_5 - \lambda)\, \mathrm{d}\lambda$$

Divergences in Schwinger parameters

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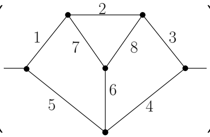
Φ () $= \Gamma(2+3\varepsilon) \int \frac{\Omega}{\varphi^{2+3\varepsilon}\psi^{-4\varepsilon}}$

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Divergences in Schwinger parameters

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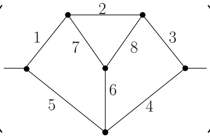
$\Phi\Big($  $\Big) = \Gamma(2+3\varepsilon)\int \frac{\Omega}{\varphi^{2+3\varepsilon}\psi^{-4\varepsilon}}$

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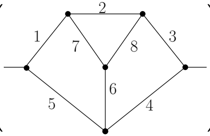
$\Phi\Bigg($ 

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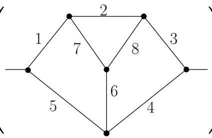
$\Phi\Big($  $\Big) = \Gamma(2+3\varepsilon)\int \frac{\Omega}{\varphi^{2+3\varepsilon}\psi^{-4\varepsilon}}$

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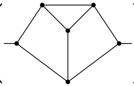
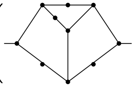
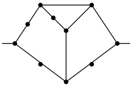
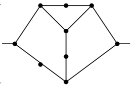
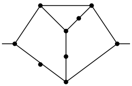
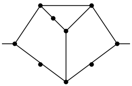
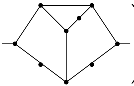
Numerator monomials correspond to squared propagators:

$\Phi_D\Big($  $\Big) = -\frac{1}{3\varepsilon}\Phi_{D+2}\Big($  $+$  $+\cdots\Big)$

$+\frac{4}{3}\Phi_{D+2}\Big($  $+$  $+\cdots+2$  $+2$  $\Big)$

Regularization of subdivergences

- Subdivergences (IR and UV) manifest as integrand $\sim \lambda^{\omega_\gamma - 1}$ with $\omega_\gamma|_{\varepsilon=0} \leq 0$, when some edges $\alpha_e \sim \lambda$ ($e \in \gamma$) get small jointly
- Well-known power counting:

$$\omega_\gamma = \begin{cases} -\operatorname{sdd}(G/\gamma) & \text{if } \gamma \text{ connects external legs (IR),} \\ \operatorname{sdd}(\gamma) & \text{otherwise (UV).} \end{cases}$$

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- Representation in terms of primitive Feynman integrals, with shifted D and a_e
- In practice, this creates too many terms. Thus use IBP!

Choose master integrals without subdivergences

Primitive (quasi-finite) master integrals

Corollary (IBP, Euclidean kinematics)

For any topology, one can choose the master integrals to be scalar and quasi-finite (free of subdivergences), given that one allows for shifted dimensions $D + 2$, $D + 4$, $\ldots$ *and dots.*

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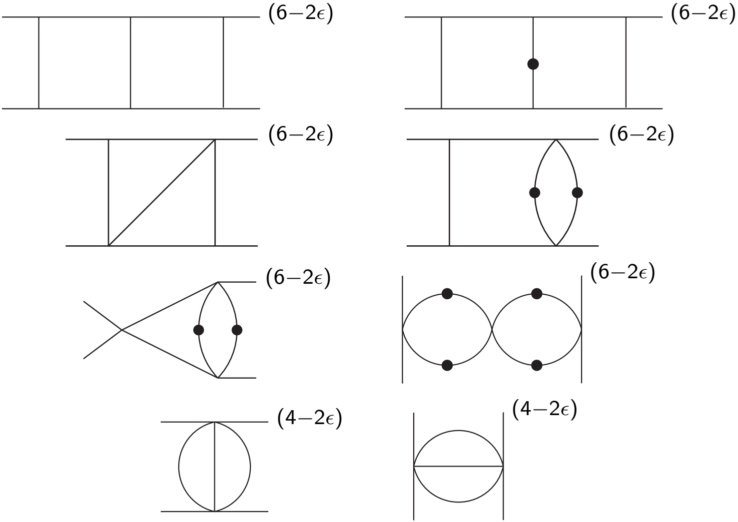
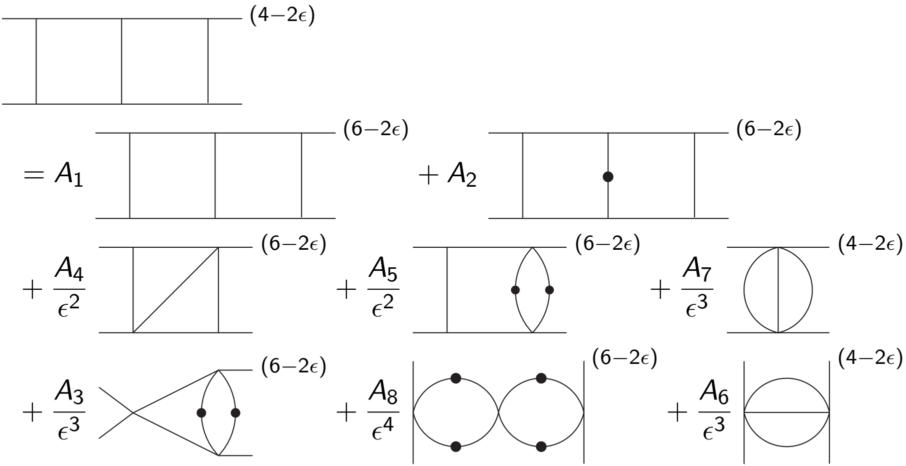
Advantages:

1. ε-expansion of the integrand
2. directly suitable for numeric quadrature
3. No splitting into non-Feynman integrals like sector decomposition
4. Empirically: Higher pole terms from subtopologies

Double box: Primitive master integrals



Double box: IBP reduction



Linearly reducible Feynman graphs

Integration with hyperlogarithms following Brown [8]

Applications by Chavez & Duhr [9], Wißbrock [1], Anastasiou et. al. [3, 2]

To compute $\int_0^\infty f \, \mathrm{d}\alpha_e$ where f is a rational linear combination of polylogarithms that depend rationally on α_e:

1. Rewrite f using hyperlogarithms:

$$f = \sum_{w,\sigma,n} \frac{G(w;\alpha_e)}{(\alpha_e - \sigma)^n} \lambda_{w,\sigma,n} \quad \text{with constants } \lambda_{w,\sigma,n} \text{ w.r.t. } \alpha_e.$$

Implemented in the `Maple` code `HyperInt` [14], completely algebraic.
Assumption: Finite integrals!

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2. Construct an antiderivative $\partial_{\alpha_e} F = f$.
3. Evaluate the limits

$$\int_0^\infty f \, \mathrm{d}\alpha_e = \lim_{\alpha_e \to \infty} F(\alpha_e) - \lim_{\alpha_e \to 0} F(\alpha_e).$$

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Linear reducibility

Precondition: For all $n < N$,

$$f_n := \left[\prod_{e=1}^{n-1} \int_0^\infty \mathrm{d}\alpha_e\right] \; \psi^{\mathrm{sdd}\, -D/2} \varphi^{-\,\mathrm{sdd}} \prod_{e\in E} \alpha_e^{a_e - 1}$$

can be written as hyperlogarithms of α_e over denominators that factor linearly in α_e.

Definition

If this holds for some ordering $e_1, \ldots, e_N$ of its edges, the Feynman graph G is called *linearly reducible*.

Linear reducibility

Precondition: For all $n < N$,

$$f_n := \left[\prod_{e=1}^{n-1} \int_0^\infty \mathrm{d}\alpha_e\right] \; \psi^{\mathrm{sdd}\, - D/2} \varphi^{-\,\mathrm{sdd}} \prod_{e \in E} \alpha_e^{a_e - 1}$$

can be written as hyperlogarithms of α_e over denominators that factor linearly in α_e.

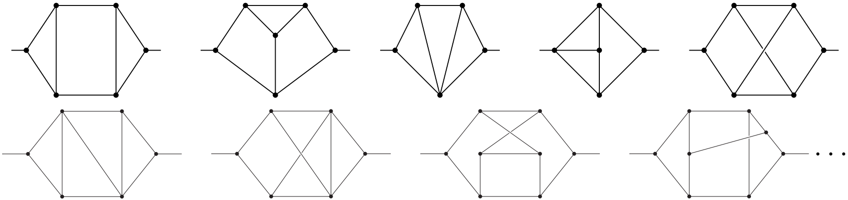
Definition

If this holds for some ordering $e_1, \ldots, e_N$ of its edges, the Feynman graph G is called *linearly reducible*.

- Combinatorial condition on the polynomials ψ and φ only; independent of ε-order and expansion point $(D, \vec{a})_{\varepsilon=0} \in 2\mathbb{N} \times \mathbb{Z}^N$
- Polynomial reduction algorithms [8, 7] available (e.g. `HyperInt`) to check sufficient criteria for linear reducibility

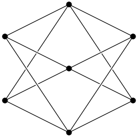
Linearly reducible massless propagators

- all up to four loops [11]: MZV and maybe alternating sums

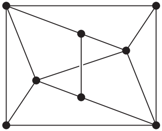


- all ϕ^4-periods up to seven loops, except for

$K_{3,4} =$



$P_{7,11} =$



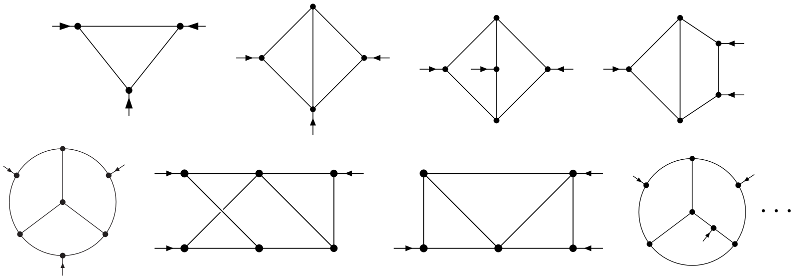
Integrable with *graphical functions*, O. Schnetz [15]. Extremely efficient (graphs up to ten loops).

Linearly reducible after change of variables. Does not give a multiple zeta value!

Linearly reducible 3-point graphs

Off-shell massless three-point integrals ($m_e = 0$ and $p_1^2, p_2^2, p_3^2 \neq 0$):

- All up to three loops [13]



Linearly reducible 3-point graphs

Off-shell massless three-point integrals ($m_e = 0$ and $p_1^2, p_2^2, p_3^2 \neq 0$):

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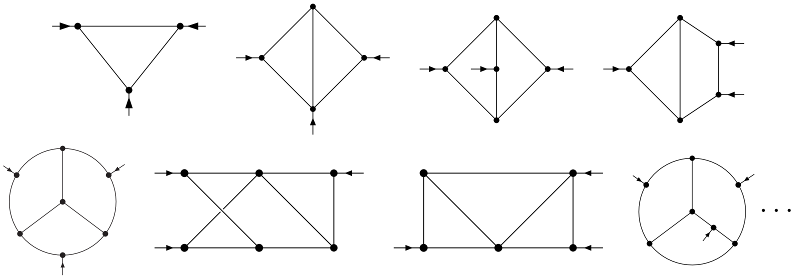
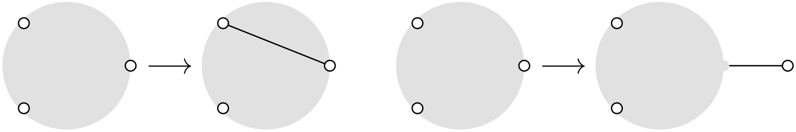


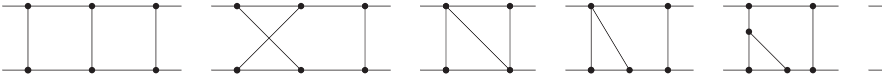
- All with vertex-width three [12]



Linearly reducible 4-point graphs

Massless on-shell four-point graphs ($m_e = p_1^2 = \cdots = p_4^2 = 0$):

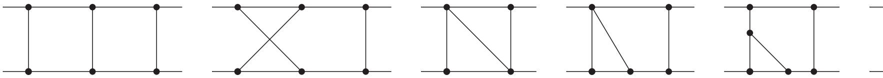
- All up to two loops [6]



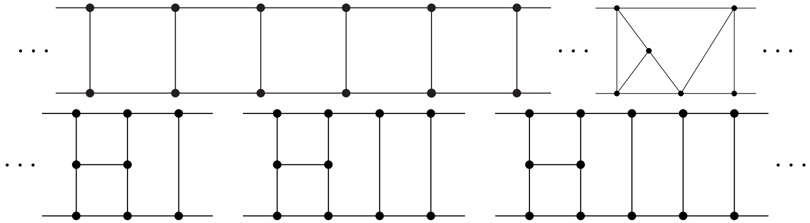
Linearly reducible 4-point graphs

Massless on-shell four-point graphs ($m_e = p_1^2 = \cdots = p_4^2 = 0$):

- All up to two loops [6]



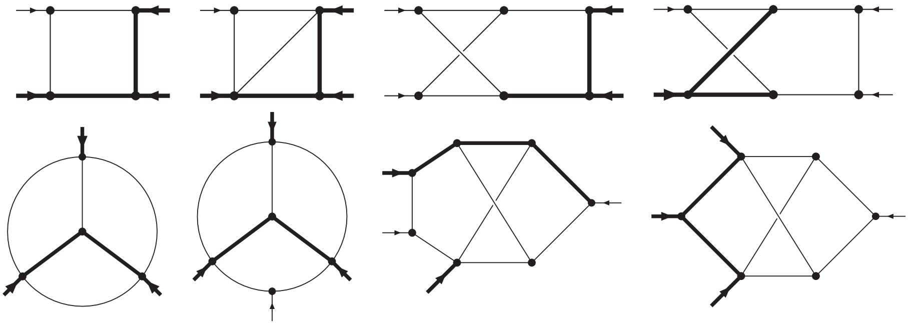
- Minors of ladder-boxes (and some generalizations [12])



Also with up to two legs off-shell.

Linearly reducible graphs with masses

Examples:



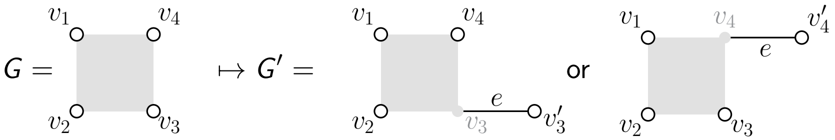
Summary

- Master integrals can be chosen to be primitive
  - existing tools for IBP-reduction can be used or extended
  - potential for direct numeric integration
- Extends exact parametric integration to divergent integrals
  - many examples of linearly reducible graphs with non-trivial kinematics
  - `Maple`™ implementation: `HyperInt` [14]
  - arbitrary ε-order, $D|_{\varepsilon=0} \in 2\mathbb{N}$, tensors, $a_e = n_e + \varepsilon \nu_e$
- So far clear for Euclidean kinematics only; possible extension to more general kinematics and phase-space integrals to be investigated

Thank you.

Start with the box and repeat, in any order:

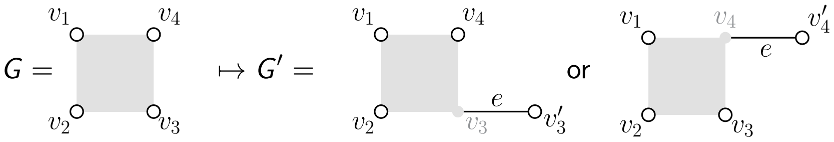
- Appending a vertex:



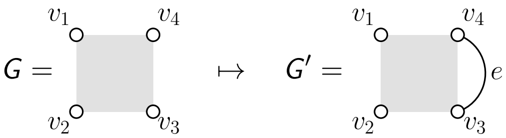
4-point recursions

Start with the box and repeat, in any order:

- Appending a vertex:



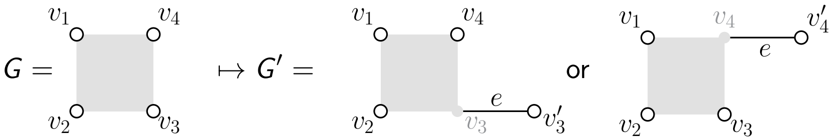
- Adding an edge:



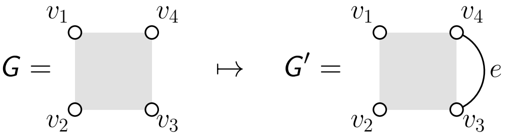
4-point recursions

Start with the box and repeat, in any order:

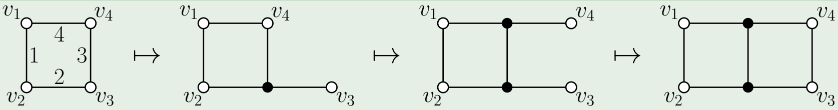
- Appending a vertex:



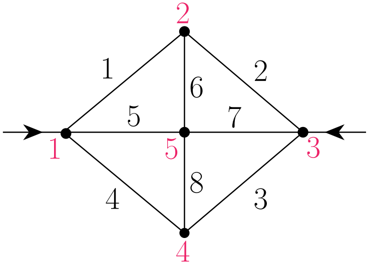
- Adding an edge:



Example



HyperInt




```
> E := [[1,2],[2,3],[3,4],[4,1],[5,1],[5,2],[5,3],[5,4]]:
> psi := graphPolynomial(E):
> phi := secondPolynomial(E, [[1,1], [3,1]]):
> sdd := nops(E)-(1/2)*4*(4-2*epsilon):
> f := series(psi^(-2+epsilon+sdd)*phi^(-sdd), epsilon=0):
> f := add(coeff(f,epsilon,n)*epsilon^n,n=0..2):
> z := [x[1],x[2],x[6],x[5],x[3],x[4],x[7],x[8]]:
> hyperInt(eval(f,z[-1]=1), z[1..-2]):
> collect(fibrationBasis(%), epsilon);
```

HyperInt

```
> E := [[1,2],[2,3],[3,4],[4,1],[5,1],[5,2],[5,3],[5,4]]:
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> f := add(coeff(f,epsilon,n)*epsilon^n,n=0..2):
> z := [x[1],x[2],x[6],x[5],x[3],x[4],x[7],x[8]]:
> hyperInt(eval(f,z[-1]=1), z[1..-2]):
> collect(fibrationBasis(%), epsilon);
```
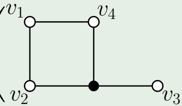
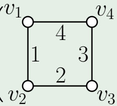

$$\left(254\zeta_7 + 780\zeta_5 - 200\zeta_2\zeta_5 - 196\zeta_3^2 + 80\zeta_2^3 - \frac{168}{5}\zeta_2^2\zeta_3\right)\varepsilon^2 + \left(-28\zeta_3^2 + 140\zeta_5 + \frac{80}{7}\zeta_2^3\right)\varepsilon + 20\zeta_5.$$

Recursion of forest functions

Example ($D = 6$ and $a_e = 1$)

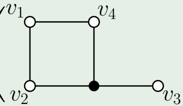
$F\Bigg($ 

$; z\Bigg) = \int_0^{z_3} F\Bigg($ 

$; z_{12}, z_{14}, z_3', z_4\Bigg) \; \mathrm{d}z_3'$

Recursion of forest functions

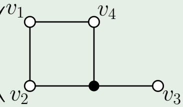
Example ($D = 6$ and $a_e = 1$)



$$F\left(\text{[graph]} ; z \right) = \int_0^{z_3} \frac{z_{12}\, \mathrm{d}z_3'}{[z_{12}(z_{14} + z_3 + z_4) + z_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$$

Recursion of forest functions

Example ($D = 6$ and $a_e = 1$)

$F\Big($ 

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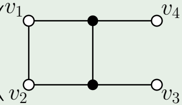
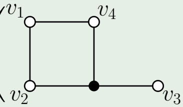
$F\Big($ 

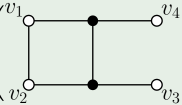
$; z\Big)$

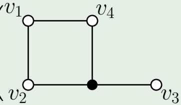
Recursion of forest functions

Example ($D = 6$ and $a_e = 1$)

$F\Big($ 

$; z\Big) = \int_0^{z_3} \frac{z_{12}\, \mathrm{d}z_3'}{[z_{12}(z_{14}+z_3+z_4)+z_3 z_4]^2} = \frac{z_3}{(z_{14}+z_4)\cdot Q}$

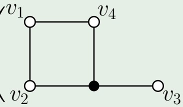
$F\Big($ 

$; z\Big) = \int_0^{z_4} F\Big($ 

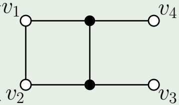
$; z_{12}, z_{14}, z_3, z_4'\Big)\ \mathrm{d}z_4'$

Recursion of forest functions

Example ($D = 6$ and $a_e = 1$)

$F\Bigg($ 

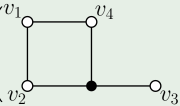
$; z\Bigg) = \int_0^{z_3} \frac{z_{12}\ \mathrm{d}z_3'}{\left[z_{12}\left(z_{14}+z_3+z_4\right)+z_3 z_4\right]^2} = \frac{z_3}{\left(z_{14}+z_4\right)\cdot Q}$

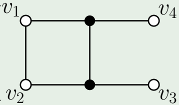
$F\Bigg($ 

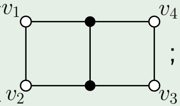
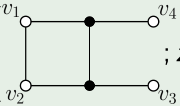
$; z\Bigg) = \frac{1}{z_{12}-z_{14}} \ln \frac{z_{12}(z_3+z_{14})(z_4+z_{14})}{z_{14}\cdot Q}$

Recursion of forest functions

Example ($D = 6$ and $a_e = 1$)

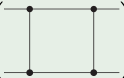
$F\Bigg($  $; z\Bigg) = \int_0^{z_3} \frac{z_{12}\, \mathrm{d}z_3'}{[z_{12}\,(z_{14} + z_3 + z_4) + z_3 z_4]^2} = \frac{z_3}{(z_{14} + z_4) \cdot Q}$

$F\Bigg($  $; z\Bigg) = \frac{1}{z_{12} - z_{14}} \ln \frac{z_{12}(z_3 + z_{14})(z_4 + z_{14})}{z_{14} \cdot Q}$

$F\Bigg($  $; z\Bigg) = \frac{1}{Q^2} \int_0^{z_{12}} F\Bigg($  $; z_{12} - x, z_{14}, z_3, z_4\Bigg)\, x \mathrm{d}x$

Recursion of forest functions

Example (kinematics: $s = (p_1 + p_2)^2$ and $x = (p_1 + p_4)^2/s$)

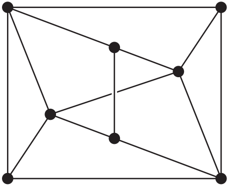
$s\Phi$ ()

$$= \int_0^\infty \frac{\mathrm{d}z_{12}}{z_{12} + x} \int_0^\infty \mathrm{d}z_3 \int_0^\infty \mathrm{d}z_4 \; \frac{z_{12}}{[z_{12}(1 + z_3 + z_4) + z_4 z_3]^2}$$

$$= \int_0^\infty \frac{\mathrm{d}z_{12}}{z_{12} + x} \int_0^\infty \frac{\mathrm{d}z_3}{(1 + z_3)(z_{12} + z_3)}$$

$$= \int_0^\infty \frac{\mathrm{d}z_{12} \;\ln z_{12}}{(z_{12} + x)(z_{12} - 1)} = \frac{\pi^2 + \ln^2 x}{2(1 + x)}$$

Massless ϕ^4 theory: primitive sixth roots of unity

$P_{7,11} =$ 

$P_{7,11}$ is not linearly reducible: After integrating ten variables, denominator

$$
\begin{aligned}
d_{10} = {} & \alpha_2\alpha_4^2\alpha_1 + \alpha_2\alpha_4^2\alpha_3 - \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_2^2\alpha_4\alpha_1 + \alpha_2^2\alpha_4\alpha_3 \\
& - 2\alpha_2\alpha_3^2\alpha_4 - \alpha_2^2\alpha_3^2 - 2\alpha_2^2\alpha_3\alpha_1 - 2\alpha_2\alpha_3^2\alpha_1 - \alpha_3^2\alpha_4^2 \\
& - 2\alpha_3^2\alpha_4\alpha_1 - \alpha_2^2\alpha_1^1 - 2\alpha_2\alpha_3\alpha_1^2 - \alpha_3^2\alpha_1^2.
\end{aligned}
$$

Changing variables $\alpha_3 = \frac{\alpha_3'\alpha_1}{\alpha_1+\alpha_2+\alpha_4}$, $\alpha_4 = \alpha_4'(\alpha_2 + \alpha_3')$ and $\alpha_1 = \alpha_1'\alpha_4'$,

$$
d_{10}' = (\alpha_2 + \alpha_3')(\alpha_2 + \alpha_2\alpha_4' - \alpha_1')(\alpha_1'\alpha_4' + \alpha_2 + \alpha_2\alpha_4' + \alpha_3'\alpha_4')
$$

factors linearly and $\alpha_1', \alpha_3', \alpha_4'$ can be integrated ($\alpha_2 = 1$).
The final integrand is $\mathrm{HPL}(\alpha_1)/(1 - \alpha_1 + \alpha_1^2)$ and gives *not a multiple zeta value*, but a polylogarithm at sixth roots of unity.

$\sqrt{3}\,\mathcal{P}(P_{7,11})$

$$= \operatorname{Im}\left(\tfrac{19\,285}{6}\zeta_9 \operatorname{Li}_2 - \tfrac{1029}{2}\zeta_7 \operatorname{Li}_4 + 240\zeta_3^2(9\operatorname{Li}_{2,3} - 7\zeta_3 \operatorname{Li}_2)\right) - \tfrac{93\,824}{9675}\pi^3\zeta_{3,5}$$

$$+ \tfrac{2592}{215}\operatorname{Im}\Big(36\operatorname{Li}_{2,2,2,5} + 27\operatorname{Li}_{2,2,3,4} + 9\operatorname{Li}_{2,2,4,3} + 9\operatorname{Li}_{2,3,2,4} + 3\operatorname{Li}_{2,3,3,3}$$

$$- 43\zeta_3(\operatorname{Li}_{2,3,3} + 3\operatorname{Li}_{2,2,4})\Big) - \tfrac{96\,393\,596\,519\,864\,341\,538\,701\,979}{790\,371\,465\,315\,684\,594\,157\,620\,000}\pi^{11}$$

$$+ \tfrac{216}{14\,755\,731\,798\,995}\operatorname{Im}\Big(2\,539\,186\,130\,125\,890\operatorname{Li}_8\zeta_3 - 1\,269\,593\,065\,062\,945\operatorname{Li}_{2,9}$$

$$-413\,965\,317\,054\,502\operatorname{Li}_6\zeta_5 - 996\,412\,983\,391\,539\operatorname{Li}_{3,8}$$

$$-546\,306\,741\,059\,841\operatorname{Li}_{4,7} - 156\,228\,639\,992\,955\operatorname{Li}_{5,6}\Big)$$

$$+ \tfrac{2592}{10\,945\,435}\pi^2\operatorname{Im}\Big(287\,205\operatorname{Li}_{2,7} - 574\,410\operatorname{Li}_6\zeta_3 + 55\,687\operatorname{Li}_{4,5} + 168\,941\operatorname{Li}_{3,6}\Big)$$

$$+ \pi\left(\tfrac{11\,613\,751}{9030}\zeta_5^2 + \tfrac{267\,067}{602}\zeta_{3,7} - \tfrac{31\,104}{215}\operatorname{Re}(3\operatorname{Li}_{4,6} + 10\operatorname{Li}_{3,7})\right)$$

Abbreviation: $\operatorname{Li}_{n_1,\dots,n_r} := \operatorname{Li}_{n_1,\dots,n_r}(e^{i\pi/3})$

References

Bibliography I

[1] J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, and F. Wißbrock.
Massive 3-loop ladder diagrams for quarkonic local operator matrix elements.
Nucl. Phys. B, 864(1):52–84, November 2012.
URL: http://www.sciencedirect.com/science/article/pii/S0550321312003355, arXiv:1206.2252, doi:10.1016/j.nuclphysb.2012.06.007.

[2] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog, and B. Mistlberger.
Real-virtual contributions to the inclusive Higgs cross-section at N^3LO.
JHEP, 2013(12):88, December 2013.
arXiv:1311.1425, doi:10.1007/JHEP12(2013)088.

Bibliography II

[3] C. Anastasiou, C. Duhr, F. Dulat, and B. Mistlberger.
Soft triple-real radiation for Higgs production at N_3LO.
JHEP, 2013(7):1–78, jul 2013.
arXiv:1302.4379, doi:10.1007/JHEP07(2013)003.

[4] T. Binoth and G. Heinrich.
An automatized algorithm to compute infrared divergent multiloop integrals.
Nucl. Phys. B, 585(3):741–759, October 2000.
arXiv:hep-ph/0004013, doi:10.1016/S0550-3213(00)00429-6.

[5] T. Binoth and G. Heinrich.
Numerical evaluation of multiloop integrals by sector decomposition.
Nucl. Phys. B, 680(1–3):375–388, March 2004.
arXiv:hep-ph/0305234,
doi:10.1016/j.nuclphysb.2003.12.023.

Bibliography III

[6] C. Bogner and M. Lüders.
Multiple polylogarithms and linearly reducible Feynman graphs.
ArXiv e-prints, February 2013.
arXiv:1302.6215.

[7] F. C. S. Brown.
On the periods of some Feynman integrals.
ArXiv e-prints, October 2009.
arXiv:0910.0114.

[8] F. C. S. Brown.
The Massless Higher-Loop Two-Point Function.
Commun. Math. Phys., 287(3):925–958, May 2009.
arXiv:0804.1660, doi:10.1007/s00220-009-0740-5.

Bibliography IV

[9] F. Chavez and C. Duhr.
Three-mass triangle integrals and single-valued polylogarithms.
JHEP, 11:114, November 2012.
arXiv:1209.2722, doi:10.1007/JHEP11(2012)114.

[10] R. N. Lee.
Presenting LiteRed: a tool for the Loop InTEgrals REDuction.
December 2012.
arXiv:1212.2685.

[11] E. Panzer.
On the analytic computation of massless propagators in dimensional regularization.
Nucl. Phys. B, 874(2):567–593, September 2013.
URL: http://www.sciencedirect.com/science/article/pii/S0550321313003003, arXiv:1305.2161, doi:10.1016/j.nuclphysb.2013.05.025.

Bibliography V

[12] E. Panzer.
Feynman integrals and hyperlogarithms.
PhD thesis, Humboldt-Universität zu Berlin, 2014.
to appear.
URL:
https://www.math.hu-berlin.de/~panzer/paper/phd.pdf.

[13] E. Panzer.
On hyperlogarithms and Feynman integrals with divergences and many scales.
JHEP, 2014(3):71, March 2014.
arXiv:1401.4361, doi:10.1007/JHEP03(2014)071.

[14] E. Panzer.
Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals.
Computer Physics Communications, 188:148–166, March 2015.
arXiv:1403.3385, doi:10.1016/j.cpc.2014.10.019.

[15] O. Schnetz.
Graphical functions and single-valued multiple polylogarithms.
ArXiv e-prints, February 2013.
arXiv:1302.6445.

[16] O. V. Tarasov.
Connection between Feynman integrals having different values of the space-time dimension.
Phys. Rev. D, 54(10):6479–6490, November 1996.
arXiv:hep-th/9606018, doi:10.1103/PhysRevD.54.6479.

[17] E. von Manteuffel, A. Panzer and R. M. Schabinger.
A quasi-finite basis for multi-loop Feynman integrals.
December 2014.
arXiv:1411.7392.