Tropical field theory

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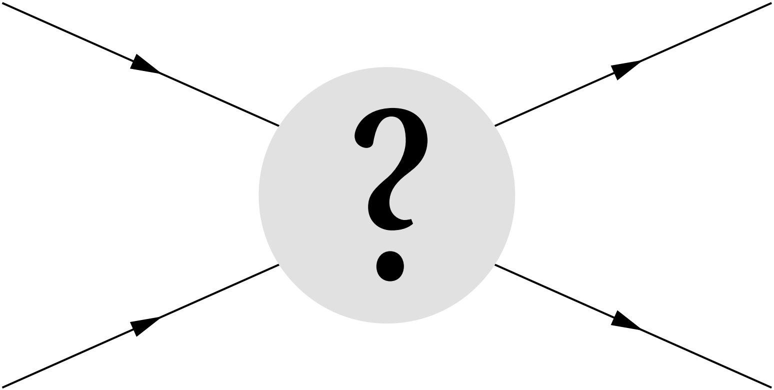
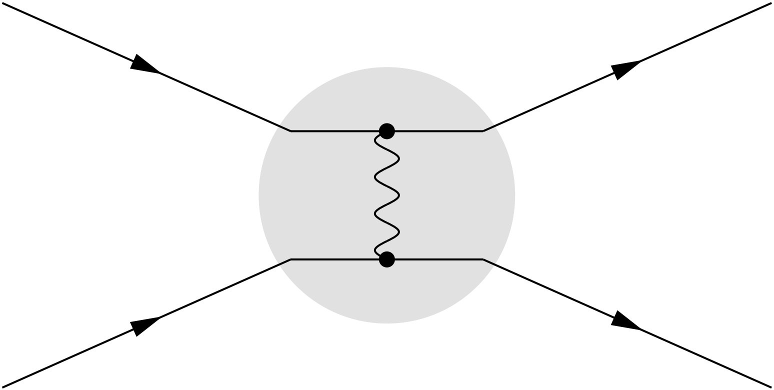
ESI Vienna

Perturbative Quantum Field Theory



Perturbative Quantum Field Theory



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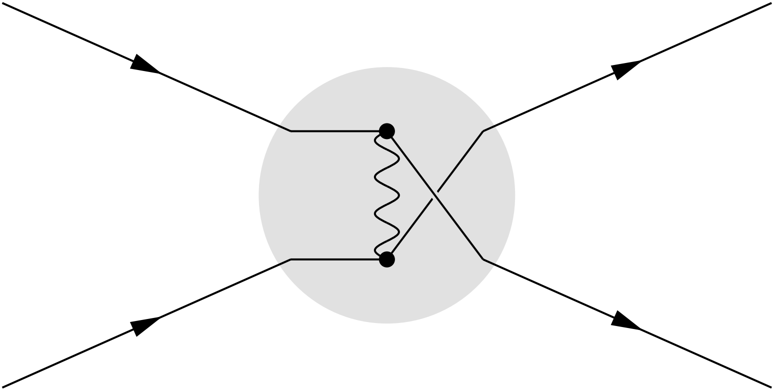

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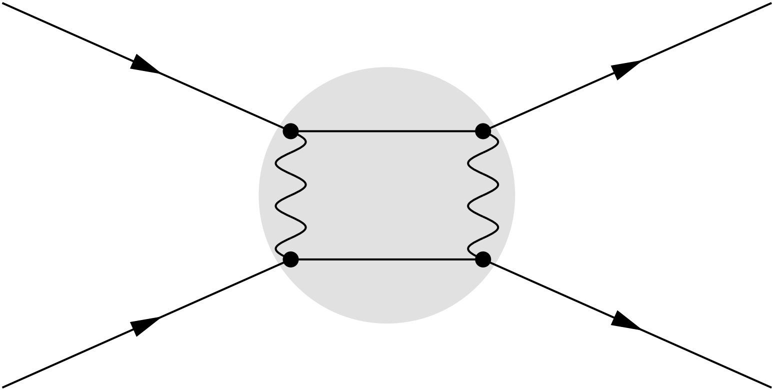

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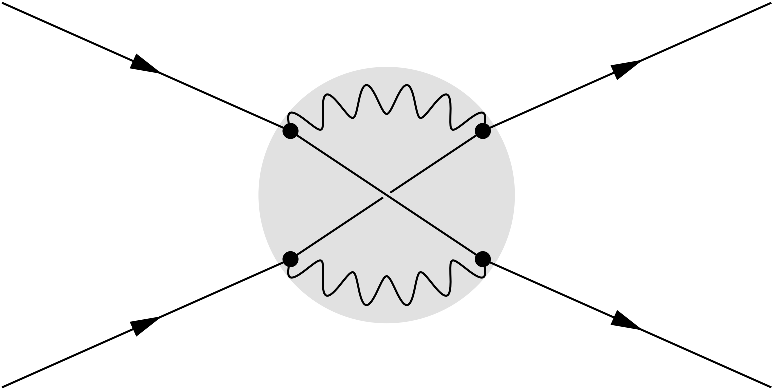

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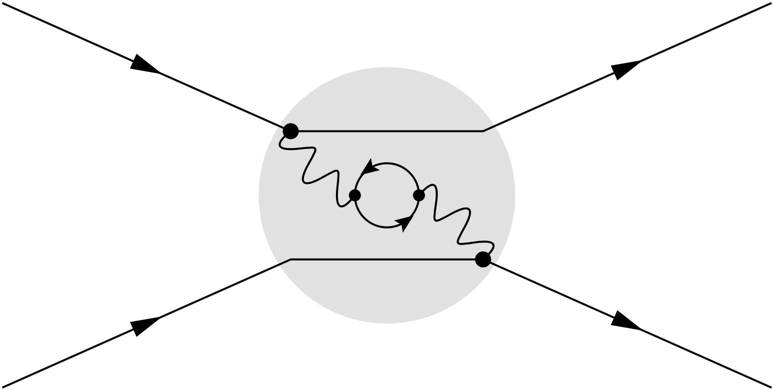

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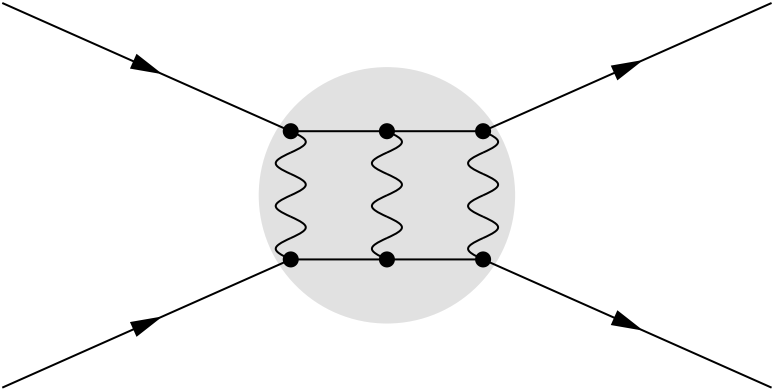

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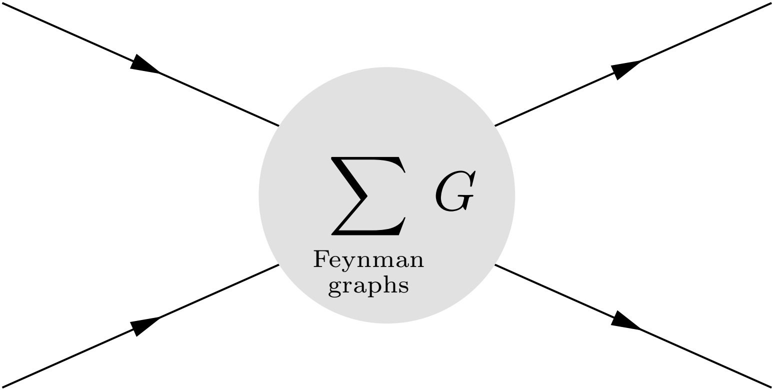

- Feynman graph $G \mapsto$ Feynman integral $\Phi(G, \{m_i^2, \vec{p}_i \cdot \vec{p}_j\})$
- compute more graphs $\sum_G \Phi(G) \Rightarrow$ higher precision

Problems:

1. $\Phi(G)$ extremely complicated

   ➡ *polylogarithms, iterated elliptic integrals, modular forms, $K3$ surfaces, Calabi-Yau manifolds, ...*

2. $\sum_G \Phi(G) = \infty$

   ➡ *factorial growth $A \cdot n! \cdot c^n \cdot n^\alpha$, resummation, Borel transformation, resurgence, ...*

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Simplifications:

1. integrable models

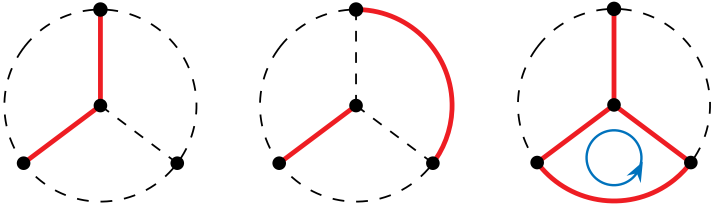
   ⇒ $\Phi(G)$ *simplify somewhat, full* $\sum_G \Phi(G)$, *convergent expansion*

2. truncated Dyson-Schwinger equations

   ⇒ *factorial growth possible, sums very restricted class of diagrams*

3. tropical limit

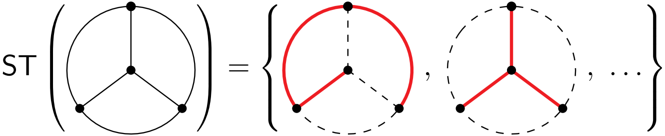
   ⇒ *all $\Phi(G)$ simplify drastically, asymptotics unchanged*



not spanning | not connected | has a loop

Definition

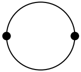
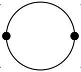
A spanning tree $T \subset G$ is a spanning, simply connected subgraph.



Definition

The graph polynomial $\mathcal{U}$ and Feynman period of G are

$$\mathcal{U} = \sum_{T \in \mathrm{ST}(G)} \prod_{e \notin T} x_e \qquad \text{and} \qquad \mathcal{P}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e \right) \frac{1}{\mathcal{U}^2|_{x_1=1}}$$

$G =$  $\Rightarrow \quad \mathcal{U} = x_1 + x_2$ and $\mathcal{P}\Big($  $\Big) = \int_0^\infty \frac{\mathrm{d}x_2}{(1+x_2)^2} = 1$

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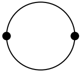
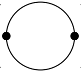
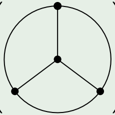
$G =$  $\Rightarrow \quad \mathcal{U} = x_1 + x_2$ and $\mathcal{P}\Big($  $\Big) = \int_0^\infty \frac{\mathrm{d}x_2}{(1+x_2)^2} = 1$

- contribute to the β-function
  ➡ *renormalization constants, running coupling, critical exponents*
- very hard to compute, even numerically

Example

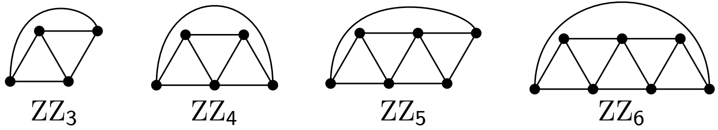
$\mathcal{P}\Big($  $\Big) = \int_{\mathbb{R}_+^5} \frac{\mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5 \mathrm{d}x_6}{(x_1 x_2 x_3 + 15 \text{ more terms})^2|_{x_1=1}} = 6\zeta(3) = 6 \sum_{n=1}^\infty \frac{1}{n^3}$

Only one *infinite* family of periods is known in ϕ^4:

Theorem (Brown & Schnetz 2012)

$$\mathcal{P}(\mathrm{ZZ}_n) = 4\frac{(2n-2)!}{n!(n-1)!}\left(1-\frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3)$$

conjectured by Broadhurst & Kreimer

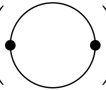


- >1000 periods are known in ϕ^4 [Broadhurst, Kreimer, Schnetz, Panzer]
- complete only up to 7 loops

Hepp bound

$$\mathcal{H}(G) = \left(\prod_{e>1} \int_0^\infty \mathrm{d}x_e\right) \frac{1}{\mathcal{U}^2_{\max}|_{x_1=1}} \quad \text{where} \quad \mathcal{U}_{\max} = \max_{T \in \mathrm{ST}} \prod_{e \notin T} x_e$$

Example:

$\mathcal{H}\Big($  $\Big)$

$$= \int_0^\infty \frac{\mathrm{d}x_2}{(\max\{1, x_2\})^2} = \int_0^1 \mathrm{d}x_2 + \int_1^\infty \frac{\mathrm{d}x_2}{x_2^2} = 2$$

Hepp bound

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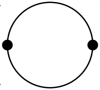
$\mathcal{H}$ () $= \displaystyle\int_0^\infty \frac{\mathrm{d}x_2}{(\max\{1, x_2\})^2} = \int_0^1 \mathrm{d}x_2 + \int_1^\infty \frac{\mathrm{d}x_2}{x_2^2} = 2$

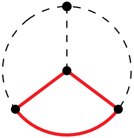
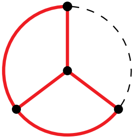
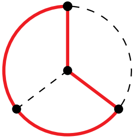
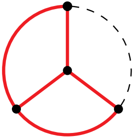
Properties:

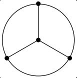
- $\mathcal{H}(G) > \mathcal{P}(G) > \mathcal{H}(G)/\left|\mathrm{ST}(G)\right|^2$ → same asymptotics up to $\mathcal{O}(c^\ell)$
- $\mathcal{H}(G) \in \mathbb{Q}_{>0}$
- computable for all G
- correlates with $\mathcal{P}(G)$
- respects symmetries of $\mathcal{P}(G)$
- generalizes to $\Phi(G, m_e^2, p_i \cdot p_j)$

$$\omega(\gamma_\ell) = |\gamma| - 2\ell$$

Theorem

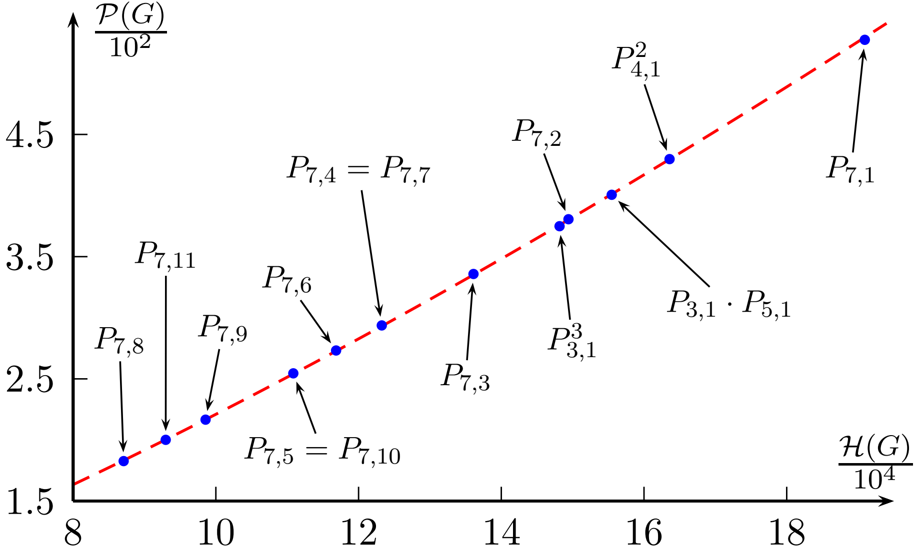
$$\mathcal{H}(G) = \sum_{\substack{\gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_\ell = G \\ \text{each } \gamma_i \text{ is 1PI}}} \frac{|\gamma_1| \cdot |\gamma_2 \setminus \gamma_1| \cdots |G \setminus \gamma_{\ell-1}|}{\omega(\gamma_1) \cdots \omega(\gamma_{\ell-1})}$$

γ_1	$\subset$	γ_2	summand	#	$\sum$
	$\subset$		$\frac{3\cdot 2\cdot 1}{1\cdot 1} = 6$	12	72
	$\subset$		$\frac{4\cdot 1\cdot 1}{2\cdot 1} = 2$	6	12

$\Rightarrow \mathcal{H}\Big($  $\Big) = 84$

7 loops in ϕ^4:

G	$\mathcal{P}(G \setminus v)$	$\mathcal{H}(G \setminus v)$
$P_{7,1}$	527.7	190952
$P_{4,1} \cdot P_{4,1}$	430.1	163592
$P_{3,1} \cdot P_{5,1}$	400.9	155484
$P_{7,2}$	380.9	149426
$P_{3,1} \cdot P_{3,1} \cdot P_{3,1}$	375.2	148176
$P_{7,3}$	336.1	136114
$\{P_{7,4}, P_{7,7}\}$	294.0	123260
$P_{7,6}$	273.5	116860
$\{P_{7,5}, P_{7,10}\}$	254.8	110864
$P_{7,9}$	216.9	98568
$P_{7,11}$	200.4	92984
$P_{7,8}$	183.0	87088



$\frac{\mathcal{P}(G)}{10^2}$
4.5
3.5
2.5
1.5
$\frac{\mathcal{H}(G)}{10^4}$
8
10
12
14
16
18
$P_{7,8}$
$P_{7,11}$
$P_{7,9}$
$P_{7,6}$
$P_{7,4} = P_{7,7}$
$P_{7,5} = P_{7,10}$
$P_{7,3}$
$P_{7,2}$
$P_{3,1}^3$
$P_{4,1}^2$
$P_{3,1} \cdot P_{5,1}$
$P_{7,1}$

Michael Borinsky:

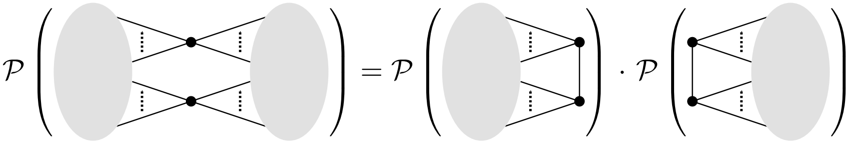
Tropical Monte Carlo quadrature for Feynman integrals

➡ numeric evaluation at large loop orders

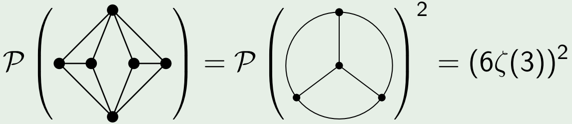
Symmetries

When is $\mathcal{P}(G_1) = \mathcal{P}(G_2)$?

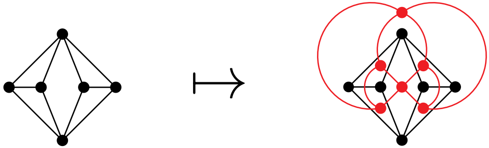
1. Product:



Example

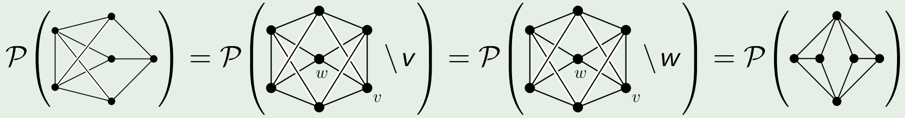


2. Planar duality: $\mathcal{P}(G) = \mathcal{P}(G^{\text{dual}})$

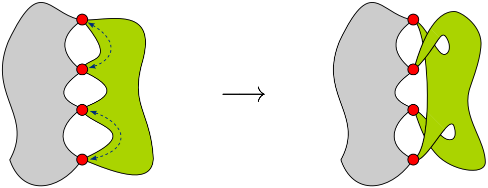


3. Completion: $\mathcal{P}(G \setminus v) = \mathcal{P}(G \setminus w)$

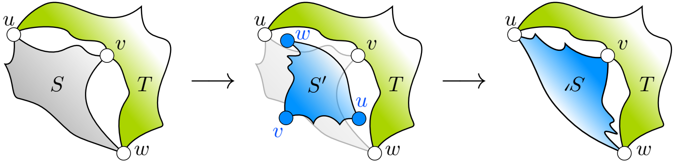
Example



4. Twist:



5. Fourier split:



Goal:

Construct simpler graph invariants with those symmetries.

1. point count [Schnetz]

$$c_2(p) = \frac{1}{p^2} \left| \left\{ \vec{x} \in (\mathbb{Z}/p\mathbb{Z})^N \colon \ \mathcal{U}(\vec{x}) = 0 \right\} \right| \quad \text{mod } p$$

2. extended graph permanent [Crump]

$P_{7,11}$

p	2	3	5	7	11	13	17	19	23
$c_2(p)$	1	0	1	-1	1	-1	1	-1	1
Perm(p)		0	1	1	1	11	5	0	22

3. $\mathcal{H}$ $(\in \mathbb{Q})$
4. $\#\{$minimal 6-cuts$\}$ $(\in \mathbb{Z})$
5. $O(-2)$ symmetry factor $(\in \mathbb{Z})$

Conjecture (ϕ^4)

$$\mathcal{H}(G_1) = \mathcal{H}(G_2) \qquad \Leftrightarrow \qquad \mathcal{P}(G_1) = \mathcal{P}(G_2)$$

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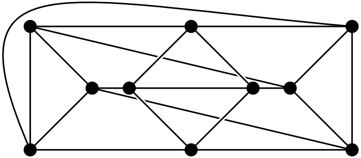
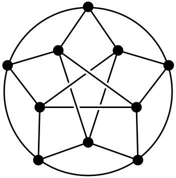
Example:

- $\mathcal{H}(P_{8,30}) = \frac{1724488}{3} = \mathcal{H}(P_{8,36})$
- $\mathcal{P}(P_{8,30}) \approx 505.5 \approx \mathcal{P}(P_{8,36})$ (exact period unknown)

$\mathcal{P}\Bigg($  $\Bigg) = \mathcal{P}\Bigg($  $\Bigg)$?

➡ **there are further symmetries of Feynman integrals**

Spanning tree polytope and its polar (relevant for sector decomposition):

$$\vec{v}_T = \vec{T} - \vec{T}^c \in \{1, -1\}^{E_G}$$

$$\mathcal{N}_G = \operatorname{conv}\{\vec{v}_T\colon\ T \in \mathrm{ST}\} \subset \mathbb{R}^{E_G}$$

$$\mathcal{N}_G^\circ = \bigcap_{T \in \mathrm{ST}} \{\vec{a}\colon\ \vec{a} \cdot \vec{v}_T \leq 1\}$$

The Hepp bound is the volume of the polar polytope

$$\mathcal{H}(G) = (E_G - 1)! \cdot \operatorname{Vol}\left(\mathcal{N}_G^\circ \cap \{a_1 = 0\}\right)$$

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The Hepp bound is the volume of the polar polytope

$$\mathcal{H}(G) = (E_G - 1)! \cdot \mathrm{Vol}\,(\mathcal{N}_G^\circ \cap \{a_1 = 0\})$$

Facets of $\mathcal{N}_G$/vertices of $\mathcal{N}_G^\circ$ are indexed by subgraphs:

$$\{\gamma \subset G\colon\ \gamma \text{ and } G/\gamma \text{ are 2-vertex connected}\}$$

Factorisation of the facets:

$$\mathcal{N}_G \cap \{\vec{\gamma}\cdot\vec{a} = \omega_\gamma\} \cong \mathcal{N}_\gamma \times \mathcal{N}_{G/\gamma}$$

Roughly, $\mathcal{N}_G$ looks like a cube, and $\mathcal{N}_G^\circ$ is a cross-polytope: very “spikey” and all volume concentrated near the centre.

Multivariate version & canonical form

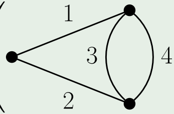
Now consider arbitrary indices:

$$\mathcal{H}(G;\vec{a}) := \left(\prod_{e>1}\int_0^\infty x_e^{a_e-1}\mathrm{d}x_e\right)\frac{1}{\mathcal{U}_{\max}^{D/2}|_{x_1=1}}$$

The dimension is fixed by $\omega(G) = \sum_e a_e - (D/2)\cdot\ell(G) \stackrel{!}{=} 0$.

Example

The flag formula generalizes to this case, e.g.



$$\mathcal{H}\left(\,\cdot\,;\vec{a}\right) = \frac{1}{a_1a_2a_3a_4}\times\left\{\frac{(a_1+a_2+a_3)a_4}{a_1+a_2+a_3-D/2} + \frac{(a_1+a_2+a_4)a_3}{a_1+a_2+a_4-D/2} + \frac{(a_3+a_4)(a_1+a_2)}{a_3+a_4-D/2}\right\}$$

Consider the Hepp bound $\mathcal{H}(G;\vec{a})$:

- it is a rational function in $\vec{a}$
- it has simple poles
- at hyperplanes $\omega(\gamma)=0$ for 1PI subgraphs γ

Factorization of residues

$$\operatorname*{Res}_{\omega(\gamma)=0} \mathcal{H}(G;\vec{a}) = \mathcal{H}(\gamma;\vec{a}_\gamma)\Big|_{\omega(\gamma)=0} \cdot \mathcal{H}(G/\gamma;\vec{a}_{G/\gamma})\Big|_{\omega(G/\gamma)=0}$$

Example: edge contraction

$$\operatorname*{Res}_{a_e=0} \mathcal{H}(G;\vec{a}) = \mathcal{H}(G/e;\vec{a}_{G/e})$$

- it is the volume of a polytope:

$$\mathcal{H}(G;\vec{a}) = (E-1)! \cdot \operatorname{Vol}\left(\left(\mathcal{N}_G + (\vec{a}-\vec{1})\right)^\circ \cap \{a_1=0\}\right)$$

$\Rightarrow$ canonical form

Summary

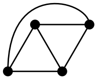
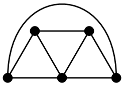
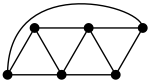
- There is a rational version of Feynman periods.
- It captures identities and gives numeric estimates.
- Volume of a polytope with factorizing residues.
- Generalizes to matroids.

Outlook

- add kinematics
- dimensional regularization
- renormalization
- tropical field theory
- asymptotics
- numerics for Feynman integrals

Theorem (Brown & Schnetz)

$$\mathcal{P}(\mathrm{ZZ}_n) = 4\frac{(2n-2)!}{n!(n-1)!}\left(1-\frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3) \sim \frac{4^n}{n\sqrt{\pi n}}.$$

$\mathrm{ZZ}_3 =$  $\mathrm{ZZ}_4 =$  $\mathrm{ZZ}_5 =$ 

Theorem (Panzer)

The Hepp bound of ZZ_n is the coefficient of x^n in the power series

$$\frac{1}{(1-x)(5x+3)}\left[\frac{5x+28}{3}-\frac{2}{1+x}-\frac{1}{x}\sqrt{\frac{1-9x}{1-x}}\right]-4x^2\log(1-x^2).$$

Asymptotics:

$$\mathcal{H}(\mathrm{ZZ}_n) \sim \frac{3^7}{2^{10}\sqrt{2\pi}}\frac{9^n}{n^{3/2}} \approx 0.852\frac{9^n}{n^{3/2}}.$$

Hepp-Period correlation