HUMBOLDT-UNIVERSITÄT ZU BERLIN

Regularization of divergent Feynman integrals and their computation with hyperlogarithms

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ETH Zürich (Hönggerberg), March 18th, 2014
Particle Physics Seminar

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Plan of the talk

1. The hyperlogarithm method of integration

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2. Examples of linearly reducible Feynman integrals

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2. Examples of linearly reducible Feynman integrals
3. Divergences and analytic regularization

Motivation: Feynman integrals in Schwinger parameters

For scalar propagators $P_e = (p_e^2 + m_e^2)^{-a_e}$, Feynman integrals become

$$\Phi(G) = \frac{\Gamma(\text{sdd})}{\prod_e \Gamma(a_e)} \int_{\mathbb{R}_+^{|E|}} F\ \Omega \quad \text{with} \quad F = \psi^{\text{sdd}-D/2} \cdot \varphi^{-\text{sdd}} \cdot \prod_{e \in E} \alpha_e^{a_e - 1}.$$

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The graph polynomials ψ, φ are sums over all spanning trees T and all spanning two-forests F with external momentum $q(T_1)$ entering T_1:

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \quad \text{and} \quad \varphi = \sum_{F = T_1 \dot\cup T_2} q^2(T_1) \prod_{e \notin F} \alpha_e + \psi \sum_e m_e^2 \alpha_e.$$

Their homogeneity is exploited in that $\Omega = \delta(H) \cdot \bigwedge_e \mathrm{d}\alpha_e$ restricts to a hyperplane H, e.g. $H = 1 - \alpha_{|E|}$. When G has $|G|$ loops,

$$\mathrm{sdd} = \sum_e a_e - \frac{D}{2}|G| \quad \text{denotes its superficial degree of divergence.}$$

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Idea: Expand $F = \sum_n \varepsilon^n F_n$ with $F_n \in \mathbb{Q}[\alpha_e, \psi^{-1}, \varphi^{-1}, \ln\psi, \ln\varphi, \ln\alpha_e]$ and compute the integrals $\int F_n\ \Omega$ along a sequence $\alpha_1, \alpha_2, \ldots, \alpha_{|E|-1}$ of edges.

Hyperlogarithms for Feynman integrals, F. Brown [9, 8, 7]

Definition

For any word $w \in \Sigma^{\times}$ in letters $\{\omega_\sigma : \ \sigma \in \Sigma\}$ over a finite set $0 \in \Sigma \subset \mathbb{C}$, the associated *hyperlogarithm* $L_w(z)$ is defined by

$$L_{\omega_0^n}(z) := \frac{\log^n z}{n!} \quad \text{for any } n \in \mathbb{N}_0 \text{ and } \quad L_{\omega_\sigma w}(z) := \int_0^z \frac{\mathrm{d}z'}{z' - \sigma} L_w(z').$$

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Alternative names

Goncharov polylogarithms, *generalized harmonic polylogarithms*:

$$L_{\omega_{\sigma_1} \ldots \omega_{\sigma_n}}(z) = G\left(\sigma_1, \ldots, \sigma_n; z\right).$$

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Example (Relation to classical and multiple polylogarithms)

For any $0 \neq \sigma \in \Sigma$ and $n \in \mathbb{N}$ note

$$L_{\omega_\sigma}(z) = \log(z - \sigma) - \log(-\sigma) = -\operatorname{Li}_1\left(\frac{z}{\sigma}\right), \quad L_{\omega_0^{n-1}\omega_\sigma}(z) = -\operatorname{Li}_n\left(\frac{z}{\sigma}\right).$$

More generally, for $\sigma_1, \ldots, \sigma_r \in \Sigma \setminus \{0\}$ and $n_1, \ldots, n_r \in \mathbb{N}$ we have

$$L_{\omega_0^{n_r-1}\omega_{\sigma_r} \cdots \omega_0^{n_2-1}\omega_{\sigma_2}\omega_0^{n_1-1}\omega_{\sigma_1}}(z) = (-1)^r \operatorname{Li}_{n_1,\ldots,n_r}\left(\frac{\sigma_2}{\sigma_1}, \ldots, \frac{\sigma_r}{\sigma_{r-1}}, \frac{z}{\sigma_r}\right).$$

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Properties

1. Algebra: $L_. : \mathbb{Q}\langle\Sigma\rangle \longrightarrow \mathcal{O}(\mathbb{C} \setminus \Sigma)$ (extended linearly) is a morphism of algebras for the *shuffle product*: $L_u \cdot L_v = L_{u \sqcup\!\sqcup v}$. For example,

$$L_{\omega_\sigma} \cdot L_{\omega_{\tau_1}\omega_{\tau_2}} = L_{\omega_\sigma\omega_{\tau_1}\omega_{\tau_2}} + L_{\omega_{\tau_1}\omega_\sigma\omega_{\tau_2}} + L_{\omega_{\tau_1}\omega_{\tau_2}\omega_\sigma}.$$

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4. Monodromies: For paths $\gamma, \eta : [0,1] \longrightarrow \mathbb{C} \setminus \Sigma$ with $\gamma(1) = \eta(0)$,

$$\int_{\gamma\star\eta} \omega_{\sigma_1} \ldots \omega_{\sigma_n} = \sum_{k=0}^{n} \int_\eta \omega_{\sigma_1} \ldots \omega_{\sigma_k} \int_\gamma \omega_{\sigma_{k+1}} \ldots \omega_{\sigma_n} \quad \text{(Chen's lemma)}$$

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5. Divergences: For any $w \in \Sigma^\times$ and $\sigma \in \Sigma \cup \{\infty\}$ exist unique $f_w^{\sigma,k}(z)$, $0 \leq k \leq |w|$, holomorphic at $z \to \sigma$ such that
$$L_w(z) = \sum_k \log^k(z - \sigma) \cdot f_w^{\sigma,k}(z).$$

Hyperlogarithms for Feynman integrals, F. Brown [9, 8, 7]

Integration

By construction and use of partial fractioning, every integrand

$$f(z) \in L(\Sigma) := \mathbb{Q}\left[z, \left\{\frac{1}{z-\sigma} : \ \sigma \in \Sigma\right\}, \{L_w : \ w \in \Sigma^\times\}\right]$$

has a primitive $F(z)$ in the enlarged algebra

$$R(\Sigma) := \mathbb{Q}\left[\Sigma \cup \left\{\frac{1}{\sigma_i - \sigma_j} : \ \sigma_i \neq \sigma_j \text{ from } \Sigma\right\}\right] \otimes L(\Sigma).$$

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Example

$$\int \frac{L_{\omega_\sigma}(z)\mathrm{d}z}{(z-\tau)^2} = -\frac{L_{\omega_\sigma}(z)}{z-\tau} + \int \frac{\mathrm{d}z}{(z-\tau)(z-\sigma)} = -\frac{L_{\omega_\sigma}(z)}{z-\tau} + \frac{L_{\omega_\tau}(z) - L_{\omega_\sigma}(z)}{\tau-\sigma}$$

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For the following integration, we must extract the dependence of

$$\int_0^\infty f(z)\mathrm{d}z = \lim_{z\to\infty} F(z) - \lim_{z\to 0} F(z)$$

on the next Schwinger variable.

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Rewriting polylogarithms as hyperlogarithms

Consider $w = \omega_{\sigma_1} \dots \omega_{\sigma_n}$ with $\sigma_1, \dots, \sigma_n \in \mathbb{Q}(t)$ depending rationally on a variable t (the next Schwinger parameter α_e to integrate):

$$\frac{\partial}{\partial t} L_w(z) = \frac{-\sigma_1'}{z - \sigma_1} L_{\omega_{\sigma_2} \dots}(z) + \sum_{i=1}^{n-1} \frac{(\sigma_i - \sigma_{i+1})'}{\sigma_i - \sigma_{i+1}} L_{\dots \not{\omega}_{\sigma_{i+1}} \dots - \dots \not{\omega}_{\sigma_i} \dots}(z) - \frac{\sigma_n'}{\sigma_n} L_{\dots \not{\omega}_{\sigma_n}}(z)$$

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Example ($\lim_{z\to\infty} L_{(\omega_{-t}-\omega_{-1})\omega_{-1}}(z)$)

$$\frac{\partial}{\partial t} \lim_{z\to\infty} \left[L_{\omega_{-t}\omega_{-1}}(z) - L_{\omega_{-1}\omega_{-1}}(z) \right] = \left[\frac{\partial}{\partial t} \log(t-1) \right] \cdot \lim_{z\to\infty} L_{\omega_{-t}-\omega_{-1}}(z)$$

$$= \frac{1}{t-1} \lim_{z\to\infty} \log \frac{(z+t)}{t(z+1)} = -\frac{1}{t-1} L_{\omega_0}(t)$$

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The constants C of integration can be computed symbolically as well in `HyperInt` [15].

Linear reducibility

This method applies only if for all $n < N$, the partial integrals

$$f_n := \left[\prod_{e=1}^{n-1} \int_0^\infty \mathrm{d}\alpha_e\right] F \quad \text{lie in} \quad L(\Sigma_n)(\alpha_n) \tag{1.1}$$

for some sets $\Sigma_n \subset \mathbb{Q}(\alpha_{n+1}, \ldots, \alpha_N)$. In particular, at the next stage the differences $\sigma_i - \sigma_j$ have to factor linearly in α_{n+1} for all $\sigma_i, \sigma_j \in \Sigma_n$.

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Polynomial reduction

A simple algorithm providing sufficient conditions for linear reducibility is given by Stembridge's procedure as explained in [9]. Much more powerful is the method of compatibility graphs developed in [8].

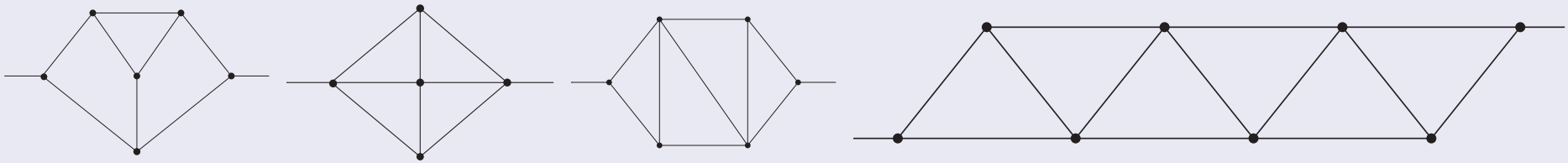
Are there linearly reducible Feynman graphs?

Single-scale integrals

Massless propagators: Coefficients of ε-expansions

Theorem (positive matrix graphs [8])

All graphs G of vertex-width $\mathrm{vw}(G) \leq 3$ are linearly reducible and their periods lie in $\mathcal{Z} := \mathrm{lin}_{\mathbb{Q}} \{\zeta(n_1, \ldots, n_r) : \ n_1, \ldots, n_r \in \mathbb{N} \text{ and } n_r > 1\}$.



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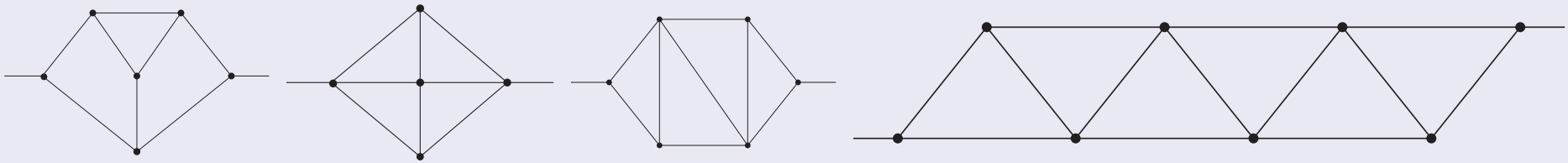
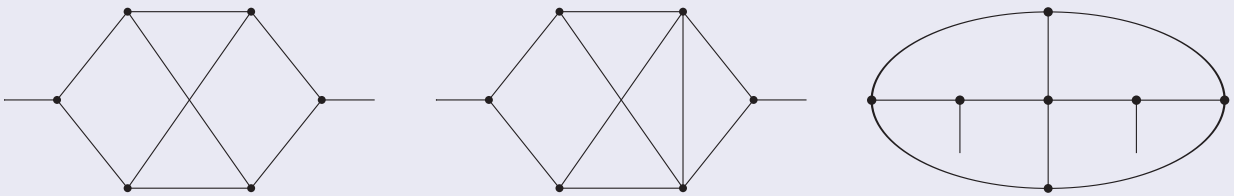


Theorem (vacuum graphs with four or five loops [14])

All massless propagators up to four loops are linearly reducible. Their periods are constrained to alternating Euler sums $\mathcal{Z}_2$:

$$\mathcal{Z}_2 := \mathrm{lin}_{\mathbb{Q}} \left\{ \mathrm{Li}_{n_1,\ldots,n_r}(\sigma_1, \ldots, \sigma_r) : \ n_i \in \mathbb{N}, \sigma_i^2 = 1 \forall i \text{ and } (n_r, \sigma_r) \neq (1,1) \right\}.$$



Massless on-shell four-point graphs

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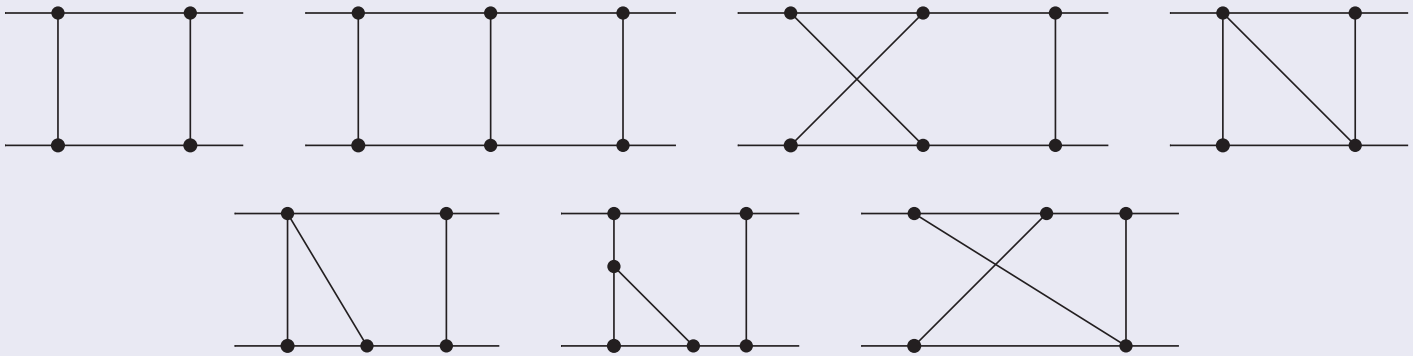
Theorem ([6])

All massless four-point on-shell graphs ($p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$) *with at most two loops are linearly reducible. In particular these include*



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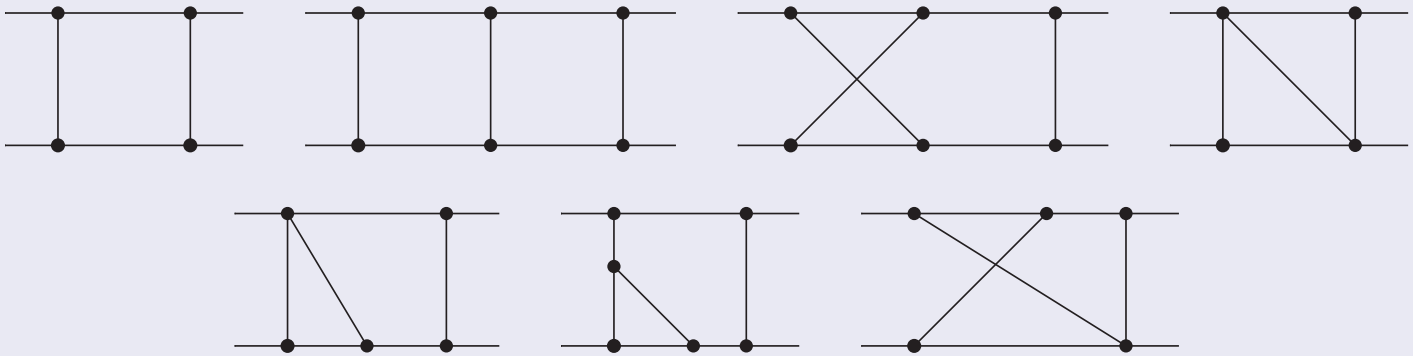
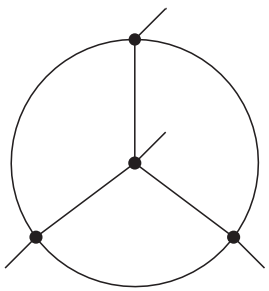


$K_4 =$ 

At three loops, there are non-reducible graphs characterized by forbidden minors like K_4.

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Theorem ([6])

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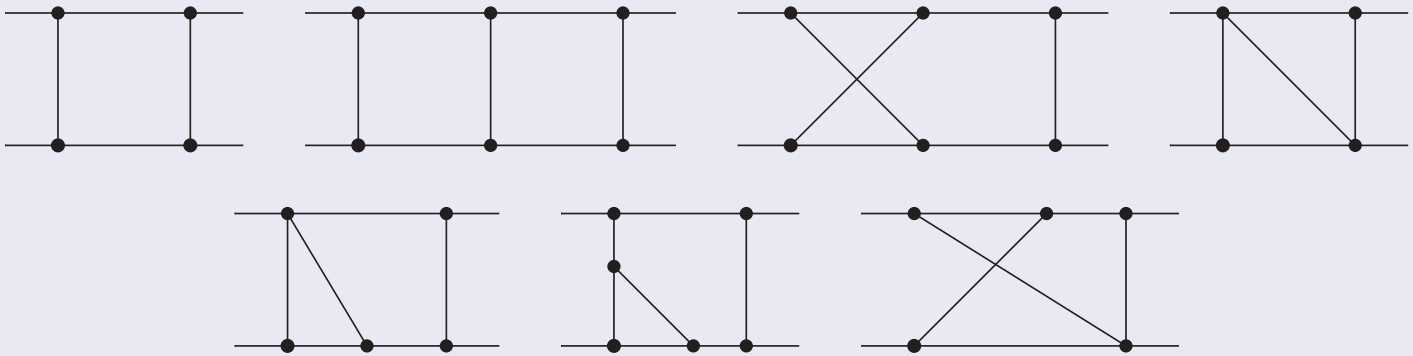
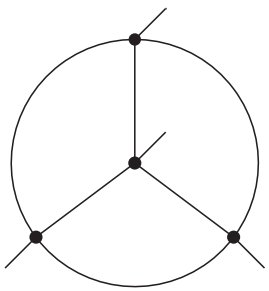


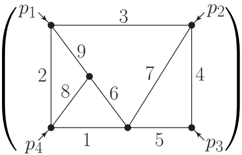
$K_4 =$ 

At three loops, there are non-reducible graphs characterized by forbidden minors like K_4.
Still, K_4 evaluates to polylogarithms [12]. Linear reducibility can be restored through a change of variables, hence parametric integration of K_4 is possible.

Massless on-shell four-point graphs

Linearly reducible four-loop example

Φ



$$= \frac{\Gamma(1+4\varepsilon)}{s^{1+4\varepsilon}} \sum_{n=-1}^{\infty} f_n\left(\frac{s}{u}\right) \cdot \varepsilon^n$$

is linearly reducible along the sequence $1, 2, 8, 6, 9, 7, 5, 4$ of edges ($\alpha_3 = 1$) with final set $\{s + u\}$, so all f_n are harmonic polylogarithms of $\frac{s}{u}$:

Massless on-shell four-point graphs

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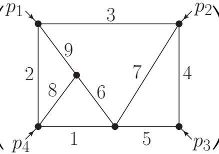
$\Phi\Big($ 

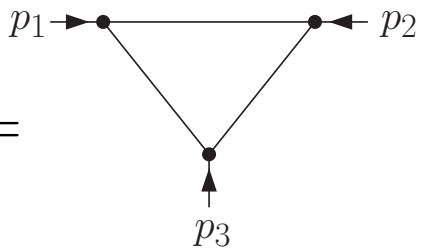
$\Big) = \dfrac{\Gamma(1+4\varepsilon)}{s^{1+4\varepsilon}} \sum_{n=-1}^{\infty} f_n\left(\dfrac{s}{u}\right)\cdot\varepsilon^n$

is linearly reducible along the sequence $1,2,8,6,9,7,5,4$ of edges ($\alpha_3 = 1$) with final set $\{s+u\}$, so all f_n are harmonic polylogarithms of $\frac{s}{u}$:

$$
\begin{aligned}
f_{-1}={}&-\tfrac{79}{70}\zeta_2^3 H_{-1}-\zeta_3\big(15\zeta_2 H_{-1,-1}-9\zeta_2 H_{-1,0}-H_{-1,-2,-1}+H_{-1,-1,-2}+6H_{-1,-1,0,0}\big)\\
&-6\zeta_3^2 H_{-1}-\tfrac{3}{2}\zeta_5(11H_{-1,-1}-5H_{-1,0})-\tfrac{3}{10}\zeta_2^2(H_{-1,-2}-17H_{-1,-1,0}-10H_{-1,-1,-1})\\
&-\zeta_2\Big(H_{-1,-2,0,0}-2H_{-1,-1,-2,0}+3H_{-1,-1,-2,-1}-H_{-1,-1,-1,0,0}+6H_{-1,-1,-3}\\
&\qquad-3H_{-1,-2,-1,-1}-2H_{-1,-1,0,0,0}\Big)+H_{-1,-2,-1,0,0,0}-H_{-1,-1,-2,-1,0,0}\\
&+H_{-1,-1,-2,0,0,0}-2H_{-1,-1,-3,0,0}+H_{-1,-2,-1,-1,0,0}
\end{aligned}
$$

Off-shell massless vertices

Consider zero internal masses $m_e = 0$ and three external momenta like in

$\Delta :=$ 

where

$$\psi = \alpha_1 + \alpha_2 + \alpha_3 \text{ and}$$
$$\varphi = p_1^2\alpha_2\alpha_3 + p_2^2\alpha_1\alpha_3 + p_3^3\alpha_1\alpha_2.$$

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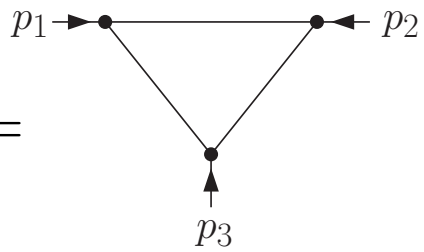
$\Delta :=$ 

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To rationalize the square-root of the Källén function we parametrize as

$$p_2^2 = p_1^2 \cdot z\bar{z} \qquad \text{and} \qquad p_3^2 = p_1^2 \cdot (1-z)(1-\bar{z}), \qquad \text{such that}$$
$$z - \bar{z} = \sqrt{p_1^2 + p_2^2 + p_3^2 - 2p_1p_2 - 2p_1p_3 - 2p_2p_3}.$$

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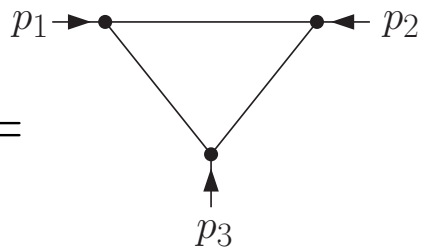
$\Delta :=$ 

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Then the triangle G_1 expands near $D = 4 - 2\varepsilon$ with $a_1 = a_2 = a_3 = 1$ as

$$\Phi(\Delta) = \frac{\Gamma(1+\varepsilon)}{z-\bar{z}} \cdot p_1^{-2(1+\varepsilon)} \cdot \sum_{n=0}^{\infty} f_n(z,\bar{z})\varepsilon^n, \qquad \text{at leading order}$$

$f_0 = 4i\Im\left\{\mathrm{Li}_2(z) + \ln|z| \cdot \ln(1-z)\right\}$ is the Bloch-Wigner dilogarithm.

Off-shell massless vertices

The functions f_n have symbols with letters in $\{z, \bar{z}, 1-z, 1-\bar{z}, z-\bar{z}\}$ and are generalized single-valued multiple polylogarithms (SVMP) [17], e.g.

$$f_1 = 4i\Im\left\{\mathrm{Li}_{1,2}\left(\frac{z}{\bar{z}}, \bar{z}\right) + 2\,\mathrm{Li}_3(z) + \mathrm{Li}_{2,1}(1,z) + \mathrm{Li}_{1,2}(1,z) \right.$$
$$\left. + \ln|z| \cdot \left[\mathrm{Li}_{1,1}\left(\frac{\bar{z}}{z}, z\right) - 2\,\mathrm{Li}_2(z) - \ln|z(1-z)| \cdot \ln(1-z)\right]\right\}.$$

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Linear reducibility at two loops was observed in [10] and we checked

Theorem ([16])

All three-loop off-shell massless three-point functions are linearly reducible. Some require additional letters $\{z\bar{z}-1, z+\bar{z}-1, z\bar{z}-z-\bar{z}\}$.

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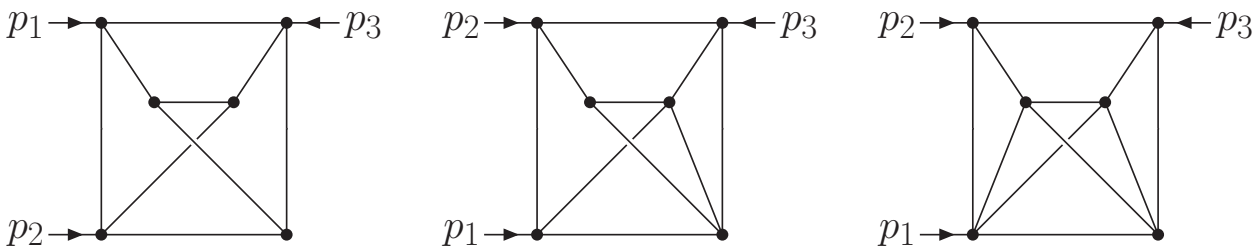
Also at higher loop numbers we found linearly reducible graphs like



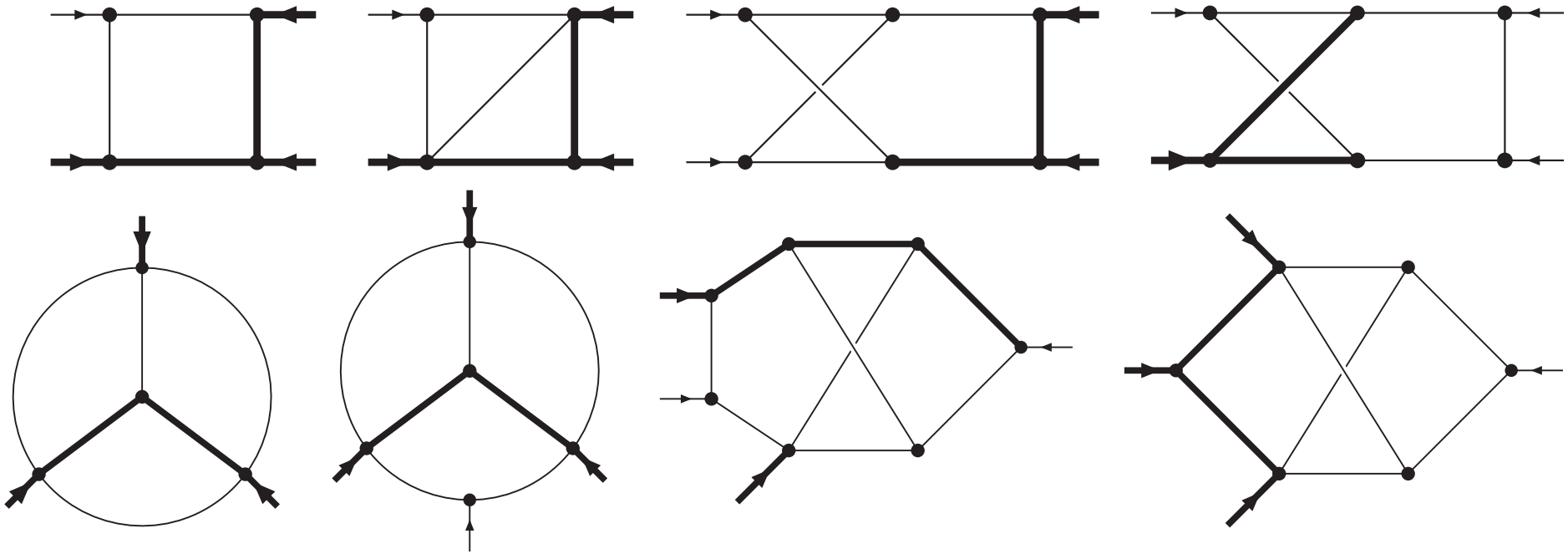
Non-trivial kinematics: up to 7 scales

Examples: massive integrals

Convention:

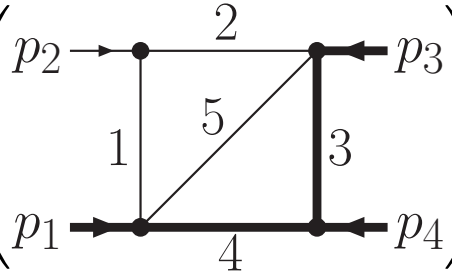
- thin lines: light-like momenta $p_e^2 = 0$, massless propagators $m_e = 0$
- thick lines: arbitrary masses m_e and external momenta p_e^2 (in particular the masses may be all different)

Then the following graphs are linearly reducible:



Non-trivial kinematics: up to 7 scales

Example: double triangle with three legs off-shell and two massive propagators

$\Phi\Bigg($ 

$\Bigg) = \Gamma(1+2\varepsilon)\int \frac{\Omega}{\varphi^{1+2\varepsilon}\psi^{1+\varepsilon}}$

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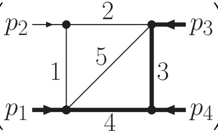

$$\Phi\left(\cdots \right) = \frac{\Gamma(1+2\varepsilon) q^{-1-2\varepsilon}}{1+p-s-u} \sum_{n=-1}^{\infty} f_n(s,u,p,m,M)\varepsilon^n$$

is linearly reducible. We simplify to $p_4^2 = 0$, let $q := p_3^2$ and set

$$p := \frac{p_1^2}{q}, s := \frac{(p_1+p_2)^2}{q}, u := \frac{(p_1+p_4)^2}{q}, M := \frac{m_3^2}{q}, m := \frac{m_4^2}{q} \text{ in}$$

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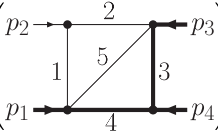

$$\Phi\left(\text{graph}\right) = \frac{\Gamma(1+2\varepsilon) q^{-1-2\varepsilon}}{1+p-s-u} \sum_{n=-1}^{\infty} f_n(s,u,p,m,M)\varepsilon^n$$

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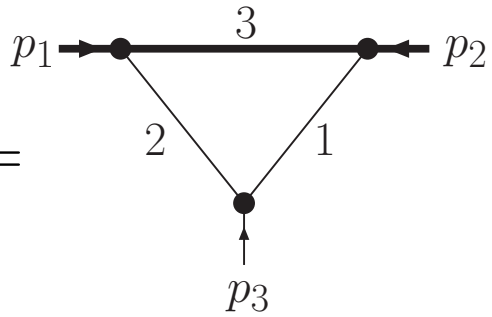
$$\begin{aligned}
f_{-1} =& \mathrm{Li}_{1,1,1}\left(\tfrac{1-u+p-s}{(1-s)(1-u)}, \tfrac{M(1-s)}{M-m}, \tfrac{M-m}{M}\right) - \mathrm{Li}_{1,1,1}\left(\tfrac{M(1-u+p-s)}{(M-m)(1-u)}, \tfrac{(u+M)(M-m)}{(-p-m+u+M)M}, \tfrac{-p-m+u+M}{u+M}\right) \\
&+ \mathrm{Li}_{1,1,1}\left(-\tfrac{u(1-u+p-s)}{(-u+p)(1-u)}, -\tfrac{(u+M)(-u+p)}{(-p-m+u+M)u}, \tfrac{-p-m+u+M}{u+M}\right) - \ln\left(\tfrac{m}{p+m}\right)\mathrm{Li}_{1,1}\left(-\tfrac{u(1-u+p-s)}{(-u+p)(1-u)}, -\tfrac{-u+p}{u}\right) \\
&- \mathrm{Li}_{1,1,1}\left(-\tfrac{u(1-u+p-s)}{(-u+p)(1-u)}, -\tfrac{M(-u+p)}{(M-m)u}, \tfrac{M-m}{M}\right) - \mathrm{Li}_{1,1,1}\left(\tfrac{1-u+p-s}{(1-s)(1-u)}, \tfrac{(1+M)(1-s)}{-m-s+1+M}, \tfrac{-m-s+1+M}{1+M}\right) \\
&+ \mathrm{Li}_{1,1,1}\left(\tfrac{M(1-u+p-s)}{(M-m)(1-u)}, \tfrac{(1+M)(M-m)}{(-m-s+1+M)M}, \tfrac{-m-s+1+M}{1+M}\right) - \ln\left(-\tfrac{p-s}{1-u}\right)\mathrm{Li}_2\left(\tfrac{p}{p+m}\right) + \ln\left(-\tfrac{p-s}{1-u}\right)\mathrm{Li}_2\left(\tfrac{s}{m+s}\right) \\
&+ \ln\left(\tfrac{m}{m+s}\right)\mathrm{Li}_{1,1}\left(\tfrac{1-u+p-s}{(1-s)(1-u)}, 1-s\right) - \ln\left(\tfrac{m}{m+s}\right)\mathrm{Li}_{1,1}\left(\tfrac{M(1-u+p-s)}{(M-m)(1-u)}, \tfrac{M-m}{M}\right) + \tfrac{1}{2}\ln\left(-\tfrac{p-s}{1-u}\right)\ln^2\left(\tfrac{m}{m+s}\right) \\
&- \tfrac{1}{2}\ln\left(-\tfrac{p-s}{1-u}\right)\ln^2\left(\tfrac{m}{p+m}\right) + \ln\left(\tfrac{m}{p+m}\right)\mathrm{Li}_{1,1}\left(\tfrac{M(1-u+p-s)}{(M-m)(1-u)}, \tfrac{M-m}{M}\right)
\end{aligned}$$

Analytic regularization of parametric integrals

Divergences in Schwinger parameters

Example: Vertex with one null momentum and one internal mass

Consider a vertex graph G with one internal mass and $p_3^2 = 0$:

$G =$ 

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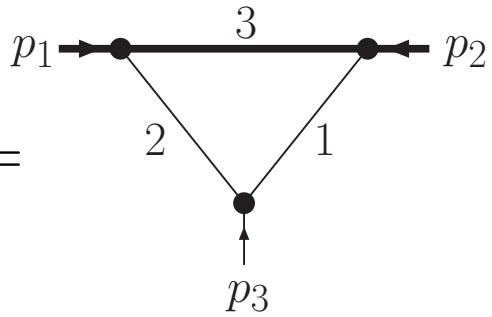
$G =$ 

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It is linearly reducible, but the parametric integral in the representation

$$\Phi(G) = \int \frac{\mathrm{d}^D k}{\pi^{D/2}} \frac{1}{(k^2 + m^2)^{a_3} (k + p_2)^{2a_1} (k - p_1)^{2a_2}}$$
$$= \frac{\Gamma(\mathrm{sdd})}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int \frac{\alpha_1^{a_1 - 1} \alpha_2^{a_2 - 1} \alpha_3^{a_3 - 1}}{\psi^{D/2 - \mathrm{sdd}} \varphi^{\mathrm{sdd}}} \Omega$$

diverges at $\varepsilon \to 0$ for $D = 4 - 2\varepsilon$ and $a_1 = a_2 = a_3 = 1$.

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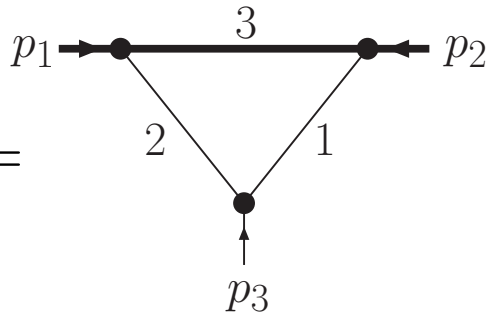
$G =$ 

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diverges at $\varepsilon \to 0$ for $D = 4 - 2\varepsilon$ and $a_1 = a_2 = a_3 = 1$. Then $\mathrm{sdd} = 1 + \varepsilon$ and the integral $\int_0^\infty \mathrm{d}\alpha_3$ diverges at the lower boundary:

$$\int \frac{\Omega}{(\alpha_1+\alpha_2+\alpha_3)^{1-2\varepsilon}\left[m^2(\alpha_1+\alpha_2+\alpha_3)+p_1^2\alpha_2+p_2^2\alpha_1\right]^{1+\varepsilon}\alpha_3^{1+\varepsilon}}$$

Divergences in Schwinger parameters: Power counting

$$F = \prod_{e \in E} \alpha_e^{a_e - 1} \cdot \psi^{\mathrm{sdd} - D/2} \cdot \varphi^{-\mathrm{sdd}}$$

Definition

For an integrand F and disjoint sets $J, K \subset E$ of edges, the rescaling

$$F_J^K := F \circ \left(\alpha_e \mapsto \begin{cases} \lambda \alpha_e, & e \in J \\ \lambda^{-1} \alpha_e, & e \in K \\ \alpha_e, & e \in E \setminus (J \cup K) \end{cases} \right)$$

defines a vanishing degree $\deg_J^K$ such that $F_J^K = \lambda^{\deg_J^K} \cdot \widetilde{F_J^K}$ where $\lim_{\lambda \to 0} \widetilde{F_J^K} \neq 0$ exists. Set $\omega_J^K := |J| - |K| + \deg_J^K$.

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Theorem (Finiteness)

When all coefficients of φ are positive and $\omega_J^K > 0$ for all disjoint $J, K \subsetneq E$ with $0 \neq J \cup K \subsetneq E$, then $\int F \, \Omega$ converges absolutely.

Analytic regularization in Schwinger parameters

A partial integration yields the relation

$$\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}\lambda^{\omega_J^K}\cdot\widetilde{F_J^K}(\lambda)=\frac{\lambda^{\omega_J^K}}{\omega_J^K}\widetilde{F_J^K}(\lambda)\Big|_{\lambda=0}^{\infty}-\frac{1}{\omega_J^K}\int_0^\infty \mathrm{d}\lambda\cdot\lambda^{\omega_J^K}\frac{\partial}{\partial\lambda}\widetilde{F_J^K}(\lambda)$$

with vanishing boundary contribution in the convergent regime.
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Example (Vertex with one mass and $p_3^2 = 0$ in $D = 4 - 2\varepsilon$)

We have $\omega_{\{3\}}^{\emptyset} = -\varepsilon$ with $\widetilde{F_{\{3\}}^{\emptyset}} = \psi^{2\varepsilon-1}\cdot\left[m^2\psi + p_2^2\alpha_1 + p_1^2\alpha_2\right]^{-1-\varepsilon}$:

$$\int\frac{\Omega}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}} = \frac{\alpha_3^{-\varepsilon}\widetilde{F_{\{3\}}^{\emptyset}}}{-\varepsilon}\Big|_{\alpha_3=0}^{\infty} + \frac{1}{\varepsilon}\cdot\int\frac{\Omega}{\alpha_3^{\varepsilon}}\frac{\partial}{\partial\alpha_3}\widetilde{F_{\{3\}}^{\emptyset}}$$

$$= \frac{1}{\varepsilon}\cdot\int\frac{\Omega\alpha_3}{\psi^{1-2\varepsilon}\varphi^{1+\varepsilon}}\left[\frac{2\varepsilon-1}{\psi} - \frac{(1+\varepsilon)\alpha_3 m^2}{\varphi}\right] \text{ when } \varepsilon < 0.$$

Regularization in Schwinger parameters

General construction

Proposition

For disjoint subsets $J, K \subsetneq E$, acting with the differential operator

$$\mathcal{D}_J^K := 1 - \frac{1}{\omega_J^K}\Big[\sum_{e\in J}\frac{\partial}{\partial\alpha_e}\alpha_e - \sum_{e\in K}\frac{\partial}{\partial\alpha_e}\alpha_e\Big]$$

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General construction

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This procedure does not introduce new singularities, i.e. the polynomial reduction and linear reducibility is not affected.

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1. If a graph G (with specification of kinematics) is *linearly reducible*, it can be integrated with hyperlogarithms:
   - to any order in ε, expansion near arbitrary even dimension $D|_{\varepsilon=0} \in 2\mathbb{N}$
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6. The package `HyperInt` [15] implements the algorithms in `Maple`^{TM}.

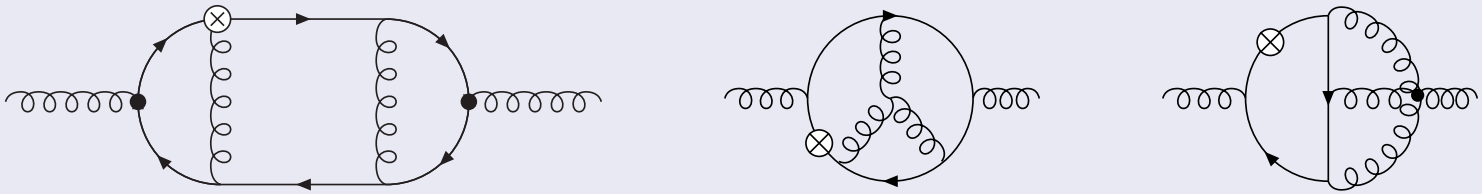
Additional slides

Single-scale integrals

Massive on-shell operator elements

Theorem ([2, 5, 1])

The following on-shell ($p^2 = 0$) propagators with equally massive fermions are linearly reducible:



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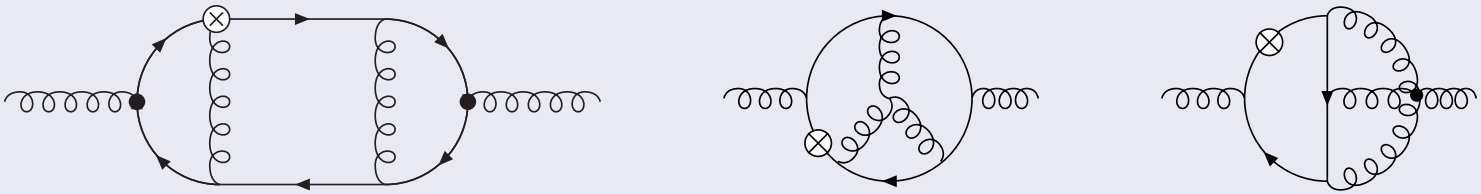
$\hat{I}_4 =$ 

In fact, the calculation used a two-scale setup to get all Mellin moments:

$$\tilde{I}_4(x) := \int \frac{\Omega}{\varphi^2} \sum_{0=j\leq N}^{\infty} x^N \cdot \frac{T_{4a}^{N-j} T_{4b}^{j}}{\psi^N} = \int \frac{\Omega}{(\psi - xT_{4a})(\psi - xT_{4b})M^2},$$

where $T_{4a}, T_{4b}, \psi \in \mathbb{Z}[\alpha_1, \ldots, \alpha_8]$ are cubic polynomials and $M := \sum_{m_e \neq 0} \alpha_e$.

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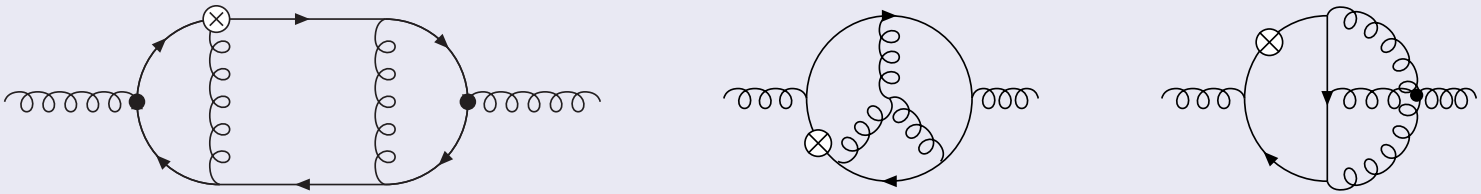
$\hat{I}_4 =$ 

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$$\tilde{I}_4(x) = \left[-\tfrac{1+x}{x} L_{\omega_{-1}} - \tfrac{1-2x-x^2}{(1-x)x^3} L_{\omega_0\omega_{-1}} + \cdots \right] \zeta(3) + 3\tfrac{1+x}{2x^3} L_{\omega_{-1}\omega_0\omega_0\omega_1} + \cdots$$
$$\in L\left(\left\{-1, 0, \tfrac{1}{2}, 1\right\}\right) \otimes \mathrm{lin}_{\mathbb{Q}} \{1, \zeta(3)\}.$$

Non-trivial kinematics: 3 variables

Examples: Conformal four-point integrals

The same functions with $z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $(1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ describe conformal four-point integrals [11]. Many of them are linearly reducible, e.g.

$$E_{12;34} = \quad = \frac{x_{23}^2 x_{24}^2}{\pi^6} \int_{\mathbb{R}^{12}} \frac{\mathrm{d}^4 x_5\, \mathrm{d}^4 x_6\, \mathrm{d}^4 x_7 \cdot x_{16}^2}{(x_{15}^2 x_{25}^2 x_{35}^2) x_{56}^2 (x_{26}^2 x_{36}^2 x_{46}^2) x_{67}^2 (x_{17}^2 x_{27}^2 x_{47}^2)}$$

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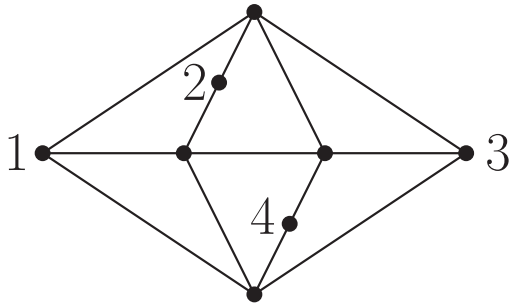
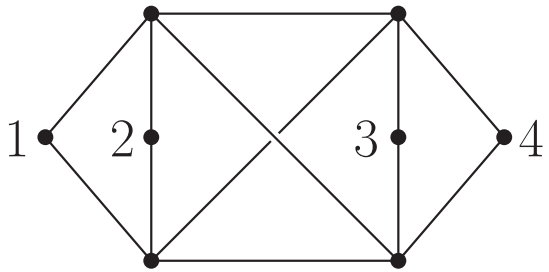
can be integrated parametrically to reproduce results of [11]. The result is a rational function in x_{ij}^2 multiplied with a polylogarithm in the variables z and $\bar{z}$.

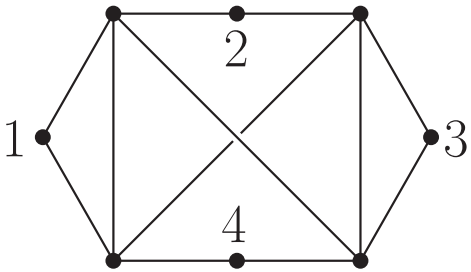
Non-trivial kinematics: 3 variables

Examples: Conformal four-point integrals

Without inverse (numerator) propagators, there are only three four-point integrals with 4 internal vertices ("loops") that are not linearly reducible:







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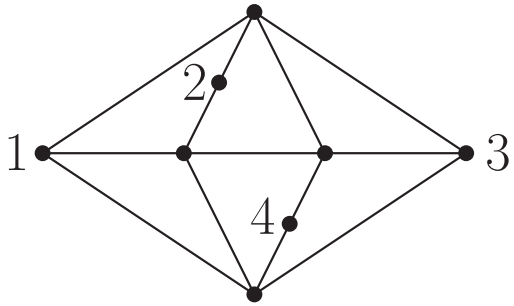
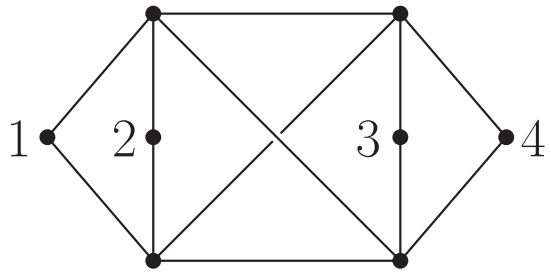
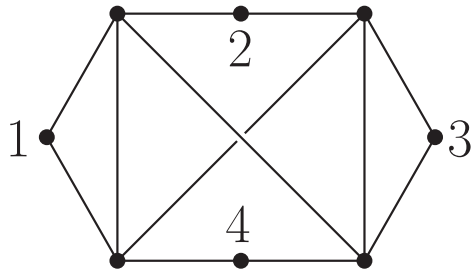
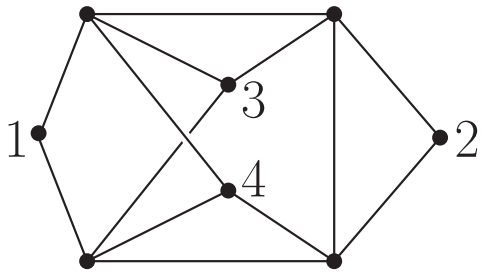
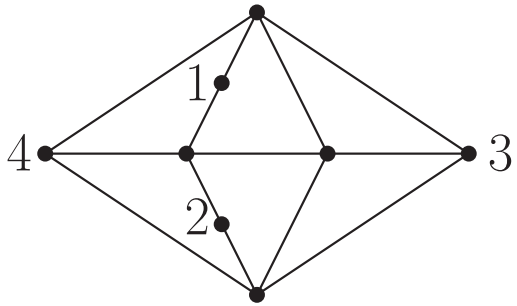


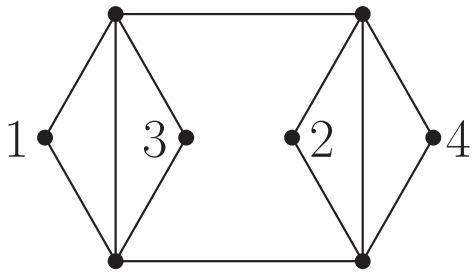




Linearly reducible examples:

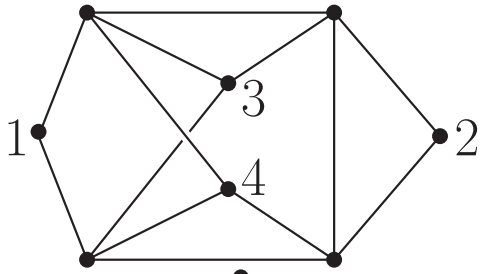
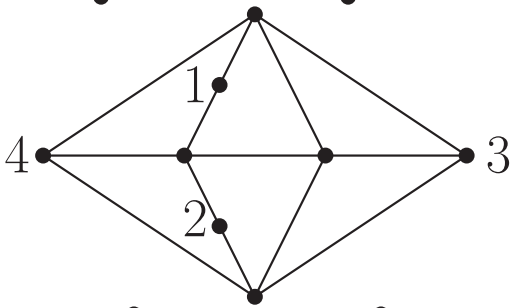
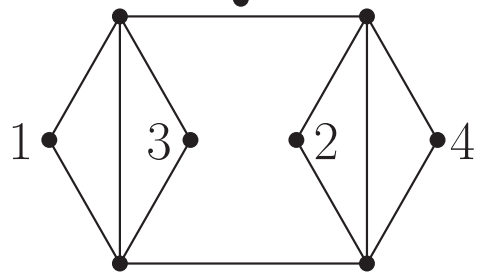






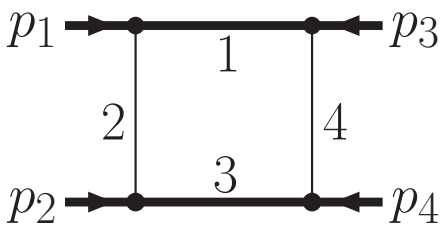
Non-trivial kinematics: 3 variables

Examples: Conformal four-point integrals

graph	denominator	weight	additional symbol letters	terms
	$x_{34}^2 x_{13}^4 x_{24}^4$ $\cdot(z-\bar{z})$ $\cdot(z+\bar{z}-2)$	8	$z-\bar{z}$ $z\bar{z}-1$ $z+\bar{z}-1$	4235
	$x_{34}^2 x_{13}^4 x_{24}^4$ $\cdot(z-\bar{z})^2$	8	$\emptyset$	107
	$x_{13}^4 x_{24}^4$ $\cdot z\bar{z}(z-\bar{z})$	7	$z-\bar{z}$	146

Extending linear reducibility

Box with two masses vis-à-vis



The on-shell massive box $p_1^2 = p_2^2 = p_3^2 = p_4^2 = -m^2$ with $m_2 = m_4 = 0$ and $m_1 = m_3 = m \neq 0$ is not linearly reducible.

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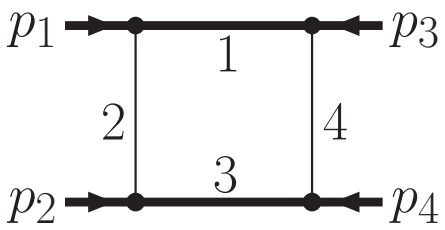

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$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \quad \text{and} \quad \varphi = s\alpha_1\alpha_3 + t\alpha_2\alpha_4 + m^2\left(\alpha_1 + \alpha_3\right)^2$$

and φ is only linear in α_2 and α_4.

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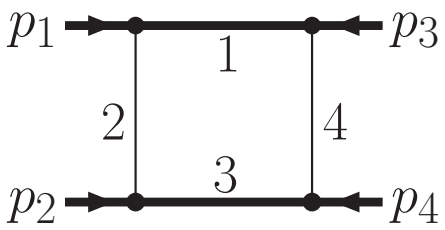

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and φ is only linear in α_2 and α_4. Reducing (integrating) α_4 we obtain

$$S_{\{4\}} = \left\{\alpha_1 + \alpha_2 + \alpha_3, s\alpha_1\alpha_3 + m^2\left(\alpha_1 + \alpha_3\right)^2, R\right\} \quad \text{where the resultant}$$
$$R := [\psi, \varphi]_{\alpha_4} = s\alpha_1\alpha_3 + m^2\left(\alpha_1 + \alpha_3\right)^2 - t\alpha_2(\alpha_1 + \alpha_2 + \alpha_3)$$

is irreducible and quadratic in all remaining Schwinger parameters, therefore prohibiting any further integration.

Extending linear reducibility

Box with two masses vis-à-vis

To proceed we change variables according to

$$\frac{s}{m^2} = \frac{(1-x)^2}{x} \quad \frac{t}{m^2} = \frac{(1-y)^2}{y} \quad \alpha_2 = \widetilde{\alpha_2}\,(\alpha_1 + x\alpha_3) \quad \alpha_4 = \widetilde{\alpha_4}\,(x\alpha_1 + \alpha_3)\,.$$

On one hand we reparametrized the kinematics via x and y to rationalize roots that would otherwise appear in the result, while afterwards

$$\varphi = \frac{m^2}{x}(\alpha_1 + x\alpha_3)(x\alpha_1 + \alpha_3) + \frac{m^2}{y}(1-y)^2\alpha_2\alpha_4$$

suggests to introduce the variables $\widetilde{\alpha_2}$ and $\widetilde{\alpha_4}$ with the effect that

$$\varphi = \frac{m^2}{xy}(\alpha_1 + x\alpha_3)(x\alpha_1 + \alpha_3)\left[y + x(1-y)^2\widetilde{\alpha}_2\widetilde{\alpha}_4\right]$$

factors linearly in these new parameters. Now $\left[\psi, y + x(1-y)^2\widetilde{\alpha}_2\widetilde{\alpha}_4\right]_{\widetilde{\alpha}_4}$ is linear in α_1 and α_3 allowing for a further integration.

Extending linear reducibility

Box with two masses vis-à-vis

Indeed we now observe linear reducibility in the new variables and obtain

$$S_{\{\widetilde{\alpha}_2,\alpha_3,\widetilde{\alpha}_4\}} = \{x+1, x-1, y+1, y-1, xy+1, x+y\}$$

as the final set in the polynomial reduction. Together with $\{x, y\}$ it defines the alphabet of the symbol of the resulting function of x and y in agreement with [13].

Extending linear reducibility

K_4

After integrating α_2 we are stuck, again because the resultant $R := [\psi, \varphi]_2 \in S_{\{2\}}$ is irreducible and quadratic in all Schwinger parameters. But its discriminant

$$D_{\alpha_3}(R) = \alpha_1^2 (\alpha_1\alpha_6 + \alpha_6\alpha_4 + \alpha_4\alpha_5 + \alpha_6\alpha_5)^2$$
$$\times \left[\alpha_6^2 t^2 \alpha_4^2 - 2t\alpha_4\alpha_6 (-t\alpha_6 + s\alpha_4 - s\alpha_6)\alpha_5 + (s\alpha_6 + s\alpha_4 + t\alpha_6)^2 \alpha_5^2\right]$$

becomes a perfect square if we introduce a variable ξ and reparametrize

$$\alpha_5 = \frac{t\alpha_4\alpha_6 (s\alpha_6 + t\alpha_6 + \xi)}{\xi (s\alpha_6 + s\alpha_4 + t\alpha_6 + \xi)}.$$

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After this transformation, R factorizes linearly in α_3 and polynomial reduction reveals linear reducibility along the sequence $\alpha_2, \alpha_3, \alpha_1, \alpha_4, \xi$ ($\alpha_6 = 1$) of integrations.

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The final set is $\{s + t\}$ and proves that to all orders we obtain harmonic polylogarithms of $\frac{t}{s}$ which we calculated explicitly up to order ε^2 and checked against [12].

Soft phase space integrals

Using Mellin-Barnes representations in an intermediate step, a technology developed in [3, 4] allowed to rewrite soft phase space integrals in terms of parametric integrals like

$$\begin{aligned}\mathcal{I}_{9,1}(\varepsilon) = &- \int_0^\infty \mathrm{d}t_1 \mathrm{d}t_2 \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \; t_1^{2-4\varepsilon} (1+t_1)^{\varepsilon-1} t_2^{1-2\varepsilon} \\ &\times x_1^{-\varepsilon} (1-x_1)^{2-4\varepsilon} x_2^{1-3\varepsilon} (1-x_2)^{-\varepsilon} x_3^{-\varepsilon} (1+t_2 x_3)^{1-3\varepsilon} (1+t_2 x_2 x_3)^{\varepsilon} \\ &\times \Big(t_1 t_2^2 x_1 x_2 x_3 + t_2^2 x_2 x_3 + t_1 t_2 x_1 x_2 + t_1 t_2 x_3 + t_2 x_2 x_3 + t_2 + t_1 + 1 \Big)^{3\varepsilon-3}.\end{aligned}$$

These are linearly reducible and could be integrated along the sequence t_1, x_1, x_2, x_3, t_2 of variables and evaluate to MZV.

References

Bibliography I

J. Ablinger, J. Blümlein, A. De Freitas, A. Hasselhuhn, S. Klein, C. Schneider, and F. Wißbrock.
New Results on the 3-Loop Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering.
ArXiv e-prints, January 2013.
arXiv:1212.5950v2.

J. Ablinger, J. Blümlein, A. Hasselhuhn, S. Klein, C. Schneider, and F. Wißbrock.
Massive 3-loop ladder diagrams for quarkonic local operator matrix elements.
Nuclear Physics B, 864(1):52–84, November 2012.
URL: http://www.sciencedirect.com/science/article/pii/S0550321312003355, arXiv:1206.2252.

Bibliography II

C. Anastasiou, C. Duhr, F. Dulat, F. Herzog, and B. Mistlberger.
Real-virtual contributions to the inclusive Higgs cross-section at N^3LO.
JHEP, 2013(12):88, December 2013.
arXiv:1311.1425.

C. Anastasiou, C. Duhr, F. Dulat, and B. Mistlberger.
Soft triple-real radiation for Higgs production at N_3LO.
JHEP, 2013(7):1–78, jul 2013.
arXiv:1302.4379.

J. Blümlein, A. De Freitas, C. Raab, F. Wißbrock, J. Ablinger, A. Hasselhuhn, M. Round, C. Schneider, and A. von Manteuffel.
Recent Results on the 3-Loop Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering.
ArXiv e-prints, July 2013.
arXiv:1307.7548.

Bibliography III

C. Bogner and M. Lüders.
Multiple polylogarithms and linearly reducible Feynman graphs.
ArXiv e-prints, February 2013.
arXiv:1302.6215.

F. C. S. Brown.
Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}(\mathbb{R})$.
Annales scientifiques de l'École Normale Supérieure, 42(3):371–489, June 2009.
arXiv:math/0606419.

F. C. S. Brown.
On the periods of some Feynman integrals.
ArXiv e-prints, October 2009.
arXiv:0910.0114.

Bibliography IV

F. C. S. Brown.
The Massless Higher-Loop Two-Point Function.
Communications in Mathematical Physics, 287:925–958, May 2009.
arXiv:0804.1660.

F. Chavez and C. Duhr.
Three-mass triangle integrals and single-valued polylogarithms.
Journal of High Energy Physics, 11:114, November 2012.
arXiv:1209.2722.

J. Drummond, C. Duhr, B. Eden, P. Heslop, J. Pennington, and V. A. Smirnov.
Leading singularities and off-shell conformal integrals.
Journal of High Energy Physics, 8:133, August 2013.
arXiv:1303.6909.

Bibliography V

J. M. Henn, A. V. Smirnov, and V. A. Smirnov.
Evaluating single-scale and/or non-planar diagrams by differential equations.
ArXiv e-prints, December 2013.
arXiv:1312.2588.

J. M. Henn and V. A. Smirnov.
Analytic results for two-loop master integrals for Bhabha scattering I.
JHEP, 1311:041, 2013.
arXiv:1307.4083.

E. Panzer.
On the analytic computation of massless propagators in dimensional regularization.
Nuclear Physics B, 874(2):567–593, September 2013.
URL: http://www.sciencedirect.com/science/article/pii/S0550321313003003, arXiv:1305.2161.

Bibliography VI

E. Panzer.
Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals.
ArXiv e-prints, March 2014.
arXiv:1403.3385.

E. Panzer.
On hyperlogarithms and Feynman integrals with divergences and many scales.
ArXiv e-prints, January 2014.
arXiv:1401.4361.

O. Schnetz.
Graphical functions and single-valued multiple polylogarithms.
ArXiv e-prints, February 2013.
arXiv:1302.6445.