Limits of Mahler measures and exact polynomials

Riccardo Pengo
(based on joint work with François Brunault, Antonin Guilloux and Mahya Mehrabdollahei)

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Unité des mathématiques pures et appliquées, École normale supérieure de Lyon
riccardo.pengo@ens-lyon.fr, riccardopengo@gmail.com
https://sites.google.com/view/riccardopengo/

Special Values of L-functions, Periods, and Fundamental Groups
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Three nice co-authors

François Brunualt
(ENS Lyon)

Antonin Guilloux
(IMJ-PRG, Paris)

Mahya Mehrabdollahei
(IMJ-PRG, Paris)

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Periods: one ring, many incarnations

La pendule à l'aile bleue
Marc Chagall (1944)

Let $\mathbb{P} \subseteq \mathbb{C}$ be the ring of periods. Then $z \in \mathbb{P}$ if and only if:

Kostsevich-Zagier (2001) $z = \int_{S_1} f_1 + i \int_{S_2} f_2$ where $f_1, f_2 \in \mathbb{Q}(z_1, \dots, z_n)$ and $S_1, S_2 \subseteq \mathbb{R}^n$ semi-algebraic;

Ayoub (2015) $z = \int_{[0,1]^n} f(\underline{z}_n)$, where $f \in \mathcal{O}(\overline{\mathbb{D}}^n) \cap \overline{\mathbb{Q}(z_1, \dots, z_n)}$

Viu-Sos (2021) $|\Re(z)|, |\Im(z)|$ are volumes of compact semi-alg. sets;

Brown (2017) $z = \langle \eta, \gamma \rangle_{(X \setminus A \bmod B)}$, where $X_{/\mathbb{Q}}$ is a smooth and proper variety, $A, B \hookrightarrow X$ are divisors with no common irreducible component, such that $A \cup B$ has SNC, while $\eta \in H^n_{\mathrm{dR}}(X \setminus A, B \setminus A)$ and $\gamma \in H_n^{\mathrm{B}}(X \setminus A, B \setminus A)$.

From each description we get a ring of *abstract periods* $\widetilde{\mathbb{P}}$ and a map $\mathrm{per} \colon \widetilde{\mathbb{P}} \twoheadrightarrow \mathbb{P}$.
Conjecture: The map $\mathrm{per} \colon \widetilde{\mathbb{P}} \twoheadrightarrow \mathbb{P}$ should be an isomorphism of rings.

Problem: Can we formulate a *topological version* of the *period conjecture*?
A possible strategy is provided by **Cresson & Viu-Sos (2019)**. In this case:

$$\widetilde{\mathbb{P}} = \frac{\mathbb{Z}[i][\, (K, d) \colon d \in \mathbb{Z}_{\geq 1},\ K \subseteq \mathbb{R}^d \text{ compact, semi-algebraic, } \dim(K) = d\,]}{\langle [(K, d)] = [(K_1, d)] + [(K_2, d)] = [(f(K), d)] \rangle}$$

where $K = K_1 \cup K_2$ with $\mathrm{codim}(K \cap K') \geq 1$, and $f \colon U \to V$ is any volume-preserving diffeomorphism between two opens $U, V \subseteq \mathbb{R}^d$ with $K \subseteq U$, whose graph Γ_f is a real algebraic set. Then, $\mathrm{per} \colon \widetilde{\mathbb{P}} \twoheadrightarrow \mathbb{P}$ is the volume map. Can one make this continuous, *e.g.* by the (Gromov)-Hausdorff distance?

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Mahler measures: from one to many variables

Stele de tubes
Arman (1966)

Mahler (1962): For $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, let $m(P) := \int_{\mathbb{T}^n} \log|P| d\mu_n$, where $\mathbb{T}^n := (S^1)^n$ and $\mu_n = \frac{1}{(2\pi i)^n}(\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n})$ is the Haar probability measure.

If $P \in \mathbb{Z}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, then $m(P) \geq 0$ and $m(P) = 0 \Leftrightarrow P = \underline{z}_n^{\mathrm{w}} \cdot \prod_{j \geq 1} \Phi_j(\underline{z}_n^{\mathrm{v}_j})^{a_j}$, where Φ_j is the j-th cyclotomic polynomial. See **Boyd (1981)**, **Smyth (1981)**.

Let $\mathcal{M}_n := m(\mathbb{Z}[\underline{z}_n^{\pm 1}] \setminus \{0\}) \subseteq \mathbb{R}_{\geq 0}$. Is it true that $\inf(\mathcal{M}_1 \setminus \{0\}) > 0$, and that $\inf(\mathcal{M}_n \setminus \mathcal{M}_{n-1}) \to +\infty$? This was asked by **Lehmer (1933)** and **Boyd (1981)**.

Boyd (1981) If $\mathcal{M}_\infty := \varinjlim_{n \geq 1} \mathcal{M}_n(\mathbb{Z}) \subseteq \mathbb{R}_{\geq 0}$ is closed, then Lehmer's question has a positive answer, because $\mathcal{M}_1 \subseteq \mathcal{M}_\infty \subseteq \overline{\mathcal{M}_1}$, and $\overline{\mathcal{M}_1} = \mathbb{R}_{\geq 0}$ if $\inf(\mathcal{M}_1 \setminus \{0\}) = 0$.

Lawton (1983) For any $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, the following equality:

$$\lim_{\rho(\mathrm{a}) \to +\infty} m(P(z_1^{a_1}, \dots, z_1^{a_n})) = m(P)$$

holds, where $\rho(\mathrm{a}) := \min\{\mathrm{b} \colon \mathrm{b} \in \mathbb{Z}^n \setminus \{0\},\ \mathrm{a} \perp \mathrm{b}\}$ for every $\mathrm{a} \in \mathbb{Z}^n$.

Smyth (2018) For any $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, the set:

$$\mathcal{M}(P) := \{m(P_A) \colon A \in \mathbb{Z}^{* \times n}\}$$

is closed, where $P_A(\underline{z}_m) := P(z_1^{a_{1,1}} \cdots z_m^{a_{m,1}}, \dots, z_1^{a_{1,n}} \cdots z_m^{a_{m,n}})$.

Moreover, $\mathcal{M}_\infty = \varinjlim_{d \geq 1} \mathcal{M}(Q_d)$, where $Q_d := \sum_{j=1}^d (z_{2j-1} - z_{2j})$.

A generalization of Lawton's theorem

Convergence
Jackson Pollock (1952)

Our idea to generalize Lawton's theorem started from the following result:
Mehrabdollahei (2020) $m(P_d) \to -18 \cdot \zeta'(-2)$, where $P_d := \sum_{0 \leq a+b \leq d} z_1^a z_2^b$.

If we believe that $\mathcal{M}_\infty$, is closed, $\{m(P_d)\}_d$ should converge to a Mahler measure! Indeed, $-18 \cdot \zeta'(-2) = m((1-z_1)(1-z_2)-(1-z_3)(1-z_4))$, as proved by **D'Andrea & Lalín (2007)**. Note that $m(P_d) = m((1-z_1)(1-z_2^{d+2})-(1-z_1^{d+2})(1-z_2))$.

Brunault, Guilloux, Mehrabdollahei & P. (2022) For $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, we have:

$$m(P) = \lim_{\rho(A) \to +\infty} m(P_A)$$

where $\rho(A) := \min\{\mathrm{v}_\infty : \mathrm{v} \in \ker(A) \setminus \{0\}\}$ for every $A \in \mathbb{Z}^{m \times n}$.

Idea of proof: As $\rho(A) \to +\infty$, the tori $\mathrm{Im}(\mu_{A^t} : \mathbb{T}^m \to \mathbb{T}^n)$ converge to $\mathbb{T}^n$ (in the Hausdorff distance). To allow for logarithmic singularities, we show that the functions $\{\log|P_A| : A \in \mathbb{Z}^{m \times n}\}$ are uniformly L^2 with respect to the measure μ_n. To do this, we use an explicit estimate on the measures $\mu_n(\{|P| \leq \varepsilon\})$, due to **Dimitrov & Habegger (2019)**.

Question: Can one give a motivic proof of our result? This requires a motivic lift of $m(P)$ to $\widetilde{m}(P) \in \widetilde{\mathbb{P}}$ (more on this later). Then, if the distance on $\widetilde{\mathbb{P}}$ is sufficiently explicit, this convergence might be provable with algebraic considerations.

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An explicit error term

Error on green,
Paul Klee (1930)

Brunault, Guilloux, Mehrabdollahei & P. (2022) For any Laurent polynomial $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ with k non-zero coefficients, and any integral matrix $A \in \mathbb{Z}^{m \times n}$ with $\rho(A) \geq \max\left\{\mathrm{diam}(P)+1,\ 7\mathrm{diam}(P)^2,\ \exp(2(k-1)\max(n,5))\right\}$, where $\mathrm{diam}(P)$ is the diameter of the Newton polytope $N_P \subseteq \mathbb{R}^n$, the following estimate:

$$|m(P_A) - m(P)| \leq 8 \cdot (98k)^{n-1} \cdot \log(\rho(A))^n \left(\frac{\mathrm{diam}(P)}{\rho(A)}\right)^{\frac{1}{k-1}}$$

holds true. We divide our proof in four steps:

1. if Q does not vanish on $\mathcal{C}_\delta := \{\underline{z} \in (\mathbb{C}^\times)^n \colon \sum_{i=1}^n |\log|z_i|| \leq \delta\} \supseteq \mathbb{T}^n$, we have:

   $$|m(Q_A) - m(Q)| \leq c_1 \cdot (\max_{\mathcal{C}_\delta} |\log|Q||) \cdot e^{-\delta \rho(A)}$$

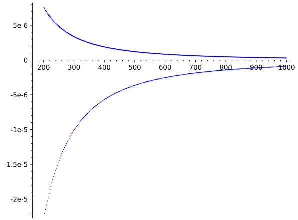
   as follows by writing $m(Q_A) - m(Q)$ as a sum of the Fourier coefficients $c_v(f)$ of a holomorphic function $f \colon \mathcal{C}_\delta \to \mathbb{C}$ which extends $\log|Q| \colon \mathbb{T}^n \to \mathbb{R}$;
2. setting $Q_\varepsilon = PP^* + \varepsilon$, where $P^*(\underline{z}_n) = \overline{P}(\underline{z}_n^{-1})$, we prove that:

   $$|m(PP^*) - m(Q_\varepsilon)|, |m(P_A P_A^*) - m((Q_\varepsilon)_A)| \leq c_2 \cdot \frac{\alpha^{1-n}}{1-\alpha} \cdot \varepsilon^{\frac{1-\alpha}{2(k-1)}}$$

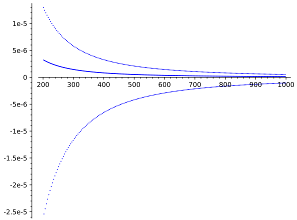
   for every $\alpha \in]0,1[$, using a new estimate for $\mu_n(\{|P| \leq \varepsilon\})$;
3. we prove that $Q_\varepsilon \neq 0$ on $\mathcal{C}_\delta$ with $\delta = c_3 \sqrt{\varepsilon}$, and $\max_{\mathcal{C}_\delta} |\log|Q|| \leq c_4 |\log(\varepsilon)|$;
4. we minimize $(2c_2) \cdot \frac{\alpha^{1-n}}{1-\alpha} \cdot \varepsilon^{\frac{1-\alpha}{2(k-1)}} + (c_1 c_4) \cdot |\log(\varepsilon)| e^{-c_3 \sqrt{\varepsilon} \rho(A)}$, choosing ε and α.

What is the true speed of convergence?

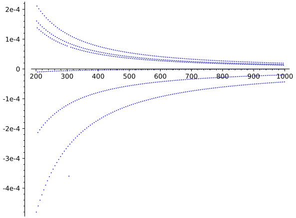
Velocità d'automobile
Giacomo Balla (1913)



$P = z_2 + (z_1 + 1)$

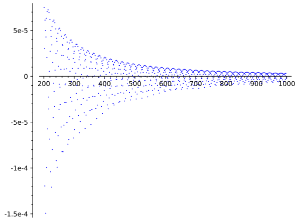


$P = (z_1 + 1)z_2 + (z_1 - 1)$

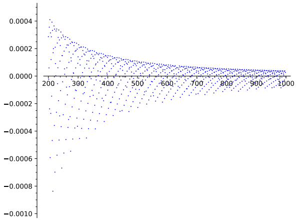


$P = z_1 z_2^2 + (z_1^2 + z_1 + 1)z_2 + z_1$

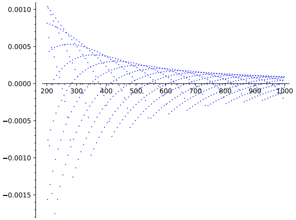
Plots of $m(P(z_1, z_1^d)) - m(P(z_1, z_2))$, yielding finitely or infinitely many smooth branches.



$P = F(z_1, z_2)$



$P = G(z_1, z_2)$



$P = H(z_1, z_2)$

Here, $F = (z_1 + 1)^4 z_2 - (z_1^2 + 1)(z_1^2 - z_1 + 1)$, $G = (z_1 + 1)z_2^2 + (z_1^2 + z_1 + 1)z_2 + z_1^2 + z_1$ and $H = (z_1^8 + z_1^6 + z_1^4 + z_1^2 + 1)(z_2^2 + 1) + (2z_1^8 - 37z_1^6 + 5z_1^5 + 70z_1^4 + 5z_1^3 - 37z_1^2 + 2)z_2$.

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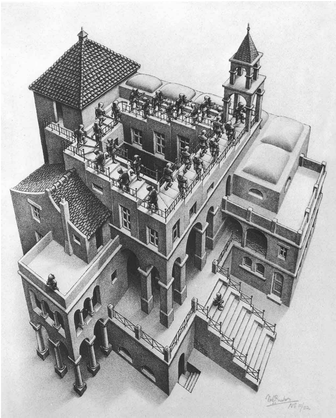
Asymptotic expansions in two variables



Ascending and descending
Maurits Cornelis Escher
(1960)

Boyd (1981) We have that $m(z_1^d + z_1 + 1) - m(z_2 + z_1 + 1) \sim c_2(d)/d^2$, where:

$$c_2(d) = \begin{cases} -\sqrt{3}\pi/6, & \text{if } d \equiv 2(3); \\ \sqrt{3}\pi/18, & \text{if } d \equiv 0, 1(3). \end{cases}$$

Condon (2012) If $P \in \mathbb{C}[\underline{z}_2] \setminus \{0\}$ and $\operatorname{Res}_{z_2}(P, \partial P/\partial z_2) \neq 0$ on $\mathbb{T}^1$, then:

$$m(P(z_1, z_1^d)) - m(P(z_1, z_2)) \approx \sum_{j=2}^{+\infty} \frac{c_j(d)}{d^j}$$

where each $c_j \colon \mathbb{R} \to \mathbb{R}$ is a linear combination of the periodic functions:

$$\{t \mapsto \mathfrak{B}_k(\langle \theta - t\varphi \rangle) \colon k \in \{2, \dots, j\},\ (e^{2\pi i\theta}, e^{2\pi i\varphi}) \in V_P(\mathbb{C}) \cap \mathbb{T}^2\}$$

where the $\mathfrak{B}_k(x)$'s are Bernoulli polynomials, and $\langle x \rangle := x - \lfloor x \rfloor$.

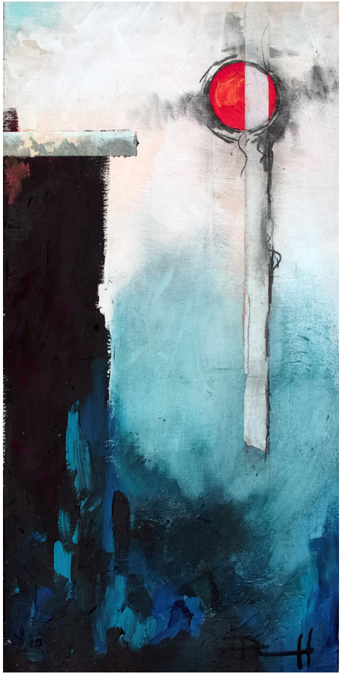
Question Let $P(z_1, z_2) = z_1 z_2^2 + (z_1^2 + z_1 + 1) z_2 + z_1$. Is it true that:

$$m(P(z_1^d, z_1)) - m(P(z_1, z_2)) \sim \frac{c_{3/2}(d)}{d^{3/2}}$$

where $c_{3/2} \colon \mathbb{R} \to \mathbb{R}$ is 6-periodic?

Upshot: The speed of convergence depends on the geometry of $V_P(\mathbb{C}) \cap \mathbb{T}^n$.

An asymptotic expansion in multiple variables



Singularity,
Sean Parnell (2015)

Let $P = (1-z_1)(1-z_2) - (1-z_3)(1-z_4)$ and $A_d = \begin{pmatrix} d+2 & 0 & 1 & 0 \\ 0 & 1 & 0 & d+2 \end{pmatrix}$.

Brunault, Guilloux, Mehrabdollahei & P. (2022) We have that:

$$m\left(\sum_{0 \leq a+b \leq d} z_1^a z_2^b\right) - m(P) = m(P_{A_d}) - m(P) \approx \frac{1}{(d+1)(d+2)}\left[-\frac{\log(d)}{2} + \sum_{k=0}^{+\infty} \frac{c_j}{d^k}\right]$$

where $c_0 := 6\big(\zeta'(-1) - \zeta'(-2)\big) + \frac{\log(2\pi)}{2} - 1$ and, for any $j \geq 1$:

$$c_j := \frac{12 \cdot (-1)^j}{j(j+1)} \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{j+1}{2k} \cdot \frac{(2^{j+1-2k}-1)(2k-1)}{(2k+1)(2k+2)} \cdot B_{2k+2} \cdot \zeta(2k)$$

Idea of proof: Write $m(P_{A_d})$ in terms of Riemann sums on $[0,1]$ for the function $f(x) = (1-2x)D(e^{2\pi i x})$, where $D(z) = \arg(1-z)\log|z| - \operatorname{Im}(\int_0^z \log(1-t)/t \, dt)$ is the Bloch-Wigner dilogarithm. This comes from the identity:

$$m(P_d) = \frac{3}{d+1} \sum_{1 \leq k \leq d+1} \frac{(d+2-2k)}{2\pi} \cdot D\left(\zeta_{d+2}^k\right) - \frac{3}{d+2} \sum_{1 \leq k \leq d} \frac{(d+1-2k)}{2\pi} \cdot D\left(\zeta_{d+1}^k\right)$$

where $\zeta_n = e^{2\pi i/n}$. To handle the endpoint logarithmic singularities of f, we use a generalized version of the Euler-Maclaurin summation formula, due to Navot (1962) and Sidi (2012).

Remark: For $j \geq 1$, we have that $c_j \in \mathbb{Q} + \mathbb{Q} \cdot \pi + \cdots + \mathbb{Q} \cdot \pi^{2\lfloor j/2 \rfloor}$.

Deninger's work and Beilinson's conjectures

Connection
Martin Whatson (2018)

Why can we write $m((1-z_1^{d+2})(1-z_2)-(1-z_1)(1-z_2^{d+2}))$ in terms of Riemann sums? Because this polynomial is *exact*.

Deninger (1997): Mahler measures are *Deligne periods*. More precisely:

$$m(P(z_1,\dots,z_n)) - m(P(z_1,\dots,z_{n-1},0)) = \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_P} \eta_n$$

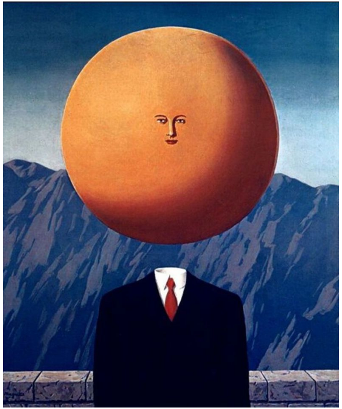
where $V_P := \{P=0\} \hookrightarrow \mathbb{G}_m^n$ and $\gamma_P := V_P(\mathbb{C}) \cap \{|z_1| = \dots |z_{n-1}| = 1, |z_n| \leq 1\}$. Moreover, η_n is a symmetric, smooth $(n-1)$-differential form on $\mathbb{G}_m^n$ which vanishes on $\mathbb{T}^n \supseteq \partial\gamma_P$, and represents the regulator $r^{\infty}_{\mathbb{G}_m^n}(\{z_1,\dots,z_n\}) \in H^n_{\mathcal{D}}(\mathbb{G}_m^n;\mathbb{R}(n))$.

If V_P is *smooth*, $\partial\gamma_P = \emptyset$ and $\{z_1,\dots,z_n\}$ extends to a smooth compactification $X_P \supseteq V_P$ (*i.e.* P is *tempered*), then the real number $m(P) - m(P(\underline{z}_{n-1},0))$ is an entry of the *Beilinson's regulator matrix* for the group $H^n_{\mathcal{M}}(X_P;\mathbb{Q}(n))$. Hence, we expect a relation between $m(P)$ and $L(H^{n-1}(X_P),n)$. For instance, the identity:

$$m\left(z_1 z_2^2 + (z_1^2 + k\cdot z_1 + 1) z_2 + z_1\right) \cdot \frac{1}{L'(E_k,0)} \overset{?}{\in} \mathbb{Q}^\times$$

was found by **Boyd (1998)**, where $E_k\colon y^2 + kxy = x^3 - 2x^2 + x$. **Bornhorn (1999)** proves that Boyd's identity follows from Beilinson's conjectures. Nowadays, many special cases are known (*e.g.* $k=1$ by **Rogers & Zudilin (2014)**). Similar identities can be shown to follow from Beilinson's conjectures, as done in **P. (2020)**.

Exact polynomials and weight drops



L'art de vivre
René F. G. Magritte (1967)

What happens when $\partial\gamma_P \neq \emptyset$? If $\gamma_P \subseteq V_P^{\mathrm{reg}}(\mathbb{C})$ and $\eta_n = d\omega$ on V_P^{reg}, Stokes gives:

$$m(P(z_1,\dots,z_n)) - m(P(z_1,\dots,z_{n-1},0)) = \frac{1}{(2\pi i)^{n-1}} \int_{\partial\gamma_P} \omega$$

and **Guilloux and Marché (2021)** say that P is *exact*. **Maillot (2004)** observes that $\partial\gamma_P \subseteq V_P(\mathbb{C}) \cap \mathbb{T}^n \subseteq W_P(\mathbb{C})$, where $W_P := V_P \cap V_{P^*}$ and $P^*(\underline{z}_n) := \overline{P(\overline{\underline{z}_n}^{-1})}$.

Hence, if $\partial\gamma_P \subseteq W_P^{\mathrm{reg}}$ and ω is closed on W_P^{reg}, with $[\omega] \in H^{n-2}_{\mathrm{dR}}(W_P^{\mathrm{reg}})$ coming from a smooth compactification $X_P^{(1)}$, the real number $m(P) - m(P(\underline{z}_{n-1},0))$ appears in the regulator matrix which should compute $L(H^{n-2}(X_P^{(1)}), n)$.

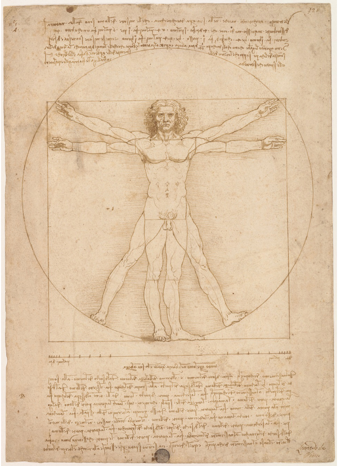
This explains the identity $m(z_1 + z_2 + 1) = L'(\chi_{-3}, -1)$, proved by **Smyth (1981)** using analytic methods. Indeed, in this case $W = \{(\zeta_3, -\zeta_3 - 1), (-\zeta_3, \zeta_3 - 1)\}$.

Smyth also proved that $m(z_1 + z_2 + z_3 + 1) = -14\zeta'(-2)$. How can we explain this? Sometimes, we can apply Stokes twice, as observed by **Lalín (2007)**. Not directly on W_P, since $\partial(\partial\gamma_P) = \emptyset$. However, if $\partial\gamma_P \not\subseteq W_P^{\mathrm{reg}}$, the pull-back of ω to a desingularization $\widetilde{W}_P$ may be exact, and $\partial\gamma_P$ may acquire a boundary. If $n = 3$, one can take $\widetilde{W}_P = \{\mathrm{Res}_{z_3}(P, P^*) = 0\}$. This explains Smyth's identity.
How much can we go on? For instance, can we explain the identity:

$$m((1+z_1)(1+z_2)(1+z_3) + (1-z_1)(1-z_2)(1+z_4)z_5) = 124 \cdot \zeta'(-4)$$

proved by **Lalín (2006)**? To do so, we need a notion of *successive exactness*.

A symmetric version of the Deninger cycle



Uomo vitruviano
Leonardo da Vinci (~1490)

Brunault & P. (2021): If $P \in \mathbb{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ and $V_0 \colon \{P \cdot P^* = 0\} \hookrightarrow \mathbb{G}_m^n$, then:

$$m(P(z_1, \dots, z_n)) - m(P(z_1, \dots, z_{n-1}, 0)) = \langle \alpha_0, \beta_0 \rangle_{V_0}$$

where $\alpha_0 = r_{V_0}^{\infty}(\{z_1, \dots, z_n\})$ and $\beta_0 \in H_{n-1}^{\mathrm{B}}(V_0)$ is a symmetrized version of γ_P. More precisely, we have essentially that $\beta_0 = V_P(\mathbb{C}) \cap \{|z_1| = \dots = |z_{n-1}| = 1\}$.

Thus, looking at the Mayer-Vietoris long exact sequence:

$$\dots \to H_{\mathrm{dR}}^{n-2}(W_P) \xrightarrow{\delta} H_{\mathrm{dR}}^{n-1}(V_0) \to H_{\mathrm{dR}}^{n-1}(V_P) \oplus H_{\mathrm{dR}}^{n-1}(V_{P^*}) \to \dots$$

we get a class $\eta_1 \in H_{\mathrm{dR}}^{n-2}(W_P)$ if $\eta_0|_{V_P} = 0$. Hence, we get:

$$m(P) - m(P(\underline{z}_{n-1}, 0)) = \langle \eta_1, \gamma_1 \rangle_{V_1}$$

where $\gamma_1 = \partial(\gamma_0)$ is obtained by looking at the adjoint Mayer-Vietoris long exact sequence in homology. Thus, say that P is *exact* if $\eta_0|_{V_P} = 0$, as before.

Historical note: Maillot points out that the relation between the involution $\underline{z}_n \mapsto \underline{z}_n^{-1}$ and the intersection $V_P(\mathbb{C}) \cap \mathbb{T}^n$ might go back to **Darboux (1875)**.

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A new approach towards successive exactness

Spiral Jetty
Robert Smithson (1970)

$$\begin{array}{ccccccc}
\dots \to H^{n-2}_{\mathrm{dR}}(V_0^{\mathrm{sing}}) & \to & H^{n-1}_{\mathrm{dR}}(V_0, V_0^{\mathrm{sing}}) & \to & H^{n-1}_{\mathrm{dR}}(V_0) & \to & H^{n-1}_{\mathrm{dR}}(V_0^{\mathrm{sing}}) = 0 \\
\downarrow & & \wr\downarrow & & \downarrow & & \\
\dots \longrightarrow H^{n-2}_{\mathrm{dR}}(D) & \longrightarrow & H^{n-1}_{\mathrm{dR}}(X, D) & \longrightarrow & H^{n-1}_{\mathrm{dR}}(X) & \longrightarrow & H^{n-1}_{\mathrm{dR}}(D) = 0
\end{array}$$

Take $X \to V_0$ to be a resolution of singularities, with $D := X \setminus V_0^{\mathrm{reg}}$ an SNCD. Write $D = D_1 \cup \cdots \cup D_r$ and $D^{(i)} = \bigsqcup_{|I|=i} D_I$, where $D_I = \bigcap_{i \in I} D_i$ (smooth). Also, set $D^{(0)} := X$. Then, we have a spectral sequence $H^q(D^{(p)}) \Rightarrow H^{p+q}(X, D)$.

Let $\tilde{\beta} \in H_{n-1}(X, D)$ coming from β_0 via $H_{n-1}(V_0) \to H_{n-1}(V_0, V_0^{\mathrm{sing}}) \cong H_{n-1}(X, D)$.

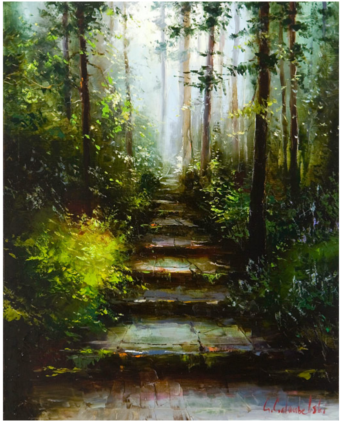
Brunault & P. (2021): Say that P is *k-exact* if there exists $\widetilde{\alpha} \in \mathrm{Fil}^k_{\mathrm{rel}}(H^{n-1}_{\mathrm{dR}}(X, D))$ lifting α. If moreover $\widetilde{\alpha} \notin \mathrm{Fil}^{k+1}_{\mathrm{rel}}$ and $\widetilde{\beta} \in \mathrm{Fil}_k^{\mathrm{rel}} \setminus \mathrm{Fil}_{k-1}^{\mathrm{rel}}$, we have that:

$$m(P(z_1, \dots, z_n)) - m(P(z_1, \dots, z_{n-1}, 0)) = \langle \mathrm{gr}^k_{\mathrm{rel}}(\widetilde{\alpha}), \mathrm{gr}_k^{\mathrm{rel}}(\widetilde{\beta}) \rangle_{(X,D)}$$

which computes $m(P)$ as an absolute period on the smooth variety $D^{(k)}$.

Example: Lalín's polynomial $P \in \mathbb{Z}[z_1, \dots, z_5]$ giving $m(P) = 124\zeta'(-4)$ should be 4-exact. This fits geometrically. Indeed, V_P has two singular lines, and blowing them up one can see that the components of D, and all the successive intersections, are rational varieties defined over $\mathbb{Q}$.

An example: the three-variable linear polynomial



Mystical Forest
Gleb Goloubetski

Let $P = z_1 + z_2 + z_3 + 1$. Recall that $m(P) = -14\zeta'(-2)$ by Smyth (1981).
Let $Z = \{z_1 = 0\} \cup \{z_2 = 0\} \cup \{z_1 + z_2 = 1\}$ and
$S = \{z_1 = 1\} \cup \{z_2 = 1\} \cup \{z_1 + z_2 = 0\}$.

We have $V_P \cong \mathbb{A}^2 \setminus Z$, thus $H^2_{\mathrm{dR}}(V_P) \cong H^1_{\mathrm{dR}}(V_P) \cong \mathbb{R}^3$, and $W_P \cong S \setminus (S \cap Z)$.
Hence, $H^1_{\mathrm{dR}}(W_P) \cong \mathbb{R}^4$ and $\mathrm{Im}(H^1_{\mathrm{dR}}(V_P) \oplus H^1_{\mathrm{dR}}(V_{P^*}) \to H^1_{\mathrm{dR}}(W_P)) \cong \mathbb{R}^3$.

Therefore, $H^1_{\mathrm{dR}}(V_0) \cong \mathbb{R}^3$ and $H^2_{\mathrm{dR}}(V_0) \cong \mathbb{R}^7$. We get a diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^1_{\mathrm{dR}}(V_0) & \longrightarrow & H^1_{\mathrm{dR}}(W_P) & \xrightarrow{\delta} & H^2_{\mathrm{dR}}(V_0, W_P) & \longrightarrow & H^2_{\mathrm{dR}}(V_0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \wr\downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1_{\mathrm{dR}}(X) & \longrightarrow & H^1_{\mathrm{dR}}(D) & \xrightarrow{\delta} & H^2_{\mathrm{dR}}(X, D) & \longrightarrow & H^2_{\mathrm{dR}}(X) & \longrightarrow & 0 \end{array}$$

whose rows are exact. Note that $D = W_P \sqcup W_P$ and $X = V_P \sqcup V_{P^*}$.

Finally, $\widetilde{\eta} \in \mathrm{Fil}^1_{\mathrm{rel}}(H^2(X, D)) = \mathrm{Fil}^2_{\mathrm{rel}}(H^2(X, D)) \cong \mathbb{R}^2$, and $D^{(2)} \cong \mathrm{Spec}(\mathbb{Q})^{\sqcup 6}$.
Thus, we should indeed expect (and we can prove) $m(P) \sim_{\mathbb{Q}^\times} \zeta'(-2)$.

Generalization: Given $P \in \mathbb{Q}[z_1, \dots, z_n]$, the Mahler measure $m(P)$ should be a mixed Tate period if and only if P is $(n-1)$-exact. Can we get every negative even ζ-value $\zeta'(-2n)$ in this way, up to a rational number?

Further steps and directions

El sueño de la razón produce monstruos
Francisco de Goya (1797-99)

Concerning limits of Mahler measures:

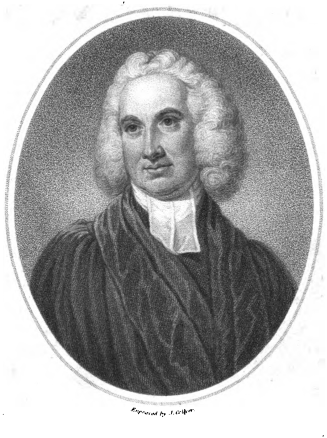
- find the explicit asymptotic expansion for $m(P_A) - m(P)$, when $\rho(A) \to +\infty$, for polynomials which do not vanish on $\mathbb{T}^n$, using the asymptotics of Fourier coefficients;
- let $Z(P,s) := \int_{\mathbb{T}^n} |P|^s d\mu_n$ for $\Re(s) > -\frac{1}{2(k-1)}$, meromorphically continued to $\mathbb{C}$. Is it true that $Z(P_A, s) \to Z(P,s)$ as $\rho(A) \to +\infty$, uniformly on compacts? Can one relate the poles of $Z(P,s)$ to the asymptotic expansion of $m(P_A) - m(P)$?

Concerning exact polynomials:

- Write $m(P)$ as a (single-valued) period for $(\overline{X} \setminus A, B \setminus (A \cap B))$, with $\overline{X}$ smooth projective. Relate this to the toric variety $\mathfrak{X}(N_P)$.
- Compare with the degeneration $P \cdot P^* = t$ for $t \to 0$.
  To do so, study $P \cdot P^* - t \in \mathbb{C}(\!(t)\!)[\underline{z}_n^{\pm 1}]$, maybe via *tropical homology*.
- Show that the identities $m(z_1 + \cdots + z_4 + 1) \stackrel{?}{=} -L'(\eta_1^3 \eta_{15}^3 + \eta_3^3 \eta_5^3, -1)$ and $m(z_1 + \cdots + z_5 + 1) \stackrel{?}{=} -8 \cdot L'(\eta_1^2 \eta_2^2 \eta_3^2 \eta_6^2, -1)$, found numerically by Rodriguez-Villegas, follow, up to a rational number, from Beilinson's conj.
- Find $(n-2)$-exact polynomials, yielding $L'(E, 2-n)$ for a (CM) ell. curve E.
- Study the co-exactness filtration on $\mathcal{M}_\infty$.
- Compute $\operatorname{trdeg}(\mathbb{Q}(\pi, m(P_1), \ldots, m(P_r))/\mathbb{Q})$, assuming the period conjecture.

ENS DE LYON

Thank you very much for your attention!



Thus on, till wisdom is pushed out of life:
Procrastination is the thief of time,
Year after year it steals, till all are fled,
And to the mercies of a moment leaves
The vast concerns of an eternal scene.
If not so frequent, would not this be strange?
That 'tis so frequent, this is stranger still.
Edward Young (1742)