Algebraicity of critical Hecke L-values

(joint work with Guido Kings)

Johannes Sprang - Universität Duisburg-Essen

Special values of L-functions, Periods and Fundamental Groups - Oxford 2022

§1. Special values of L-functions, ...

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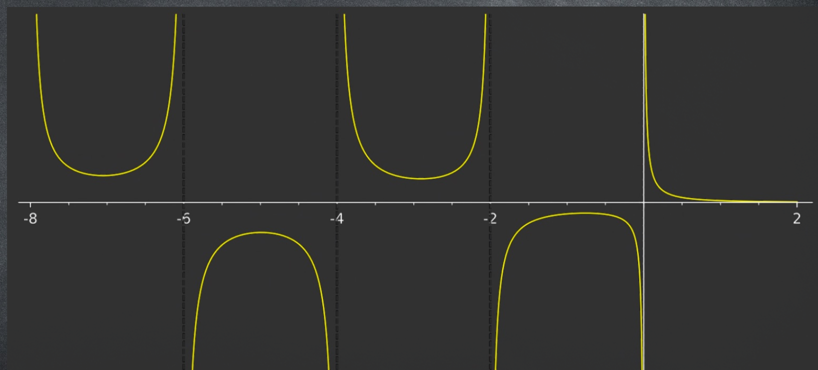

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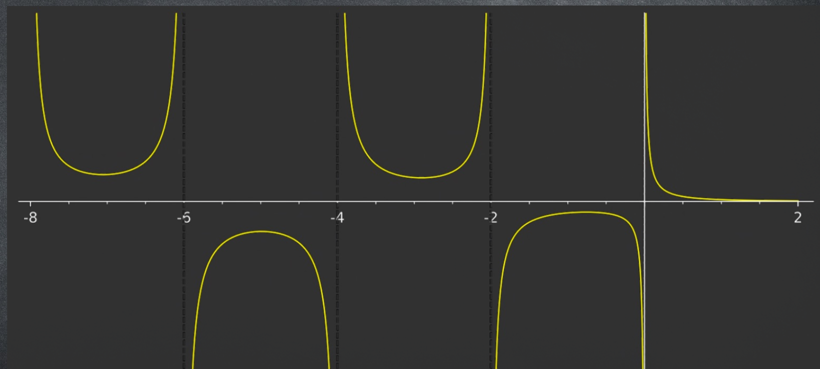
Rem.: Euler's thm shows that the critical zeta values are algebraic up to powers of $(2\pi i)$.



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Upshot: To study the algebraicity of critical Hecke L-values up to explicit periods, we have to deal with two cases:

- totally real fields
- totally imaginary fields

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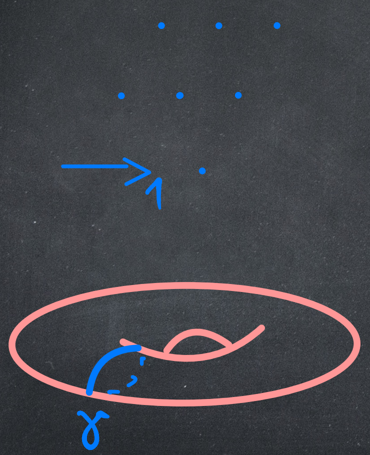

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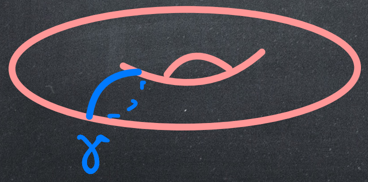

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Needs alg. moduli space.

§2. Special values of L-functions, Periods ...

Thm. (Siegel-Klingen): Let K be a totally real field. For a critical Hecke character χ of K, we have:

$$\frac{L(\chi,0)}{(2\pi i)^{w_\chi}} \in \overline{\mathbb{Q}}^\times$$

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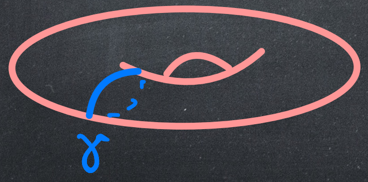

$\Omega_\sigma := \int_\gamma \omega_\sigma$, $\quad \gamma \triangleq 1 \in \mathcal{O}_K \subseteq \mathbb{C}^\Sigma$

§3. Special values of L-functions, Periods and Fundamental Groups

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Motivation: Polylogarithm on $\mathbb{G}_m \setminus \{1\}$

Ramakrishnan, Deligne: $\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\} = \mathbb{G}_m \setminus \{1\}$

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$$(*) \quad \begin{cases} d\,\mathrm{Li}_{k+1}(z) = \mathrm{Li}_k(z)\,\dfrac{dz}{z}, & k \geq 1 \\[2ex] d\,\mathrm{Li}_1(z) = \dfrac{dz}{1-z} \end{cases}$$

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has a fundamental system of solutions on $\{ z \in \mathbb{C} : |z| < 1 \}$

$$F(z) = \left(\begin{array}{c|cccc} 1 & \mathrm{Li}_1(z) & \mathrm{Li}_2(z) & \cdots & \mathrm{Li}_n(z) \\ \hline 0 & 2\pi i & 2\pi i \log z & \cdots & \frac{2\pi i}{n!}(\log z)^{n-1} \\ \vdots & & (2\pi i)^2 & \cdots & \frac{(2\pi i)^2}{(n-1)!}(\log z)^{n-2} \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 0 & (2\pi i)^n \end{array} \right)$$

We may restate

$$(*) \qquad \begin{cases} d\,\mathrm{Li}_{k+1}(z) = \mathrm{Li}_k(z)\,\dfrac{dz}{z}, & k \geq 1 \\[2ex] d\,\mathrm{Li}_1(z) = \dfrac{dz}{1-z} \end{cases}$$

as follows:

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$\rightsquigarrow$ local system of solutions $\underline{\mathrm{Pol}}^n$ on $\mathbb{G}_m(\mathbb{C}) \setminus \{1\} = \mathbb{C} \setminus \{0,1\}$

$\underline{Pol}$ is a non-trivial extension

$$0 \longrightarrow \underline{\mathcal{L}og}^n \longrightarrow \underline{Pol}^n \longrightarrow \underline{\mathbb{C}} \longrightarrow 0 \in \mathrm{Ext}^1_{\mathbb{G}_m(\mathbb{C})\setminus\{1\}}(\underline{\mathbb{C}}, \underline{\mathcal{L}og}^n)$$

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↳ has monodromy representation

$$\mathbb{C}[\pi_1(\mathbb{C}^\times)]/\mathcal{J}^{n+1} \quad , \quad \mathcal{J} = \ker\left(\mathbb{C}[\pi_1(\mathbb{C}^\times)] \xrightarrow{\text{aug}} \mathbb{C}\right).$$

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$\pi_1(\mathbb{C}^\times) = \gamma \cdot \mathbb{Z}$

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This cohomological interpretation admits generalizations to

$\rightarrow$ other realizations (ℓ-adic, de Rham, syntomic, ...)

$\rightarrow$ more genaral base schemes, e.g. abelian varieties.

For an abelian variety A/S get

$$\mathrm{pol}^n_{dR} \in H^{2g-1}_{dR}(A \setminus A[D], \mathcal{L}og^n_{dR})$$

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$$i : \mathrm{Spec}(K) \hookrightarrow Y(N)$$

$$i^* t^* \mathrm{pol}_{dR} \in \prod_{k \geq 0} H^1_{dR}(\mathrm{Spec}\, K, i^* \mathrm{Sym}^k \mathcal{H}^1_{dR}) = 0$$

Key idea: „Hodge-de Rham" s.s:

$$H^{g-1}(A \setminus A[D], \mathcal{L}og^n_{dR} \otimes \Omega^g_{A/S}) \longrightarrow H^{2g-1}_{dR}(A \setminus A[D], \mathcal{L}og^n_{dR})$$

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„coherent refinement" $\underset{\mathrm{EK}^n}{\Psi}$ $\longmapsto$ $\underset{\mathrm{pol}^n_{dR}}{\Psi}$

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$$\Big\updownarrow$$

$$\frac{1}{\Omega_T} \sum_{\mathcal{O}_K^\times \setminus \mathcal{O}_K}{}' \frac{1}{(\lambda + t)^\beta}$$

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<u>Applying</u> $\nabla : \mathcal{L}og^n_{dR} \to \mathcal{L}og^n_{dR} \otimes \Omega^1_{A/S}$ gives:

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For $A/\overline{\mathbb{Q}}$ with $A(\mathbb{C}) = \mathbb{C}^\Sigma/\mathcal{O}_K$ and $\Gamma \subseteq \mathcal{O}_K^\times$ of finite index

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$$t^* \nabla^{\circ a} \mathrm{EK}^n_\Gamma \longleftrightarrow \frac{1}{\Omega_T} \cdot \sum_{\mathcal{O}_K^\times \setminus \mathcal{O}_K}{}' \frac{(\overline{\lambda + t})^\alpha}{(\lambda + t)^\beta}, \quad |\alpha| = a$$

Thank you for your attention!