

# On Polynomial Systems Arising from a Weil Descent

## (Extended version)

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**Abstract.** In the last two decades, many computational problems arising in cryptography have been successfully reduced to various systems of polynomial equations. In this paper, we revisit a class of polynomial systems introduced by Faugère, Perret, Petit and Renault. Based on new experimental results and heuristic evidence, we conjecture that their degrees of regularity are only slightly larger than the original degrees of the equations, resulting in a very low complexity compared to generic systems. We then revisit applications to the elliptic curve discrete logarithm problem (ECDLP) for binary curves, to the factorization problem in  $SL(2, \mathbb{F}_{2^n})$  and to other discrete logarithm problems. As a main consequence, our heuristic analysis suggests that an index calculus variant due to Diem requires a *subexponential* number of bit operations  $O(2^{cn^{2/3} \log n})$  over the binary field  $\mathbb{F}_{2^n}$ , where  $c$  is a constant smaller than 2. According to our estimations, generic discrete logarithm methods are outperformed for any  $n > N$  where  $N \approx 2000$ , but elliptic curves of currently recommended key sizes ( $n \approx 160$ ) are not immediately threatened. The analysis can be easily generalized to other extension fields.

## 1 Introduction

While linear systems of equations can be efficiently solved with Gaussian elimination, polynomial systems are much harder to solve in general. After their introduction by Buchberger [14], Gröbner bases have become the most popular way to solve polynomial systems of equations, in particular since the development of fast algorithms like F<sub>4</sub> [28] and F<sub>5</sub> [29]. Polynomial systems arising in cryptography tend to have a special structure that

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<sup>\*</sup> Supported by an F.R.S.-FNRS postdoctoral research fellowship at Université catholique de Louvain, Louvain-la-Neuve.

<sup>\*\*</sup> This work was partly supported by the Belgian State's IAP program P6/26 BCRYPT.

simplifies their resolution. In the last twenty years, many cryptographic challenges have been first reduced to polynomial systems of equations and then solved with fast and sometimes *dedicated* Gröbner basis algorithms [46,32,41,11,24,25,34,13,33].

## Our contribution

In this paper, we revisit a particular class of polynomial systems introduced by Faugère et al. [36,37], together with their cryptographic applications. These systems naturally arise by deploying a multivariate polynomial equation over an extension field into a system of polynomial equations over the ground prime field (a technique commonly called *Weil descent*). As pointed out in [36,37], the block structure (called *multi-homogeneous structure* in [37]) of the resulting equations and the presence of an abnormally high number of low degree equations suggest that they can be solved more efficiently than generic systems.

Our first contribution is a new complexity analysis of these systems. We first observe that polynomial systems arising from a Weil descent are a natural generalization of a well-known family of polynomial systems appearing in the cryptanalysis of HFE [52,46,20,32,41,26,11,24,25]. Starting from this observation, we extend various experimental and theoretical results on HFE to the more general class of polynomial systems arising from a Weil descent. Following [41], we identify a subsystem with a small number of variables from which we derive an upper bound on the degree of regularity. We also compute the first fall degrees [24,26] and we experimentally observe very small values for the degrees of regularity [32]. These results suggest that the degrees of regularity of these systems are only slightly larger than the degrees of their equations, essentially as small as they could be.

Following [37], we subsequently study an elliptic curve discrete logarithm algorithm of Diem [23] in the case of binary fields. Based on our heuristic analysis of polynomial systems arising from a Weil descent, we conjecture that the elliptic curve discrete logarithm problem can be solved over the binary field  $\mathbb{F}_{2^n}$  in *subexponential time*

$$O(2^c n^{2/3} \log n),$$

where  $c$  is a constant smaller than 2. For  $n$  prime, this problem was previously thought to have complexity  $O(2^{n/2})$ .

Finally, we discuss further applications of polynomial systems arising from a Weil descent, including the factorization problem in  $SL(2, \mathbb{F}_{2^n})$ ,

HFE and other discrete logarithm problems. The various applications of polynomial systems arising from a Weil descent make our analysis particularly useful to cryptography. Although we focus on characteristic 2 in this paper, most of our results can be easily extended to other characteristics.

## Outline

The remaining of this paper is organized as follows. Section 2 contains most of the notations and definitions used in the paper. Section 3 provides general background on algebraic cryptanalysis with Gröbner bases. Section 4 contains our new analysis of polynomial systems arising from a Weil descent. The application to Diem’s algorithm is detailed in Section 5 and other applications are discussed in Section 6. Finally, Section 7 concludes the paper.

## 2 Definitions and notations

We mostly follow the notations introduced in [36]. For any “small” prime  $p$  and any  $n \in \mathbb{Z}$ , we write  $\mathbb{F}_{p^n}$  for the finite field with  $p^n$  elements. We see the field  $\mathbb{F}_{p^n}$  as an  $n$ -dimensional vector space over  $\mathbb{F}_p$  and we let  $\{\theta_1, \dots, \theta_n\}$  be a basis for  $\mathbb{F}_{p^n}/\mathbb{F}_p$ . With some abuse of notations, we use bold letters for all elements, variables and polynomials over  $\mathbb{F}_{p^n}$  and normal letters for all elements, variables and polynomials over  $\mathbb{F}_p$ .

If  $x_1, \dots, x_N$  are variables defined over a field  $\mathbb{K}$ , we write  $R := \mathbb{K}[x_1, \dots, x_N]$  for the ring of polynomials in these variables. Given a set of polynomials  $f_1, \dots, f_\ell \in R$ , the *ideal*  $I(f_1, \dots, f_\ell) \subset R$  is the set of polynomials  $\sum_{i=1}^\ell g_i f_i$ , where,  $g_1, \dots, g_\ell \in R$ . We write  $\text{Res}_{x_i}(f_1, f_2)$  for the *resultant* of  $f_1, f_2 \in R$  with respect to the variable  $x_i$ . A *monomial* of  $R$  is a power product  $\prod_{i=1}^k x_i^{e_i}$  where  $e_i \in \mathbb{N}$ . A *monomial ordering* for  $R$  is an ordering  $>$  such that  $m_1 > m_2 \Rightarrow m_1 m_3 > m_2 m_3$  for any monomials  $m_1, m_2, m_3$  and  $m > 1$  for any monomial  $m$ . The *leading monomial*  $LM(f)$  of a polynomial  $f \in R$  for a given ordering is equal to its largest monomial according to the ordering. Its *leading term* is the corresponding term. For any polynomial  $f \in R$ , we denote the set of monomials of  $f$  by  $\text{Mon}(f)$ .

We measure the memory and time complexities of algorithms by respectively the number of bits and bit operations required. Actual experimental results are given in megabytes and seconds. We write  $O$  for the “big O” notation: given two functions  $f$  and  $g$  of  $n$ , we say that  $f = O(g)$

if there exist  $N, c \in \mathbb{Z}^+$  such that  $n > N \Rightarrow f(n) \leq cg(n)$ . Similarly, we write  $o$  for the “small o” notation: given two functions  $f$  and  $g$  of  $n$ , we say that  $f = o(g)$  if for any  $\epsilon > 0$ , there exists  $N \in \mathbb{Z}$  such that for any  $n > N$ , we have  $|f(n)| \leq \epsilon|g(n)|$ . Finally, we write  $\omega$  for the *linear algebra constant*. Depending on the algorithm used for linear algebra, we have  $2.376 \leq \omega \leq 3$ .

### 3 Background on polynomial system resolution

Whereas linear systems can be solved with Gaussian elimination, polynomial systems of equations are usually solved with Gröbner bases.

#### 3.1 Gröbner basis algorithms

Let  $R$  be a polynomial ring and let  $>$  be a fixed monomial ordering for this ring. A *Gröbner basis* [14,21] of an ideal  $I(f_1, \dots, f_\ell) \subset R$  is a basis  $\{f'_1, \dots, f'_{\ell'}\}$  of this ideal such that for any  $f \in I(f_1, \dots, f_\ell)$ , there exists  $i \in \{1, \dots, \ell'\}$  such that  $\text{LT}(f'_i) | \text{LT}(f)$ .

The first Gröbner basis algorithm was provided by Buchberger in his PhD thesis [14]. Lazard [48] later observed that computing a Gröbner basis is essentially equivalent to performing linear algebra on *Macaulay matrices* at a certain degree.

**Definition 1 (Macaulay Matrix [49,50]).** Let  $R$  be a polynomial ring over a field  $K$  and let  $\mathcal{B}_d := \{m_1 > m_2 > \dots\}$  be the sorted set of all monomials of degree  $\leq d$  for a fixed monomial ordering. Let  $F := \{f_1, \dots, f_\ell\} \subset R$  be a set of polynomials of degrees  $\leq d$ . For any  $f_i \in F$  and  $t_j \in \mathcal{B}_d$  such that  $\deg(f_i) + \deg(t_j) \leq d$ , let  $g_{i,j} := t_j f_i$  and let  $c_{i,j}^k \in K$  be such that  $g_{i,j} = \sum_{m_k \in \mathcal{B}} c_{i,j}^k m_k$ . The Macaulay matrix  $\mathcal{M}_d(F)$  of degree  $d$  is a matrix containing all the coefficients  $c_{i,j}^k$ , such that each row corresponds to one polynomial  $g_{i,j}$  and each column to one monomial  $m_k \in \mathcal{B}_d$ .

The idea behind Lazard’s observation is *linearization*: new equations for the ideal are constructed by algebraic combinations of the original equations, every monomial term appearing in the new equations is treated as an independent new variable, and the system is solved with linear algebra. Gröbner basis algorithms like F<sub>4</sub> [28] and F<sub>5</sub> [29] successively construct Macaulay matrices of increasing sizes and remove linear dependencies in the rows until a Gröbner basis is found. Moreover, they optimize the computation by avoiding monomials  $t_j$  that would produce trivial linear

combinations such as  $f_1f_2 - f_2f_1 = 0$ . The complexity of this strategy is determined by the cost of linear algebra on the largest Macaulay matrix occurring in the computation.

### 3.2 Degree of regularity and first fall degree

The degree of the largest Macaulay matrix appearing in a Gröbner basis computation with the algorithm F5 is called the *degree of regularity*  $D_{reg}$ . For a “generic” sequence of polynomials  $f_1, \dots, f_\ell \in R$  (with  $\ell \leq n$ ), this degree is equal to  $1 + \sum_{i=1}^{\ell} (\deg(f_i) - 1)$  [7]. The degree of regularity can be precisely estimated in the case of *regular* and *semi-regular* sequences [7,9] and (assuming a variant of Fröberg conjecture) in a few other cases [30,12]. However, precisely estimating this value for other classes of systems (in particular for the various structured systems appearing in cryptanalysis problems) may be a very difficult task.

In practice, the degree of regularity may often be approximated by the first degree at which a non trivial *degree fall* occurs during a Gröbner basis computation.

**Definition 2.** Let  $R$  be a polynomial ring over a field  $K$  and let  $F := \{f_1, \dots, f_\ell\} \subset R$ . The first fall degree of  $F$  is the smallest degree  $D_{firstfall}$  such that there exist polynomials  $g_i \in R$  with  $\max_i(\deg(f_i) + \deg(g_i)) = D_{firstfall}$ , satisfying  $\deg(\sum_{i=1}^{\ell} g_i f_i) < D_{firstfall}$  but  $\sum_{i=1}^{\ell} g_i f_i \neq 0$ .

We have  $D_{reg} \geq D_{firstfall}$ . For many classes of polynomial systems, the two definitions lead to very close numbers. Although this is not true in general (counter-examples can be easily produced), it seems to be true for “random systems” and “most real-life systems of equations” [41, p. 350] including HFE and its variants [32,41,26,24,25,12]. This can intuitively be explained by the observation that an extremely large number of relations with a degree fall occur at the degree  $D_{firstfall}$  or the degree  $D_{firstfall} + 1$  in these contexts, and these low degree relations can in turn be combined to produce lower degree relations [26, p. 561], until a Gröbner basis is finally found. In fact, the *first fall degree* has even sometimes been called *degree of regularity* in the cryptography community [26,24,25].

### 3.3 Algebraic cryptanalysis

Many polynomial systems arising in cryptanalysis are very far from generic ones. In fact, their special structures often induce lower degrees of regularity, hence much better time complexities. Gröbner basis techniques have

successfully attacked many cryptosystems, including HFE and its variants [52,46,32,41,11,24,25], the Isomorphism of Polynomials [34,13] and some McEliece variants [33]. In many cases, the resolution of these systems could be accelerated using *dedicated* Gröbner basis algorithms that exploited the particular structures. As was first pointed out in [36,37], this is also the case for polynomial systems arising from a Weil descent.

## 4 Polynomial systems arising from a Weil descent

Let  $n, n', m$  be positive integers and let  $V$  be a vector subspace of  $\mathbb{F}_{2^n}/\mathbb{F}_2$  with dimension  $n'$ . Let  $\mathbf{f} \in \mathbb{F}_{2^n}[\mathbf{x}_1, \dots, \mathbf{x}_m]$  be a multivariate polynomial with degrees bounded by  $2^t - 1$  with respect to all variables. In [36,37], Faugère et al. considered the following problem:

$$\text{Find } \mathbf{x}_i \in V, i = 1, \dots, m, \text{ such that } \mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_m) = \mathbf{0}. \quad (1)$$

The constraints  $\mathbf{x}_i \in V, i = 1, \dots, m$  are called *linear constraints*. From now on, we assume that  $mn' \approx n$  such that Problem (1) has about one solution on average. We also assume  $n' \geq t$ . The *multilinear* case ( $t = 1$ ) was first considered in [36] and later extended in [37].

### 4.1 A polynomial system

Following [36,37], Problem (1) can be reduced to a system of polynomial equations. Let  $\{\theta_1, \dots, \theta_n\}$  be a basis of  $\mathbb{F}_{2^n}$  over  $\mathbb{F}_2$  and let  $\{\mathbf{v}_i | i = 1, \dots, n'\}$  be a basis of  $V$  over  $\mathbb{F}_2$ . We define  $m \cdot n'$  variables  $x_{ij}$  over  $\mathbb{F}_2$  such that  $\mathbf{x}_i = \sum_{j=1}^{n'} x_{ij} \mathbf{v}_j$  and we group them into  $m$  *blocks of variables*  $X_i := \{x_{ij} | j = 1, \dots, n'\}$ . By substituting each  $\mathbf{x}_i$  in  $\mathbf{f}$ , decomposing in the basis  $\{\theta_1, \dots, \theta_n\}$  and reducing by the *field equations*  $x_{ij}^2 - x_{ij} = 0$ , we obtain

$$\mathbf{0} = \mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_m) = \mathbf{f} \left( \sum_{j=1}^{n'} x_{1j} \mathbf{v}_j, \dots, \sum_{j=1}^{n'} x_{mj} \mathbf{v}_j \right) = [\mathbf{f}]_1^\downarrow \theta_1 + \dots + [\mathbf{f}]_n^\downarrow \theta_n$$

for some  $[\mathbf{f}]_1^\downarrow, \dots, [\mathbf{f}]_n^\downarrow \in \mathbb{F}_2[x_{11}, \dots, x_{mn'}]$  that depend on  $\mathbf{f}$  and on the vector subspace  $V$ . Problem (1) can therefore be reformulated as finding a solution to the (algebraic) system

$$[\mathbf{f}]_1^\downarrow = 0, \dots, [\mathbf{f}]_n^\downarrow = 0. \quad (2)$$

Due to the bounds on the degrees of  $\mathbf{f}$ , this system has a *block structure*: the degrees of all polynomials  $[\mathbf{f}]_k^\downarrow$  are bounded by  $t$  with respect to all blocks of variables. The resolution of System (2) can therefore be greatly accelerated using *block-structured* Gröbner basis algorithms [31,36,37].

## 4.2 Known results and link to HFE

The analysis of Problem (1) was initiated by Faugère, Perret, Petit and Renault who identified the block structure [36,37]. These authors also showed that some well-chosen algebraic combinations of the equations of System (2) produce equations of particularly small degree. Moreover, they tried to linearize the problem by explicitly adding these new equations to System (2).

In this paper, we observe that a particular instance of Problem (1) had previously been studied in the cryptography literature. Indeed, the well-known problem of inverting HFE [52,32,41] leads to a particular instance of System (2), where the polynomial  $\mathbf{f}$  is univariate ( $m = 1$ ) and the linear constraints are trivial ( $V = \mathbb{F}_{2^n}$ ).<sup>1</sup> Interestingly, although the polynomial  $\mathbf{f}$  used in HFE has a particular shape (it leads to quadratic equations over  $\mathbb{F}_2$ ), we will see that this shape has generically little influence on the complexity of Problem (1).

Ten years of research on HFE systems have shown that their degrees of regularity are abnormally low compared to generic systems, resulting in very efficient attacks. Although no definitive proof of these results has been published yet, the experimental observations of [32] are now being supported by theoretical evidence such as the isolation of a subsystem with less variables [41], the existence of many low degree equations [19], first fall degree computations [24,26] and complexity results on the Min-Rank problem [12]. In this paper, we generalize some of these results to polynomial systems arising from a Weil descent.

## 4.3 Experimental observations

We start our analysis of these systems with an experimental study of their degree of regularity for various parameters  $n, m, n', t$ . For each set of parameters, we generate a random vector space  $V$  of dimension  $n'$  and a random multivariate polynomial  $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_m)$  with degree bounded by  $2^t - 1$  with respect to each variable. We then perform a Weil descent on this polynomial and we append the field equations to the system. Finally, we apply the Magma function *Groebner* to the result and we collect the maximal degree  $D$  reached during the computation, as given by the *Verbose* output of the Magma function. We repeat each experiment 100 times.

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<sup>1</sup> In HFE contexts, the attacker is not given  $\mathbf{f}$  but only a “hidden” version of System (2). This can be ignored in the complexity analysis of Gröbner basis algorithms since the hiding transformation only consists of a linear combinations of the equations and a linear change of variables [52,41].

Table 1: Average maximal degree reached in Gröbner Basis experiments, average computation time (in seconds) and maximal memory requirements (in MB) for random polynomials

$t$	$n$	$n'$	$m$	$mt + 1$	$D_{av}$	Time	Mem.	$t$	$n$	$n'$	$m$	$mt + 1$	$D_{av}$	Time	Mem.
1	6	3	2	3	3.1	0	10	2	6	3	2	5	5.1	0	10
1	6	2	3	4	3.8	0	10	2	6	2	3	7	6.7	0	10
1	8	4	2	3	3.0	0	11	2	8	4	2	5	5.1	0	11
1	12	6	2	3	3.6	0	11	2	9	3	3	7	7.2	0	12
1	12	4	3	4	4.2	0	11	2	12	4	3	7	7.1	1	38
1	12	3	4	5	5.3	0	14	2	12	3	4	9	9.3	2	95
1	12	2	6	7	7.4	1	23	2	15	5	3	7	7.0	12	263
1	15	5	3	4	4.1	5	20	2	16	8	2	5	5.1	13	36
1	15	3	5	6	6.3	7	114	3	6	3	2	7	6.6	0	10
1	16	8	2	3	3.0	14	25	3	12	6	2	7	7.0	1	31
1	16	4	4	5	5.3	16	98	3	12	4	3	10	10.1	9	70
1	16	2	8	9	9.6	69	3388	3	12	3	4	13	12.6	70	113
1	18	9	2	3	3.0	85	74	3	15	5	3	10	10.0	118	2371
1	18	6	3	4	4.1	86	89	3	16	8	2	7	7.0	23	253
1	18	3	6	7	7.4	233	5398	3	16	4	4	13	13.2	1891	20135
1	20	10	2	3	3.0	487	291	4	8	4	2	9	8.7	1	11
1	20	5	4	5	6.2	515	733	4	12	4	3	13	12.6	199	116
1	20	4	5	6	6.2	669	3226	4	15	5	3	13	13.1	2904	6696

Table 1 reports the average value of  $D$  for these experiments, as well as the average computation time and the maximal memory used (all experiments were done on an Intel Xeon CPU X5500 processor running at 2.67 GHz, with 24 GB RAM). As is often the case in Gröbner basis computations, our experiments were limited more by the memory requirements than by the computation time.

For all parameter sets, the maximal degrees occurring during Gröbner basis computations were much smaller than the degrees of regularity of regular or semi-regular systems with the same degrees. In fact, our experiments suggest that the degree of regularity of System (2) is not much higher than the value  $mt + 1$ . In other words since the original equations have degree  $mt$ , the degree of regularity is essentially as small as it could be. The even lower values obtained for all parameter sets such that  $t = n'$  can be explained by a probable degeneracy in the degrees of the equations. Taking  $m = 1$ , we recover known experimental results on HFE [32].

#### 4.4 Heuristic upper bound on $D_{reg}$

The success of Gröbner basis algorithms on HFE-like cryptosystems comes from the particularly low degrees of regularity of the corresponding sys-

tems [32]. The first theoretical explanation for these low degrees was provided by Granboulan et al.'s [41] and relies on the construction of a modified system with less variables and higher or equal degree of regularity. To extend this analysis to all polynomial systems arising from a Weil descent, we first introduce a new modeling of Problem (1).

We now suppose that  $\{\theta_1, \dots, \theta_n\}$  is a normal basis of  $\mathbb{F}_{2^n}$  over  $\mathbb{F}_2$ , such that  $\theta_i := \theta^{2^{i-1}}$  for some  $\theta \in \mathbb{F}_{2^n}$ . Let  $v_{ij} \in \mathbb{F}_2$  such that  $\mathbf{v}_i = \sum_{j=1}^n v_{ij} \theta_j$ . We define  $nm$  auxiliary binary variables  $y_{ij}$  such that  $\mathbf{x}_i = \sum_{j=1}^n y_{ij} \theta_j$ . Proceeding to a Weil descent as in Section 4.1, we obtain a new system<sup>2</sup>

$$[\mathbf{f}]_1^{\downarrow y} = 0, \dots, [\mathbf{f}]_n^{\downarrow y} = 0 \quad (3)$$

in the variables  $y_{ij}$ , to which we add  $m(n+n')$  field equations  $y_{ij}^2 - y_{ij} = 0$  and  $x_{ij}^2 - x_{ij} = 0$ , as well as  $mn$  linear equations

$$y_{ij} = \sum_{k=1}^n x_{ik} v_{kj} \quad (4)$$

modeling the linear constraints. The resulting system of  $m(n+n')$  variables and  $n+m(n+n')+mn$  equations is equivalent to System (2) (with the field equations) through the substitution (4), hence they have the same degree of regularity.

Following Granboulan et al. [41], we perform additional modifications on this system to obtain a new system with less variables and higher or equal degree of regularity. First, we observe that linear equations do not contribute to the degree of regularity and can therefore be removed without affecting it. The resulting system is composed of  $n+mn$  equations containing only the variables  $y_{ij}$  and  $mn'$  field equations  $x_{ij}^2 - x_{ij} = 0$ . Without decreasing the degree of regularity, we can focus on the first part containing Equations (3) and the field equations  $y_{ij}^2 - y_{ij} = 0$ .

In the next step, we observe that the degree of regularity of this system is not affected if we see the variables  $y_{ij}$  over  $\mathbb{F}_{2^n}$  rather than over  $\mathbb{F}_2$ . Thanks to the field equations, the set of solutions is not affected by this change either. We then apply an invertible linear transformation on Equations (3), defined by

$$F_i := \sum_{j=1}^n \theta^{2^{i+j}} [\mathbf{f}]_j^{\downarrow y}, \quad i = 1, \dots, n.$$

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<sup>2</sup> We add a subscript  $y$  to the arrows in System (3) to stress that the Weil descent is done on the  $y_{ij}$  variables and to distinguish this system from System (2).

This transformation implies  $F_i = F_1^{2^{i-1}}$ . Finally, we perform a linear change of variables defined by

$$z_{ij} := \sum_{k=1}^n \theta^{2^{j+k-1}} y_{ik}, \quad i = 1, \dots, m, \quad j = 1, \dots, n.$$

Since this corresponds to setting  $z_{i1} = \mathbf{x}_i$ ,  $z_{i2} = \mathbf{x}_i^2$ , ...,  $z_{in} = \mathbf{x}_i^{2^{n-1}}$ , each  $F_k$  only depends (linearly) on  $z_{ij}$ ,  $k \leq j \leq t+k-1$ . A last linear transformation changes the field equations into  $z_{ij}^2 = z_{i,j+1}$  and  $z_{i,n}^2 = z_{i,1}$ . The resulting system

$$\begin{cases} F_i = 0, & i = 1, \dots, n \\ z_{ij}^2 = z_{i,j+1}, & i = 1, \dots, m, \quad j = 1, \dots, n-1 \\ z_{i,n}^2 = z_{i,1} & i = 1, \dots, m \end{cases} \quad (5)$$

has the same degree of regularity as System (2).

Since  $F_2 = F_1 \cdot F_1$  modulo the field equations, the polynomial  $F_2$  can be expressed at the degree  $2mt$  as an algebraic combination of  $F_1$  and the field equations. Similarly, all polynomials  $F_i$ ,  $i \geq 2$  can be recovered at degree  $2mt$  from algebraic combination of  $F_1$  and the field equations. Therefore, the degree of regularity of the original system is smaller than the maximum of  $2mt$  and the degree of regularity of the system

$$\begin{cases} F_1 = 0, \\ z_{ij}^2 = z_{i,j+1}, \quad i = 1, \dots, m, \quad j = 1, \dots, n-1 \\ z_{i,n}^2 = z_{i,1} \quad i = 1, \dots, m. \end{cases} \quad (6)$$

Finally like [41], we bound this last degree by the degree of regularity of the subsystem

$$\begin{cases} F_1 = 0, \\ z_{ij}^2 = z_{i,j+1}, \quad i = 1, \dots, m, \quad j = 1, \dots, t-1. \end{cases} \quad (7)$$

Assuming that System eq:system5 behaves like a generic system with the same degrees and the same number of variables<sup>3</sup>, its degree of regularity can be bounded by  $mt - 1 + m(t-1) + 1 = m(2t-1)$  using Macaulay's bound. Under this heuristic assumption, we conclude that the degree of regularity of System (3) is bounded by  $2mt$ .

We point out that the value  $2mt$  is already much below the degree of regularity of a generic system of equations (or even a generic *binary*

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<sup>3</sup> A similar assumption of semi-regularity is needed in [41] to apply Bardet's theorem.

system of equations) with the same degrees [7,8]. Still, our experiments suggest that this bound is not even tight. To explain these experimental values, we first recall the low degree equations identified in [36,37].

#### 4.5 Low degree equations

A key step in the cryptanalysis of HFE has been the observation that an abnormally large number of low degree algebraic combinations of the original equations have a degree lower than expected. This observation explains to a large extent the good performances of Gröbner basis algorithms on HFE instances. Explicit formulae for some of the low degree equations occurring in HFE were recently given by Ding and Hodges in [24].

Similarly, some algebraic combinations of Equations (2) have an abnormally low degree. For completeness, we give an example of these low degree equations, identified by Faugère et al. in [36,37]. Let us first write the monomial  $\mathbf{x}_1$  as  $\sum_{i=1}^n [\mathbf{x}_1]_i^\downarrow \theta_i$ , where each polynomial  $[\mathbf{x}_1]_i^\downarrow \in \mathbb{F}_2[x_{11}, \dots, x_{1,n'}]$  has degree 0 or 1. Defining  $a_{ijk} \in \mathbb{F}_2$  such that  $\theta_i \theta_j = \sum_k a_{ijk} \theta_k$ , we obtain

$$\mathbf{x}_1 \mathbf{f} = \left( \sum_{i=1}^n [\mathbf{x}_1]_i^\downarrow \theta_i \right) \left( \sum_{j=1}^n [\mathbf{f}]_j^\downarrow \theta_j \right) = \sum_{i,j,k=1}^n a_{ijk} [\mathbf{x}_1]_i^\downarrow [\mathbf{f}]_j^\downarrow \theta_k.$$

Decomposing  $\mathbf{x}_1 \mathbf{f}$  according to the basis  $\{\theta_1, \dots, \theta_n\}$ , we see that every polynomial  $[\mathbf{x}_1 \mathbf{f}]_k^\downarrow$  can be written (modulo the field equations) as an algebraic combination of the polynomials in (2)

$$[\mathbf{x}_1 \mathbf{f}]_k^\downarrow = \sum_{i,j=1}^n a_{ijk} [\mathbf{x}_1]_i^\downarrow [\mathbf{f}]_j^\downarrow = \sum_{j=1}^n p_{ik}(x_{11}, \dots, x_{1,n'}) [\mathbf{f}]_j^\downarrow \quad (8)$$

where each polynomial  $p_{ik}(x_{11}, \dots, x_{1,n'}) := \sum_{i=1}^n a_{ijk} [\mathbf{x}_1]_i^\downarrow$  has degree 0 or 1. On the other hand, since the polynomial  $\mathbf{x}_1 \mathbf{f}$  has degree at most  $2^t$  in the variable  $\mathbf{x}_1$ , the degree of each polynomial  $[\mathbf{x}_1 \mathbf{f}]_i^\downarrow$  is still bounded by  $t$  with respect to every block of variables  $X_i$  and its total degree is  $mt$  (instead of  $mt + 1$  as expected from Equation (8)). Similarly deploying  $\mathbf{mf}$  for various monomials  $\mathbf{m} \in \mathbb{F}_{2^n}[\mathbf{x}_1, \dots, \mathbf{x}_{n'}]$ , many more low degree equations can be generated [36,37]. The existence of these equations is very specific to polynomial systems arising from a Weil descent.

Removing linear dependencies (such as the *vectorial dependencies* identified in [37]) still leaves us with a lot of new “low” degree equations that can be added to the system, therefore increasing the probability to

obtain linear equations at a pretty low degree with Gröbner basis algorithms. In fact, since the new equations are algebraic combinations of the original ones, they do not even need to be explicitly added to the system, but Gröbner basis algorithms will recover them “blindly” and quickly benefit from their existence.

In [36,37], Faugère et al. used these low degree equations in a linearization strategy “à la Lazard [48]” to solve System (2). By a combinatorial analysis on the number of equations and monomials at increasing degrees, they derived a bound on the degree of regularity of System (2). We point out that the bound given in [37] is very far from the actual value suggested by our experiments, but the linearization algorithm described in [37] is clearly suboptimal<sup>4</sup>. Indeed, this algorithm only includes equations generated as above, and ignores the majority of algebraic combinations of Equations (2) which do not directly result from a Weil descent on a polynomial  $\mathbf{gf}$ . A better linearization strategy should take all these algebraic combinations into account. The corresponding combinatorial analysis seems to be much harder than the analysis presented in [36,37], but we expect it to provide a rigorous explanation for the actual degrees of regularity observed in Section 4.3.

#### 4.6 First fall degree

As a first step in that direction, we observe that the low degree equations identified in [36,37] provide a very strong bound on the first fall degree of System (2).

**Proposition 1.** *The first fall degree of System (2) is at most  $mt + 1$ .*

*Proof.* By definition, the proof amounts to showing the existence of a polynomial  $g \neq 0$  with degree at most  $mt$  that can be written as  $g(x_{11}, \dots, x_{mn'}) = \sum_{i=1}^n p_i(x_{11}, \dots, x_{m,n'}) [\mathbf{f}]_i^\downarrow$  for some polynomials  $p_i \in K[x_{11}, \dots, x_{mn'}]$  of degree 1. In fact, Equation (8) shows that we can take  $g := [\mathbf{x}_1 \mathbf{f}]_k^\downarrow$  for any  $k$ .

This proposition provides a heuristic explanation for the degrees of regularity observed in Section 4.3 since the first fall degree is often a good approximation of the degree of regularity. As recalled in Section 3.2, this heuristic assumption is “classical” in algebraic cryptanalysis, and it has in particular been verified for various HFE-like systems [41,26,24].

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<sup>4</sup> On the other hand, the analysis of [36] is wrong since it ignores the vectorial dependencies identified in [37].

**Assumption 1.** Let  $n, m, t, n' \in \mathbb{Z}$ . Let  $\mathbf{f}$  be generated as in our experiments. For all but a negligible fraction of the resulting systems, we have  $D_{\text{reg}} = D_{\text{firstfall}} + o(D_{\text{firstfall}})$ .

The assumption intuitively makes sense for System (2): as we saw in Section 4.5, not only one but many degree falls are occurring at degree  $D_{\text{firstfall}}$  and the next ones (each monomial  $\mathbf{m}$  leads to new degree falls).

#### 4.7 Heuristic complexity bounds for Problem (1)

Given the degree of regularity, the complexity of Problem 1 simply follows from the cost of linear algebra.

**Proposition 2.** If Assumption 1 holds, Problem 1 can be solved with standard Gröbner basis algorithms (like F4 or F5) in time  $O(n^{\omega D})$  and memory  $O(n^{2D})$ , where  $\omega$  is the linear algebra constant and  $D \approx mt$ .

In the univariate case, this estimation reduces to  $D \approx t$  which perfectly matches known cryptanalysis results on HFE algebraic systems [32,41]. Interestingly, the special shape of HFE polynomials (they deploy to quadratic equations over  $\mathbb{F}_2$ ) seems to have no impact on the degree of regularity (although further restrictions on the shape may have an impact as pointed out in [24]). In the multilinear case, the estimation provided by Proposition 2 becomes  $D \approx m$  which matches the experimental data of [36].

As observed in [36,37], the block structure of System (2) can be exploited to accelerate its resolution. According to our analysis, the maximal degree appearing in the computation is approximately equal to the initial degree of Equations (2) and can be naturally distributed among the  $m$  blocks. Therefore, a dedicated Gröbner basis algorithm can be designed to exploit the sparsity induced by the block structure and reduce the time and memory complexities of solving Problem (1).

**Proposition 3.** If Assumption 1 holds, Problem 1 can be solved with block Gröbner basis algorithms in time  $O((n')^{\omega D})$  and memory  $O((n')^{2D})$ , where  $\omega$  is the linear algebra constant and  $D \approx mt$ .

Additional heuristic methods like hybrid approaches (consisting in mixing exhaustive search and polynomial system resolution [59,10]) may lead to substantial complexity improvements in practice, as was described in [36] for the multilinear case.

## 5 Index calculus for elliptic curves

We now turn to the main application (so far) of Problem (1). As pointed out in [37], an instance of Problem (1) appears in the relation search step of an index calculus algorithm for elliptic curves proposed by Diem [23]. Given a cyclic (additive) group  $G$ , a generator  $P$  of this group and another element  $Q$  of  $G$ , the discrete logarithm problem asks for computing an integer  $k$  such that  $Q = kP$ . Groups typically used in cryptography include the multiplicative groups of finite fields, groups of points on elliptic curves and hyperelliptic curves and Jacobians of higher genus curves. Index calculus algorithms [47,27] with *subexponential* complexities have long been obtained for the multiplicative groups of finite fields [1,18,2,5,43] and more recently for the Jacobian groups of hyperelliptic curves [3,39,38]. On the other hand, the best algorithms for solving elliptic curve discrete logarithms remained generic algorithms until very recently.

In 2004, Semaev introduced his *summation polynomials* and identified their potential application to build index calculus algorithms on elliptic curves [57] over prime fields  $\mathbb{F}_p$ . These ideas were independently extended by Gaudry [40] and Diem [22] to elliptic curves over composite fields  $\mathbb{F}_{p^n}$ . Following this approach, Gaudry [40] and later Joux and Vitse [44,45] obtained index calculus algorithms running faster than generic algorithms for any  $p$  and any  $n \geq 3$ . On the other hand, Diem [22,23] identified some families of curves with a subexponential time index calculus algorithm by letting  $p$  and  $n$  grow simultaneously in an appropriate way. As far as was known at the moment, the two families of elliptic curves recommended by standards [51] (elliptic curves over prime fields  $\mathbb{F}_p$  or over binary fields  $\mathbb{F}_{2^n}$  with  $n$  prime) remained immune to these attacks. In 2012, Faugère et al. [37] observed that the computation of the relations in an algorithm of Diem for binary fields [23] could be reduced to special instances of Problem (1).

### 5.1 Diem's variant of index calculus

Let  $K$  be a finite field and let  $E$  be an elliptic curve over  $K$  defined by the equation

$$E : y^2 + xy = x^3 + \mathbf{a}_2x^2 + \mathbf{a}_6 \quad (9)$$

for some  $\mathbf{a}_2, \mathbf{a}_6 \in \mathbb{F}_{2^n}$ . Semaev's *summation polynomials*  $\mathbf{S}_r$  are multivariate polynomials satisfying  $\mathbf{S}_r(\mathbf{x}_1, \dots, \mathbf{x}_r) = \mathbf{0}$  for some  $\mathbf{x}_1, \dots, \mathbf{x}_r \in \bar{K}$  if and only if there exist  $\mathbf{y}_1, \dots, \mathbf{y}_r \in \bar{K}$  such that  $(\mathbf{x}_i, \mathbf{y}_i) \in E(\bar{K})$  and  $(\mathbf{x}_1, \mathbf{y}_1) + \dots + (\mathbf{x}_r, \mathbf{y}_r) = P_\infty$  [57]. The summation polynomials

of the curve (9) can be recursively computed as  $\mathbf{S}_2(\mathbf{x}_1, \mathbf{x}_2) := \mathbf{x}_2 + \mathbf{x}_1$ ,  $\mathbf{S}_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \mathbf{x}_1^2 \mathbf{x}_2^2 + \mathbf{x}_1^2 \mathbf{x}_3^2 + \mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3 + \mathbf{x}_2^2 \mathbf{x}_3^2 + \mathbf{a}_6$  and for any  $r \geq 4$ , any  $k, 1 \leq k \leq r - 3$ ,

$$\mathbf{S}_r(\mathbf{x}_1, \dots, \mathbf{x}_r) := \text{Res}_{\mathbf{X}}(\mathbf{S}_{r-k}(\mathbf{x}_1, \dots, \mathbf{x}_{m-k-1}, \mathbf{X}), \mathbf{S}_{k+2}(\mathbf{x}_{r-k}, \dots, \mathbf{x}_r, \mathbf{X})). \quad (10)$$

For  $r \geq 2$ , the polynomial  $\mathbf{S}_r$  is symmetric and has degree  $2^{r-2}$  in every variable  $\mathbf{x}_i$  [57].

Summation polynomials were used by Gaudry [40], Joux and Vitse [44] and Diem [22,23] to compute relations in index calculus algorithms for elliptic curves over composite fields. The following variant is an adaptation of Diem [23].

1. **Factor Basis definition.** Fix two integers  $m, n' < n$  with  $mn' \approx n$  and a vector space  $V \subset \mathbb{F}_{2^n}/\mathbb{F}_2$  of dimension  $n'$ . Let  $\mathcal{F}_V := \{(\mathbf{x}, \mathbf{y}) \in E(K) | \mathbf{x} \in V\}$  be the *factor basis*.
2. **Relation search.** Find about  $2^{n'}$  relations  $a_i P + b_i Q = \sum_{j=1}^m P_{ij}$  with  $P_{ij} \in \mathcal{F}_V$ . For each relation,
  - (a) Compute  $R_i := a_i P + b_i Q$  for random integers  $a_i, b_i$ .
  - (b) Solve Semaev's polynomial  $\mathbf{S}_{m+1}(\mathbf{x}_1, \dots, \mathbf{x}_m, (R_i)_x)$  with the constraints  $\mathbf{x}_i \in V$ .
  - (c) If there is no solution, go back to (a).
3. **Linear Algebra.** Perform linear algebra on the relations to recover the discrete logarithm value.

In previous works [40,22,23,44], a Weil descent was applied to Semaev's polynomials and the resulting systems were solved with resultants or Gröbner basis algorithms. In these works, the complexity of the relation search step was derived from the complexity of solving generic systems. However as pointed out in [36,37] and further demonstrated in Section 4 of the present paper, polynomial systems arising from a Weil descent are very far from generic ones.

## 5.2 A new complexity analysis

We now revisit Diem's algorithm [23] and its analysis by [37] in accordance with our new analysis of Problem (1). Let  $n, m, n'$  be integer numbers. Before starting Diem's algorithm, the  $(m+1)$ th summation polynomial must be computed. Using Collins' evaluation/interpolation method [17] for the resultant of Equation (10), this can be done in time approximately

$2^{t_1}$  where<sup>5</sup>

$$t_1 \approx m(m+1). \quad (11)$$

We then compute about  $2^{n'}$  relations. To obtain these relations, we solve special instances of Problem (1) where

$$\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_m) := \mathbf{S}_{m+1}(\mathbf{x}_1, \dots, \mathbf{x}_m, (a_i P + b_i Q)_x) \quad (12)$$

has degree  $2^{m-1}$  with respect to every variable. Since Semaev's polynomials are clearly not random ones, we perform additional experiments.

In our experiments, we apply Diem's algorithm to a randomly chosen binary curve  $E : y^2 + xy = x^3 + \mathbf{a}_2 x^2 + \mathbf{a}_6$  defined over  $\mathbb{F}_{2^n}$ , where  $n \in \{11, 17\}$ . We first fix  $m \in \{2, 3\}$  and  $n' := \lceil n/m \rceil$ . We then generate a random vector space  $V$  of dimension  $n'$  and a random point  $R$  on the curve such that Equation (12) has solutions. As in Section 4.3, we finally use the *Groebner* function of Magma to solve Semaev's equation  $\mathbf{S}_{m+1}(\mathbf{x}_1, \dots, \mathbf{x}_m, R_x) = \mathbf{0}$  with the linear constraints. We repeat this experiment 100 times for each parameter set, then we repeat all our experiments with the Koblitz curve  $E : y^2 + xy = x^3 + x^2 + 1$ . The average value of the maximal degrees reached during the computation, the average computation time and the maximal memory requirements are reported in Table 2.

Table 2: Average maximal degree reached in Gröbner Basis experiments, average computation time (in seconds) and maximal memory requirements (in MB) for Semaev polynomials. (R): Random curves. (K): Koblitz curves.

$E$	$n$	$n'$	$m$	$t$	$mt + 1$	$D_{av}$	Time	Mem.
K 11	6	2	2	5	3.0	0	11	
K 11	4	3	3	10	7.1	1	15	
K 17	9	2	2	5	4.0	0	15	
K 17	6	3	3	10	7.2	132	2133	
$E$	$n$	$n'$	$m$	$t$	$mt + 1$	$D_{av}$	Time	Mem.
R 11	6	2	2	5	3.0	0	11	
R 11	4	3	3	10	7.1	1	15	
R 17	9	2	2	5	4.0	0	16	
R 17	6	3	3	10	7.1	130	2136	

In all cases, the maximal degrees reached in the computations were even *below* the first fall degree bound given by Proposition (1). This phenomenon is probably due to the sparsity of Semaev's polynomials and will be exploited in future work (in particular, the degree of  $\mathbf{S}_{m+1}$  with

<sup>5</sup> To compute  $\mathbf{S}_{m+1}$ , we apply Collins' algorithm on  $S_k$  where  $k = \lceil \frac{m+3}{2} \rceil$ . This polynomial has degree  $2^{\lceil (m-1)/2 \rceil}$  in each variable. Following Collins, Theorem 9, we have  $t_1 \leq 2(m+1)m/2 = m(m+1)$ .

respect to every variable is  $2^{m-1}$  but bounded by  $2^m - 1$  in the analysis of Section 4). From now on in the analysis, we ignore this difference and analyze Semaev’s polynomials as the random polynomials of Section 4.

**Assumption 2.** *Assumption 1 still holds if  $\mathbf{f}$  is generated from Semaev’s polynomials as in the experiments of this section.*

Under Assumption (2), Step 2(b) of Diem’s algorithm can be solved using a dedicated Gröbner basis algorithm taking advantage of the block structure, in a time  $(n')^{\omega D}$ , where  $D \approx (m^2 + 1)$  and  $\omega$  is the linear algebra constant. Once the  $x$  components of a relation have been computed, the  $y$  components can be found by solving  $m$  quadratic equations and testing each possible combination of the solutions. This requires a time roughly  $2^m$ , that can be neglected. On average, the probability that a point  $R_i := a_i P + b_i Q$  can be written as a sum of  $m$  points from the factor basis can be heuristically approximated by  $\frac{2^{mn' - n}}{m!}$  [23]. Assuming  $mn' \approx n$ , the total cost of the relation search step can therefore be approximated by  $2^{t_2}$ , where

$$t_2 \approx m \log m + n' + \omega(m^2 + 1) \log n'. \quad (13)$$

The last step of Diem’s algorithm consists in (sparse) linear algebra on a matrix of rank about  $2^{n'}$  with about  $m$  elements of size about  $n$  bits per row. This step takes a time approximately equal to  $mn2^{\omega'n'} = 2^{t_3}$ , where

$$t_3 \approx \log m + \log n + \omega'n' \quad (14)$$

and  $\omega'$  is the *sparse* linear algebra constant. If Assumption (2) holds and if  $mn' \approx n$ , the total time taken by Diem’s algorithm can be estimated by  $T := 2^{t_1} + 2^{t_2} + 2^{t_3}$ , where  $t_1, t_2, t_3$  are defined as above.

### 5.3 On the hardness of ECDLP in characteristic 2

We now use Formulas (11) to (14) to evaluate the hardness of the elliptic curve discrete logarithm problem over the field  $\mathbb{F}_{2^n}$  for “small” values of  $n$ . In our estimations, we use  $\omega = \log(7)/\log(2)$  and  $\omega' = 2$ . We consider  $n \in \{50, 100, 160, 200, 500, 1000, 2000, 2500, 5000, 10^4, 2 \cdot 10^4, 5 \cdot 10^4, 10^5, 2 \cdot 10^5, 5 \cdot 10^5, 10^6\}$  and  $m \in \{2, \dots, n/2\}$ . For every pair of values, we compute values  $t_1$ ,  $t_2$  and  $t_3$  with Equations (11), (13) and (14) respectively. Finally, we approximate the total running time of Diem’s algorithm by  $2^{t_{max}}$  where  $t_{max} := \max(t_1, t_2, t_3)$ . For every value of  $n$ , Table 3 presents the data corresponding to the value  $m$  for which  $t_{max}$  is minimal. We point out that the numbers obtained here have to be

taken cautiously since they all rely on Assumption 2 and involve some approximations.

Table 3: Complexity estimates for Diem's algorithm in characteristic 2

$n$	$m$	$n'$	$t_1$	$t_2$	$t_3$	$t_{max}$	$n$	$m$	$n'$	$t_1$	$t_2$	$t_3$	$t_{max}$
50	2	25	6	92	57	92	2000	4	500	20	936	1013	1013
100	2	50	6	131	108	131	2500	5	500	30	1166	1014	1166
160	2	80	6	171	168	171	5000	6	833	42	1857	1682	1857
200	2	100	6	195	209	209	10000	7	1429	56	2919	2873	2919
500	3	167	12	379	344	379	20000	9	2222	90	4810	4462	4810
1000	4	250	20	638	512	638	50000	12	4167	156	9105	8353	9105

According to our estimations, Diem's version of index calculus (together with a sparse Gröbner basis algorithm) beats generic algorithms for any  $n \geq N$ , where  $N$  is an integer close to 2000. An actual attack for current cryptographically recommended parameters ( $n \approx 160$ ) seems to be out of reach today, but the numbers in [37] suggest that medium-size parameters could be reachable with additional Gröbner basis heuristics like the hybrid method [10]. Large prime variations [38] of Diem's algorithm may also lead to substantial improvements in practice. This will be investigated in further work.

Letting  $n$  grow and fixing  $n' := n^\alpha$  and  $m := n^{1-\alpha}$  for a positive constant  $\alpha < 1$ , we obtain

$$\begin{aligned} t_1 &\approx n^{2(1-\alpha)}, \\ t_2 &\approx (1-\alpha)n^{1-\alpha} \log n + n^\alpha + \alpha\omega n^{2(1-\alpha)} \log n, \\ t_3 &\approx (2-\alpha) \log n + \omega'n^\alpha \end{aligned}$$

Taking  $\alpha := 2/3$ , the relation search dominates the complexity of the index calculus algorithm and we deduce the following result.<sup>6</sup>

**Proposition 4.** *Under Assumption 2, the discrete logarithm problem over  $\mathbb{F}_{2^n}$  can asymptotically be solved in time  $O(2^{cn^{2/3}} \log n)$ , where  $c := 2\omega/3$  and  $\omega$  is the linear algebra constant.*

In particular if the Gaussian elimination algorithm is used for linear algebra, we have  $\omega = 3$  and  $c = 2$ . We stress that Proposition 4 holds even when  $n$  is prime. Until now, the best complexity estimates obtained in that case corresponded to generic algorithms that run in time  $2^{n/2}$ .

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<sup>6</sup> Note that the weaker bound  $D_{reg} \leq 2mt$  derived in Section 4.4 with Macaulay's bound also leads to a subexponential complexity but with a constant  $c = 4\omega/3$ .

## 6 Further applications of Problem (1)

### 6.1 Factoring elements in $SL(2, \mathbb{F}_{2^n})$

The factorization problem in a non Abelian (multiplicative) group  $G$  is the following one: given a set of generators  $\mathcal{S} := \{s_1, \dots, s_k\}$  for this group and an element  $h \in G$ , the problem asks for a decomposition  $h = \prod_{i=1}^N s_{m_i}$  as a product of the generators. The preimage security of *Cayley hash functions*, an interesting family of cryptographic hash functions with natural parallelism, directly relies on this problem [58,16,53,55]. The problem becomes potentially hard when additional restrictions are put on the length  $N$  of the product. For a family of groups of increasing size, the standard computational assumption is that no product of polynomial length can be computed in polynomial time, the complexity parameter being the logarithm of the size of the groups. The mere existence of these products in general depends on a famous conjecture of Babai on the diameter of Cayley graphs [6,42].

Using a sequence of reductions introduced in [54], Faugère et al. [36] reduced the factorization problem in  $SL(2, \mathbb{F}_{2^n})$  to a particular instance of Problem (1) with  $t = 1$ , where

$$\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_m) := (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) \left[ \prod_{i=1}^m \begin{pmatrix} \mathbf{x} + \mathbf{x}_i & 1 \\ 1 & 0 \end{pmatrix} \right] (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) \quad (15)$$

for some  $\mathbf{x} \in \mathbb{F}_{2^n}$ . The first fall degree of the corresponding system is at most  $m+1$ . We remark that the polynomial  $\mathbf{f}$  is not totally random since  $\mathbf{f}(\mathbf{x}_1, \dots, \mathbf{x}_m) = \mathbf{f}(\mathbf{x}_m, \dots, \mathbf{x}_1)$ , so Assumption 1 needs to be adapted to this case. The experimental data presented in [36] supports the corresponding assumption. Combining [54,36] and the analysis of Section (4), we easily deduce that for any  $\mathcal{S}$  and any  $h$ , a polynomial length factorization of  $h$  can be computed in probabilistic subexponential time. Since our estimation of the degree of regularity is smaller than in [36], we can also derive new smaller complexity estimates for this problem.

### 6.2 HFE and other discrete logarithm problems

As we pointed out above, System (2) can also be seen as a generalization of HFE systems. These systems have been intensively studied in the literature [32,41,26,24,25,12], and Assumption (1) has been widely verified in this case. As recalled above, the main specificity of HFE polynomials with respect to “random” polynomials is that they lead to quadratic

polynomials over the prime field. Interestingly, the polynomial systems arising from “generic” HFE polynomials seem to have the same degree of regularity as if they arised from random polynomials with the same degrees. It is however known that further restrictions on  $\mathbf{f}$  may lower the degree of regularity [24].

Besides ECDLP, factoring in  $SL(2, \mathbb{F}_{2^n})$  and HFE, the analysis of this paper can be applied to analyze index calculus algorithms over a wide variety of groups, including the Jacobian of higher genus curves. (These additional applications had also been identified by Faugère et al. [37,35]). In particular, discrete logarithm problems in the field  $\mathbb{F}_{2^n}$  can be reduced to an instance of Problem (1) and then solved in heuristic time  $O(2^{\frac{1}{2}\omega n^{1/2} \log n})$ . The complexity of this approach does not compete with Coppersmith’s algorithm [18] but is comparable to Adleman’s first index calculus algorithm [1].

## 7 Conclusion and perspectives

In this paper, we revisited the complexity of solving a class of polynomial systems previously considered by Faugère et al. [36,37]. These systems appear when a multivariate polynomial over an extension field is deployed via a Weil descent into a system of polynomial equations over the ground prime field. We observed that these systems can be seen as natural extensions of HFE systems and we generalized various results on HFE. Based on experimental results and heuristic arguments, we conjectured that the degree of regularity of polynomial systems arising from a Weil descent are only slightly larger than their original degrees, and we deduced new heuristic bounds on their resolution. Interestingly, our bounds nicely generalize previous bounds on HFE.

The most proeminent consequence of our analysis so far concerns the elliptic curve discrete logarithm problem (ECDLP) over binary fields. Indeed, our heuristic analysis suggests that ECDLP can be solved in *subexponential* time  $O(2^{cn^{5/3} \log n})$  over the binary field  $\mathbb{F}_{2^n}$ , where  $c$  is a constant smaller than 2. This complexity is obtained with an index calculus algorithm due to Diem [22] and a block-structured Gröbner basis algorithm. In practice, our estimations predict that the resulting algorithm is faster than generic algorithms (previously thought to be the best algorithms for this problem) for any  $n$  larger than  $N$ , where  $N$  is an integer approximately equal to 2000. In particular, binary elliptic curves of currently recommended sizes ( $n \approx 160$ ) are not immediately threatened.

Besides ECDLP in characteristic 2, the systems introduced in [36,37] have a wide range of applications. We briefly discussed the factorization problem in  $SL(2, \mathbb{F}_{2^n})$ , HFE systems and other discrete logarithm problems. We leave a refinement of our analysis to the particular polynomials appearing in these applications to further work, similarly to what was done in [24] for HFE in odd characteristic.

Our complexity estimates are based on heuristic assumptions that differ from other index calculus algorithms, but are common in algebraic cryptanalysis. The polynomial systems appearing in the cryptanalysis of HFE have been intensively studied in the last 15 years, yet we have no definitive proof for their commonly admitted complexity. Our paper broadens the interest of these researches to all polynomial systems arising from a Weil descent and to their various applications. We leave further experimental and theoretical investigation of our heuristic assumptions to further work.

To conclude this paper, we point out that most of our results generalize quite easily to other fields, resulting in comparable asymptotic complexities.

*Acknowledgements* We are indebted to Sylvie Baudine and the anonymous reviewers for their help in improving this paper. We also thank Jean-Charles Faugère, Ludovic Perret and Guénaël Renault for their useful comments on a preliminary version of this paper. Finally, Christophe Petit would like to thank Daniel Augot for his hospitality since this paper was partially written at LIX.

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