On a particular case of the bisymmetric equation for quasigroups

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February 18, 2014

Abstract

We characterize the solutions of the equation

$$D(G(x,y),G(u,v)) = G(D(x,u),T(y,v)) \tag{1}$$

where D, G and T are quasigroups. We also discuss the particular case when $D=T$.

1 Introduction and Notations

A *quasigroup* on a set Q is an operation $(\cdot): Q\times Q\to Q$ such that for any $a,b\in Q$, there are unique x,y such that $a\cdot x=b$ and $y\cdot a=b$. In this paper, we use small letters for elements of Q and capital letters for quasigroups. We use greek letters for permutations on Q. If $x\in Q$ and α is a permutation on Q, we write $\alpha(x)$ for the image of x by α. We write $\beta\alpha$ for the composition of α and β, where α is applied first.

Two quasigroups $\oplus$ and $\otimes$ on a same set Q are *isotopic* if there exist three permutations α,β,γ of Q such that for any $x,y\in Q$, we have $x\otimes y=(x\alpha\oplus y\beta)\gamma^{-1}$. When $(Q,+)$ is an Abelian group and α is a permutation on Q, we say that α is *additive* for $+$ if for any $x,y\in Q$, we have $\alpha(x+y)=\alpha(x)+\alpha(y)$. When α and β are two permutations on the same set Q, we say that α and β commute if for all $x\in Q$, we have $\alpha\beta(x)=\beta\alpha(x)$.

Functional equations on quasigroups have been previously considered in [1, 2, 3]. In [1], Aczél, Belousov and Hosszú studied various quasigroup equations, including the generalized bisymmetry equation

$$A(B(x,y),C(u,v)) = D(E(x,u),F(y,v)).$$

They showed that for any solution of this equation, all the quasigroups A,B,C,D,E,F are isotopic to the same Abelian group. Here, we show that the additional constraints $B=C=D$, $A=E$ imply some additivity and commutativity properties.

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Mathematics Subject Classification: $20N05$, $05B15$
Keywords and phrases: quasigroup, functional equation

2 Our Results

Let G, D, T satisfying (1). From Theorem 3 in Aczél, Belousov, Hosszú [1], there exist an Abelian group $+$ and 6 permutations $\psi, \epsilon, \delta, \varphi, \beta, \gamma$ such that

$$G(x,y) = \psi(x) + \epsilon(y), \quad D(x,y) = \delta(x) + \varphi(y), \quad T(x,y) = \epsilon^{-1}(\beta(x) + \gamma(y)). \tag{2}$$

Let $-$ be such that $x + y = z \Leftrightarrow x = z - y$, and let e be the neutral element of $+$.

Proposition 1 *Let G, D, T be three quasigroups. These quasigroups satisfy*

$$D(G(x,y), G(u,v)) = G(D(x,u), T(y,v))$$

if and only if there exist an Abelian group $+$, two constants k_1, k_2 and four permutations $\hat{\psi}, \hat{\delta}, \hat{\varphi}, \epsilon$ such that the three permutations $\hat{\psi}$, $\hat{\delta}$ and $\hat{\varphi}$ are additive for $+$, the permutation $\hat{\psi}$ commutes with both $\hat{\delta}$ and $\hat{\varphi}$, and

$$\begin{aligned} G(x,y) &= \hat{\psi}(x) + \epsilon(y) + k_1, \\ D(x,y) &= \hat{\delta}(x) + \hat{\varphi}(y) + k_2, \\ T(x,y) &= \epsilon^{-1}\left(\hat{\delta}\epsilon(x) + \hat{\varphi}\epsilon(y) + k_3\right), \end{aligned}$$

where $k_3 := \hat{\delta}(k_1) + \hat{\varphi}(k_1) - k_1 + k_2 - \hat{\psi}(k_2)$.

When we additionally impose $T = D$, we get

Proposition 2 *Let G, D be two quasigroups. These quasigroups satisfy*

$$D(G(x,y), G(u,v)) = G(D(x,u), D(y,v)) \tag{3}$$

if and only if there exist an Abelian group $+$, two constants k_1, k_2 and four permutations $\hat{\psi}, \hat{\delta}, \hat{\varphi}, \hat{\epsilon}$, all of them additive for $+$, such that both $\hat{\psi}$ and $\hat{\epsilon}$ commute with both $\hat{\delta}$ and $\hat{\varphi}$,

$$\hat{\delta}(k_1) + \hat{\varphi}(k_1) + k_2 = \hat{\psi}(k_2) + \hat{\epsilon}(k_2) + k_1$$

and

$$\begin{aligned} G(x,y) &= \hat{\psi}(x) + \hat{\epsilon}(y) + k_1, \\ D(x,y) &= \hat{\delta}(x) + \hat{\varphi}(y) + k_2. \end{aligned}$$

3 Proof of Proposition 1

Proving that any G, D, T defined as in Proposition 1 satisfy Equation (1) is a straighforward check. We now prove that any solution of Equation (1) is as in Proposition 1.

From Equations (1) and (2), we get

$$\delta(\psi(x) + \epsilon(y)) + \varphi(\psi(u) + \epsilon(v)) = \psi(\delta(x) + \varphi(u)) + \beta(y) + \gamma(v). \tag{4}$$

When $x = \psi^{-1}(e)$, Equation (4) gives

$$\delta\epsilon(y) - \beta(y) = \psi(\delta\psi^{-1}(e) + \varphi(u)) + \gamma(v) - \varphi(\psi(u) + \epsilon(v)).$$

Since this equation must be satisfied for any y, u, v, the left and right terms must be equal to a constant value c_1. We deduce

$$\delta\epsilon(y) - \beta(y) = c_1. \tag{5}$$

Taking $y = \beta^{-1}(e)$, we get

$$c_1 = \delta\epsilon\beta^{-1}(e).$$

Similarly when $u = \psi^{-1}(e)$, Equation (4) gives

$$\varphi\epsilon(v) - \gamma(v) = \psi(\delta(x) + \varphi\psi^{-1}(e)) + \beta(y) - \delta(\psi(x) + \epsilon(y))$$

hence

$$\varphi\epsilon(v) - \gamma(v) = c_2, \tag{6}$$

where

$$c_2 = \varphi\epsilon\gamma^{-1}(e).$$

Susbtituting Equations (5) and (6) in Equation (4), we get

$$\delta(\psi(x) + \epsilon(y)) + \varphi(\psi(u) + \epsilon(v)) = \psi(\delta(x) + \varphi(u)) + \delta\epsilon(y) - c_1 + \varphi\epsilon(v) - c_2.$$

We deduce the following functional equation in δ, ψ and φ only

$$\delta(\psi(x) + y) + \varphi(\psi(u) + v) = \psi(\delta(x) + \varphi(u)) + \delta(y) + \varphi(v) - c_1 - c_2. \tag{7}$$

Taking $v = e$ and $x = \delta^{-1}(e)$, we get

$$\psi\varphi(u) - \varphi\psi(u) = \delta\left(\psi\delta^{-1}(e) + y\right) - \delta(y) - \varphi(e) + c_1 + c_2,$$

which implies

$$\psi\varphi(u) - \varphi\psi(u) = c_3, \tag{8}$$

where

$$c_3 = \psi\varphi\psi^{-1}\varphi^{-1}(e).$$

Similarly substituting $y = e$ and $u = \varphi^{-1}(e)$ in Equation (7), we get

$$\psi\delta(x) - \delta\psi(x) = \varphi\left(\psi\varphi^{-1}(e) + v\right) - \delta(e) - \varphi(v) + c_1 + c_2,$$

which implies

$$\psi\delta(x) - \delta\psi(x) = c_4, \tag{9}$$

where

$$c_4 = \psi\delta\psi^{-1}\delta^{-1}(e).$$

Equation (7) may be re-written as

$$\delta\left(\delta^{-1}(x) + \delta^{-1}(y)\right) + \varphi\left(\varphi^{-1}(u) + \varphi^{-1}(v)\right) = \psi\left(\delta\psi^{-1}\delta^{-1}(x) + \varphi\psi^{-1}\varphi^{-1}(u)\right) + y + v - c_1 - c_2.$$

Using Equations (8) and (9), this leads to

$$\delta\left(\delta^{-1}(x) + \delta^{-1}(y)\right) + \varphi\left(\varphi^{-1}(u) + \varphi^{-1}(v)\right) = \psi\left(\psi^{-1}(x + c_4) + \psi^{-1}(u + c_3)\right) + y + v - c_1 - c_2. \tag{10}$$

Since $+$ is Abelian, we can swap x and y or u and v without changing the left-hand term of Equation (10). We therefore obtain the following functional equation in ψ only:

$$\psi\left(\psi^{-1}(x \oplus c_4) + \psi^{-1}(u \oplus c_3)\right) + y + v = \psi\left(\psi^{-1}(y \oplus c_4) + \psi^{-1}(v \oplus c_3)\right) + x + u.$$

Replacing x by $\psi(x) - c_4$, u by $\psi(u) - c_3$, y by $\psi(y) - c_4$ and v by $\psi(v) - c_3$, we get

$$\psi(x+u) - \psi(x) - \psi(u) = \psi(y+v) - \psi(y) - \psi(v),$$

hence

$$\psi(x \oplus u) - \psi(x) - \psi(u) = c_5 \tag{11}$$

for a constant c_5 such that

$$c_5 = \psi(e+e) - \psi(e) - \psi(e) = e - \psi(e).$$

Using Equation (11), Equation (10) becomes

$$\delta(\delta^{-1}(x) + \delta^{-1}(y)) + \varphi(\varphi^{-1}(u) + \varphi^{-1}(v)) = x + y + u + v + c_4 + c_3 - \psi(e) - c_1 - c_2$$

or

$$\delta(x+y) - \delta(x) - \delta(y) = \varphi(u) + \varphi(v) - \varphi(u+v) + c_4 + c_3 - \psi(e) - c_1 - c_2. \tag{12}$$

This implies

$$\delta(x+y) - \delta(x) - \delta(y) = c_6 \tag{13}$$

where $c_6 = e \ominus \delta(e)$. On the other hand, Equation (12) also implies

$$\varphi(u) + \varphi(v) - \varphi(u+v) = c_7 \tag{14}$$

where $c_7 = \varphi(e)$. Let now

$$\hat{\psi} := \psi - \psi(e).$$

Equation (11) implies

$$\hat{\psi}(x \oplus u) = \psi(x \oplus u) - \psi(e) = \psi(x) + \psi(u) - 2\psi(e) = \hat{\psi}(x) + \hat{\psi}(u), \tag{15}$$

in other words $\hat{\psi}$ is additive for $+$. Similarly, Equations (13) and (14) imply that $\hat{\delta} := \delta - \delta(e)$ and $\hat{\varphi} := \varphi - \varphi(e)$ are additive. Equation (8) and the additivity of $\hat{\varphi}$ and $\hat{\psi}$ now imply

$$\hat{\psi}\hat{\varphi}(u) + \hat{\psi}\varphi(e) + \psi(e) = \hat{\varphi}\hat{\psi}(u) + \hat{\varphi}\psi(e) + \varphi(e) + c_3.$$

For $u = e$, it follows that

$$\hat{\psi}\varphi(e) + \psi(e) = \hat{\varphi}\psi(e) + \varphi(e) + c_3$$

hence Equation (8) eventually implies that

$$\hat{\psi}\hat{\varphi}(u) = \hat{\varphi}\hat{\psi}(u),$$

in other words $\hat{\psi}$ and $\hat{\varphi}$ commute. Similarly, Equation (9) implies that $\hat{\psi}$ and $\hat{\delta}$ commute. By Equations (5) and (6), we have

$$\beta(x) + \gamma(y) = \delta\epsilon(x) - c_1 + \varphi\epsilon(y) - c_2 = \hat{\delta}\epsilon(x) + \hat{\varphi}\epsilon(y) + \delta(e) + \varphi(e) - c_1 - c_2.$$

Defining $k_1 := \psi(e)$, $k_2 := \delta(e) + \varphi(e)$ and $k_3 := \delta(e) + \varphi(e) - c_1 - c_2$, we deduce from Equation (2) that

$$\begin{aligned} G(x,y) &= \hat{\psi}(x) + \epsilon(y) + k_1, \\ D(x,y) &= \hat{\delta}(x) + \hat{\varphi}(y) + k_2, \\ T(x,y) &= \epsilon^{-1}\left(\hat{\delta}\epsilon(x) + \hat{\varphi}\epsilon(y) + k_3\right), \end{aligned}$$

with $\hat{\psi}$, $\hat{\delta}$ and $\hat{\varphi}$ with the properties required. Using the additivity of $\hat{\delta}$, $\hat{\varphi}$ and $\hat{\psi}$, we compute

$$\begin{aligned} D(G(x,y),G(u,v)) &= \hat{\delta}\left(\hat{\psi}(x)+\epsilon(y)+k_1\right)+\hat{\varphi}\left(\hat{\psi}(u)+\epsilon(v)+k_1\right)+k_2 \\ &= \hat{\delta}\hat{\psi}(x)+\hat{\delta}\epsilon(y)+\hat{\delta}(k_1)+\hat{\varphi}\hat{\psi}(u)+\hat{\varphi}\epsilon(v)+\hat{\varphi}(k_1)+k_2 \end{aligned}$$

and

$$\begin{aligned} G(D(x,u),T(y,v)) &= \hat{\psi}\left(\hat{\delta}(x)+\hat{\varphi}(u)+k_2\right)+(\hat{\delta}\epsilon(y)+\hat{\varphi}\epsilon(v)+k_3)+k_1. \\ &= \hat{\psi}\hat{\delta}(x)+\hat{\psi}\hat{\varphi}(u)+\hat{\psi}(k_2)+\hat{\delta}\epsilon(y)+\hat{\varphi}\epsilon(v)+k_3+k_1. \end{aligned}$$

Since $\hat{\psi}$ commutes with both $\hat{\varphi}$ and $\hat{\delta}$, we deduce

$$\begin{aligned} G(D(x,u),T(y,v)) &= \hat{\delta}\hat{\psi}(x)+\hat{\varphi}\hat{\psi}(u)+\hat{\psi}(k_2)+\hat{\delta}\epsilon(y)+\hat{\varphi}\epsilon(v)+k_3+k_1 \\ &= D(G(x,y),G(u,v))+\hat{\psi}(k_2)+k_3+k_1-\hat{\delta}(k_1)-\hat{\varphi}(k_1)-k_2. \end{aligned}$$

Equation (1) then implies

$$k_3=\hat{\delta}(k_1)+\hat{\varphi}(k_1)-k_1+k_2-\hat{\psi}(k_2).$$

This concludes the proof of Proposition 1.

4 Proof of Proposition 2

Proving that any G, D, T defined as in Proposition 2 satisfy Equation (3) is a straighforward check. We now prove that any solution of Equation (3) is as in Proposition 2. By Proposition 1, we have

$$G(x,y)=\hat{\psi}(x)+\hat{\epsilon}(y)+k_1, \qquad D(x,y)=\hat{\delta}(x)+\hat{\varphi}(y)+k_2$$

for permutations $\hat{\psi}, \hat{\delta}, \hat{\varphi}, \hat{\epsilon}$ such that $\hat{\psi}$, $\hat{\delta}$ and $\hat{\varphi}$ are additive for $+$, and moreover $\hat{\psi}$ commutes with both $\hat{\delta}$ and $\hat{\varphi}$. By symmetry of D and G in Equation (3), $\hat{\epsilon}$ must also be distributive for $+$ and it must commute with both $\hat{\delta}$ and $\hat{\varphi}$. As in the proof of Proposition 1, we compute

$$D(G(x,y),G(u,v))=\hat{\delta}\hat{\psi}(x)+\hat{\delta}\epsilon(y)+\hat{\delta}(k_1)+\hat{\varphi}\hat{\psi}(u)+\hat{\varphi}\epsilon(v)+\hat{\varphi}(k_1)+k_2.$$

Similarly, we have

$$G(D(x,y),D(u,v)) = \hat{\psi}\hat{\delta}(x)+\hat{\psi}\hat{\varphi}(u)+\hat{\varphi}(k_2)+\hat{\epsilon}\hat{\delta}(y)+\hat{\epsilon}\hat{\varphi}(v)+\hat{\epsilon}(k_2)+k_1.$$

Equation (3) then leads to

$$\hat{\delta}(k_1)+\hat{\varphi}(k_1)+k_2=\hat{\psi}(k_2)+\hat{\epsilon}(k_2)+k_1.$$

This concludes the proof of Proposition 2.

Acknowledgement. We thank the anonymous referee for his comments that helped us improve both the content of our main results and their presentation.

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