

Geometric Postulation of a Smooth Function
and the Number of Rational Points

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1. Introduction

This paper is devoted to giving refinements and extensions of some of the results of Bombieri and the author [1] obtaining upper bounds for the number of integral lattice points on the graphs of functions. Consider a sufficiently smooth function $f(x)$ with graph Γ , and a positive integer d . The main device of that paper was to consider integral points on Γ that do not lie on any real algebraic curve of degree d . The Main Lemma of [1] shows that such points cannot be too close together relative to certain norms of the function.

We pursue here two different goals relative to the Main Lemma. The first is to obtain local conditions on the function $f(x)$ that control the multiplicity of the intersection of Γ with any algebraic curve of degree d . This is essentially an investigation into the *hypotheses* of the Main Lemma, and constitutes the Geometric Postulation of the title. Indeed, in sections 2 and 3 we will obtain such conditions for any linear space of real algebraic curves.

The applications to integral points are given in section 5. For example, we show that if $f(x) \in C^{104}$ on $[0, 1]$ and

$$W(f, 2) = f'' \begin{vmatrix} f''' & 3f'' & 0 \\ f^{iv} & 4f''' & 6f'' \\ f^v & 5f^{iv} & 20f''' \end{vmatrix}$$

is nowhere zero then, for every $\varepsilon > 0$,

$$|t\Gamma \cap \mathbb{Z}^2| \leq c(f, \varepsilon)t^{\frac{1}{2}+\varepsilon}$$

where $t\Gamma$ is the homothetic dilation of Γ by a factor $t \geq 1$ (that is the graph of $y = tf(x/t), x \in t[0, 1]$). The same bound was proved in [1] under the assumption that f was C^∞ and strictly convex.

The second goal is an improvement of the Main Lemma itself that allows us to strengthen some of the results obtained therefrom. Thus, for example, let $f(x)$ be a transcendental analytic function on a closed bounded interval I , and Γ the graph of f . It was shown in [1] that

$$|t\Gamma \cap \mathbb{Z}^2| \leq c(f, \varepsilon)t^\varepsilon$$

for every $\varepsilon > 0$. In particular, if $t = N$ is an integer, the same bound $c(f, \varepsilon)N^\varepsilon$ applies to the number of rational points on Γ of denominator N . We will obtain a bound of the same form for the number of rational points on Γ of *height* less than or equal to N , by which we mean points

$$P = (x, y) = \left(\frac{a}{b}, \frac{c}{d}\right), \quad a, b, c, d \in \mathbb{Z}, \quad |a|, |b|, |c|, |d| \leq N.$$

In fact we will prove a version for points of Γ with coordinates in a number field, and allow also (non-homothetic) dilations of Γ . The new Main Lemma is proved in section 4, and the applications are given in section 6.

The reason that our applications are divided over two sections has to do with a second device of [1] that does not generalize to the context of the new Main Lemma. This is the recurrence argument that was used to eliminate the dependence on the norms of the functions in certain cases and obtain uniform estimates.

Thus while we can show that the hypotheses $f(x) \in C^{105}[0, N], N \geq 1, |f'| \leq 1$ and

$f^{(105)}$ $W(f, 2)$ non-vanishing imply

$$|\Gamma \cap \mathbb{Z}^2| \leq c(\varepsilon)N^{\frac{1}{2}+\varepsilon},$$

for any $\varepsilon > 0$, we cannot give a similarly uniform estimate for the rational points of height $\leq N$ on Γ when $f \in C^{105}[0, 1]$ with $|f'| \leq 1$ and $f^{(105)}$ $W(f, 2)$ non-vanishing.

Returning to the geometric postulation of $f(x)$, let L_d denote the space of real algebraic plane curves of degree $\leq d$. The space L_d forms a real projective space of dimension $\frac{1}{2}(d+1)(d+2) - 1 = D - 1$. Thus, $D - 1$ points in the plane always lie on a curve in L_d , while D points in general do not. A function $f(x)$, defined on an interval I and possessing $D - 1$ derivatives will be called *d-averse* if no D points on the graph Γ of $f(x)$ lie on a curve in L_d (counting multiplicity). Thus, for example, the exponential function is *d-averse* for every d .

In section 2 we give a sufficient local condition for a function with $D - 1$ derivatives to be *d-averse*. Our result follows from a mean value theorem of Pólya [3]. This condition, the non-vanishing of a certain finite number of Wronskian determinants, generalizes the non-vanishing of $f''(x)$ as a sufficient condition for the graph Γ of a function $f(x)$ with two derivatives to intersect any line at most twice.

Of course, the non-vanishing of $f''(x)$ is also a necessary condition, and this raises the following question: If the graph Γ of a sufficiently smooth function $f(x)$ intersects a curve in L_d in D points, is there necessarily another curve in L_d intersecting Γ in a D -fold point? The latter condition is controlled by the vanishing of a single Wronskian. In section 3, we give an affirmative answer to this question for $d = 2$. The Wronskian is then the above exhibited $W(f, 2)$.

These considerations are related to the notion of disconjugacy for the solutions of a (linear) differential equation, the subject of the above paper of Pólya. While curves of a given degree are the solutions of a certain (non-linear) differential equation, our question is about the disconjugacy of the polynomials of degree d in x and $f(x)$.

One can also consider the postulation of a function for non-linear spaces of algebraic curves. If V is such a space, of dimension $D - 1$, one can ask whether a function possessing $D - 1$ derivatives on an interval, and intersecting a member of V in D points, counting multiplicity, necessarily intersects another member of V in a D -fold point. Note that it is no longer necessarily the case that any $D - 1$ points in the plane lie on a member of V . We can answer the above question affirmatively for the space of parabolas. We do not give the proof as it is similar to, but more complicated than, the proof for conics (Theorem 2), and we have no applications for the result.

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2. Geometric Postulation

Let $M = \{m_1(x, y), m_2(x, y), \dots, m_D(x, y)\}$ be a finite subset of $\mathbb{R}[x, y]$, linearly independent over \mathbb{R} , consisting of D elements. With M we associate the space L_M of real algebraic plane curves defined by equations

$$\sum \lambda_i m_i(x, y) = 0, \quad \lambda_i \in \mathbb{R}.$$

Thus L_M is a real projective space of dimension $D - 1$. We let M_d denote the set of all monomials in x and y of degree $\leq d$, and put $L_d = L_{M_d}$.

Suppose that $f(x)$ has n derivatives and graph Γ , and that C is an algebraic curve defined by $g(x, y) = 0$. For $k \leq n + 1$, we will say that Γ and C intersect with multiplicity k at a point $x = a$ if

$$\left. \frac{d^i g(x, f(x))}{dx^i} \right|_{x=a} = 0, \quad i = 1, \dots, k - 1.$$

For $m_i \in M$, write $m_i(x)$ for $m_i(x, f(x))$, and $m_i^{(j)}(x)$ for the j -th derivative of $m_i(x)$ with respect to x .

Lemma 1. *Suppose that $f(x)$ possesses n derivatives on an interval I . Suppose that x_1, x_2, \dots, x_h are distinct points of I , and k_1, k_2, \dots, k_h are positive integers not exceeding $n + 1$. Then there is a curve in L_M intersecting the graph Γ of $f(x)$ with multiplicity k_i at x_i for $i = 1, 2, \dots, h$ if and only if*

$$\text{rank} \begin{pmatrix} m_1(x_1) & m_2(x_1) & \dots & m_D(x_1) \\ m_1'(x_1) & m_2'(x_1) & \dots & m_D'(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ m_1^{(k_1-1)}(x_1) & m_2^{(k_1-1)}(x_1) & \dots & m_D^{(k_1-1)}(x_1) \\ m_1(x_2) & m_2(x_2) & \dots & m_D(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ m_1^{(k_2-1)}(x_2) & m_2^{(k_2-1)}(x_2) & \dots & m_D^{(k_2-1)}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ m_1^{(k_h-1)}(x_h) & m_2^{(k_h-1)}(x_h) & \dots & m_D^{(k_h-1)}(x_h) \end{pmatrix} < D.$$

In particular, if $n \geq D - 1$, Γ has a D -fold intersection with a member of L_D at x if and only if the Wronskian determinant

$$\det(m_j^{(i-1)}(x)) = 0.$$

Proof. Both statements are equivalent to the existence of a vector orthogonal to the space spanned by the rows of the matrix in \mathbb{R}^D . \square

Definition. Suppose $f(x)$ is $D - 1$ times differentiable on an interval I with graph Γ . We call $f(x)$ M -averse or L_M -averse if Γ never intersects a curve in L_M in D points counting multiplicity. We call $f(x)$ M -disconjugate if there is a sequence $V_1 < V_2 < \dots < V_D = \text{span}(M)$ of subspaces of $\text{span}(M)$, with V_i of dimension i (i.e. a complete flag of $\text{span}(M)$) such that $f(x)$ is V_i -averse for $i = 1, \dots, D - 1$.

For functions $\phi_1, \phi_2, \dots, \phi_n$ possessing $n - 1$ derivatives on an interval,

$$W(\phi_1, \phi_2, \dots, \phi_n)$$

will denote the $n \times n$ Wronskian determinant. Our local condition for M -aversity follows readily from the following theorem of Pólya [3]. (See also [4, II part 5, problem 99].)

Proposition 1. *Suppose the functions $\phi_1, \phi_2, \dots, \phi_n$ possess derivatives of order $n - 1$ on an interval $I = [a, b]$ and satisfy:*

$$W(\phi_1) > 0, W(\phi_1, \phi_2) > 0, \dots, W(\phi_1, \dots, \phi_{n-1}) > 0$$

for $x \in I$. Suppose that x_1, \dots, x_h are distinct points of I , and k_1, \dots, k_h are positive integers satisfying:

$$a \leq x_1 < x_2 < \dots < x_h \leq b, \quad k_1 + k_2 + \dots + k_h = n.$$

Let Δ be the $n \times n$ determinant:

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & \dots & \phi_n(x_1) \\ \phi_1'(x_1) & \phi_2'(x_1) & \dots & \phi_n'(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(k_1-1)}(x_1) & \phi_2^{(k_1-1)}(x_1) & \dots & \phi_n^{(k_1-1)}(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \dots & \phi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{(k_h-1)}(x_h) & \phi_2^{(k_h-1)}(x_h) & \dots & \phi_n^{(k_h-1)}(x_h) \end{vmatrix}.$$

There exists an intermediate point x , with $x_1 < x < x_k$, such that the value assumed at x by the Wronskian $W(\phi_1, \dots, \phi_n)$ is $<, =, > 0$ according as Δ is $<, =, > 0$. \square

An immediate consequence of the above and lemma 1 is:

Theorem 1. *A sufficient condition for a function $f(x)$ possessing $D - 1$ derivatives on an interval I to be M -averse is the non-vanishing of the D Wronskian determinants*

$$W(m_1(x, f(x)), \dots, m_k(x, f(x))), \quad k = 1, \dots, D$$

throughout the interval I . \square

Since the M -aversivity of $f(x)$ depends only on the space L_M , but the Wronskians appearing in the theorem depend (except for the last one) on the choice of M as an ordered basis of the space it spans, our conclusion is really the following: Suppose $f(x)$ is M -disconjugate. Then $f(x)$ is M -averse if and only if it has no D -fold intersections with curves in L_M .

Definition. Let $M = \{m_1, \dots, m_D\}$ be a finite, ordered, linearly independent subset of $\mathbb{R}[x, y]$, and $f(x)$ a function with $D - 1$ derivatives. Set

$$W(f, M) = W(m_1(x), \dots, m_D(x)),$$

and

$$\Pi W(f, M) = \prod_{i=1}^D W(m_1(x), \dots, m_i(x)).$$

We define $W(f, d)$ to be $W(f, M_d)$, and $\Pi W(f, d)$ to be $\Pi W(f, M_d)$, where we take $M_d = \{1, x, y, x^2, xy, y^2, \dots, y^d\}$ as an ordered set.

3. Conics

Let $f(x)$ be a function possessing 5 derivatives on an interval I . Our object here is to prove:

Theorem 2. *Suppose that the graph Γ of $f(x)$ intersects a conic C in 6 points counting multiplicity. Then there is another conic C^* intersecting Γ in a 6-fold point intermediate to the intersections of C .*

From this there follows by lemma 1:

Corollary. *A necessary and sufficient condition for $f(x)$ to be 2-averse is that*

$$W(f, 2) = f'' \begin{vmatrix} f''' & 3f'' & 0 \\ f^{iv} & 4f''' & 6f'' \\ f^v & 5f^{iv} & 20f''' \end{vmatrix}$$

be non-vanishing throughout I . \square

It is sufficient to prove the theorem assuming $f''(x) \neq 0$, for if $f''(a) = 0$ for $a \in I$ then the tangent to Γ at a is a triple intersection, and then the conic consisting of this line taken twice has a 6-fold intersection with Γ . We therefore begin with several lemmas concerning conics and convex curves. In lemmas 2 through 8, we assume that $h(x)$ is a function possessing 5 derivatives on an interval I with graph H , and that $h''(x) \neq 0$.

Lemma 2. *A conic intersecting H in 5 points is non-singular.*

Proof. Since H intersects any line at most twice, a conic through 5 points of H cannot be a pair of lines. \square

Lemma 3. *Suppose B and C are non-singular conics, each intersecting H at $(x, h(x))$ with multiplicity $m \leq 6$. Then B and C intersect each other with multiplicity $\geq m$ at $(x, h(x))$.*

Proof. If $m = 0$, the conclusion is trivial, while if $m = 1$ the conclusion follows because the point $(x, h(x))$ lies on both curves. Suppose that $m > 1$, and that the conics

B and C are defined by equations $p(x, y) = 0$ and $q(x, y) = 0$ respectively. Since $m > 1$, it follows that B and C do not have vertical tangents at $(x, h(x))$, so that $p_y(x, h(x))$ and $q_y(x, h(x))$ are non-zero. Hence, $p(x, y) = 0$ and $q(x, y) = 0$ can be solved for y in a neighbourhood of x , giving functions $P(x)$ and $Q(x)$ respectively. The multiplicity m condition now implies that $P^{(i)}(x) = h^{(i)}(x) = Q^{(i)}(x)$ for $i = 0, \dots, m - 1$. \square

Lemma 4. *Suppose that B and C are conics intersecting H in 5 points, just 4 of which are common. Then B and C have no other intersections.*

Proof. By lemma 2, B and C are non-singular, hence by lemma 3 the 4 points of mutual intersection are 4 points of intersection of B and C . By Bézout's theorem, B and C can have no further intersections unless they share a component. But $B \neq C$ since they do not share the other intersection with H , and neither curve can be a product of lines. \square

Lemma 5. *Let P, Q, R be non-collinear points in the plane. The lines PQ, QR, RP divide the plane into 7 regions. Let T be the interior of the triangle PQR , A the region touching T only at P , B the region touching T only at Q , and C the region touching T only at R . Then a simple, or simple closed, convex curve S having a tangent at each point that contains the points P, Q, R can never enter the interiors of the regions A, B, C, T .*

Proof. Suppose S contains a point Z that lies in A . Since S must go through Q and R , it cannot remain in A . Being convex, S cannot meet PQ except in P or Q , or PR except in P or R , hence S must leave A through P . Since S cannot be tangent to PQ or PR at P , it must enter T . Similarly, S must leave T through Q or R , entering B or C , respectively, but then cannot intersect R or Q respectively. The argument for T is similar: indeed, if S has a point in T then it must also enter one of the regions A, B, C . \square

Lemma 6. *Suppose a conic intersects H in 5 or more points. Then all the intersections are with one connected component of the conic.*

Proof. We need only consider H intersecting a hyperbola C in 5 points, since a conic through 5 points of H is non-singular by lemma 2, and ellipses and parabolas are connected. We must have three points at least lying on one branch of the hyperbola. Call them P, Q, R . If any of these points is a multiple intersection, then H must lie entirely on the same side of the tangent it shares there with the hyperbola as the hyperbola. The conclusion then follows because the other branch of the hyperbola lies on the other side of this tangent. If the points P, Q, R are distinct, then they are non-collinear, and we apply lemma 5. The conclusion now follows because the other branch of the hyperbola lies in one of the region A, B, C of lemma 5. \square

Lemma 7. *Let F be the family of conics intersecting H in four given points P_i , not necessarily distinct. Then F is a pencil, that is, a one-dimensional linear space of conics.*

Proof. Clearly F is a linear space of conics of dimension at least 1. Let Q be a point of H distinct from the P_i . If the dimension of F is > 1 , then there are at least two distinct conics of F through Q . But this is impossible because these curves cannot share a component, and so cannot intersect in 5 points. \square

Lemma 8. *Let F be the pencil of conics through four given points P_i of H . There exists a point Z in the plane such that no member of F intersecting H in an additional point intersects Z .*

Proof. Suppose first that among the P_i at least 3 are distinct. Call them P, Q, R , and take Z to be a point in the interior of the triangle T of lemma 5. I claim that Z has

the desired property, for the connected component of a conic going through 5 points of H is convex and simple, or simple-closed, and hence by lemma 5 cannot go through Z , while, in the case of a hyperbola, the other branch is in the interior of one of the regions A, B, C of lemma 5. If at most two of the P_i are distinct, then certainly one of them is a double intersection; in this case, take Z to be a point on the tangent of H at this point. Any conic through the P_i and one further point is not a pair of lines, by lemma 2, and shares the tangent with H used to choose Z , which, by Bézout's theorem, it may not intersect again. \square

Key Corollary. *There is a member of the pencil F having no further intersections with H .*

Proof. The conic through the point Z of the lemma has this property. \square

The above corollary is all we shall require of the preceding lemmas in the proof of the theorem.

Proof of Theorem 2. As already remarked, we can assume that $f''(x) \neq 0$. Suppose that C is a conic intersecting Γ in 6 points P_1, P_2, \dots, P_6 , not necessarily distinct, with $P_i = (x_i, y_i)$ and $x_1 \leq x_2 \leq \dots \leq x_6$. We can also assume that the intersections of Γ with any conic do not accumulate in the interior of I , or our conclusion is immediate.

We describe operations $T_i, i = 1, \dots, 5$ that transform an intersection $(C, P_j), P_j = (x_j, y_j)$, consisting of a conic C and an intersection $P_j, j = 1, \dots, 6$, to another intersection $(C', P'_j), P'_j = (x'_j, y'_j)$ also of multiplicity 6. If $P_i = P_{i+1}$, T_i is the identity map, while if $P_i \neq P_{i+1}$, we will have the properties:

1. $x_i < x'_i = x'_{i+1} < x_{i+1}$;

2. $x'_j \leq x_j$ for $j \geq i + 2$;
3. $x'_j \geq x_j$ for $j \leq i - 1$.

To describe T_i , we suppose that $P_i \neq P_{i+1}$. Let F be the pencil of conics through the four points $P_j, j \neq i, i + 1$, and, appealing to the key corollary, let $B : p(x, y) = 0$ be a member of F having no further intersections with Γ . Let $q(x, y) = 0$ be the equation of C . Thus $p(x, y)$ and $q(x, y)$ generate the pencil. Since

$$\begin{vmatrix} q(x_i, f(x_i)) & p(x_i, f(x_i)) \\ q(x_{i+1}, f(x_{i+1})) & p(x_{i+1}, f(x_{i+1})) \end{vmatrix} = 0,$$

but $p(x, f(x)) \neq 0$ on $[x_i, x_{i+1}]$, it follows from Pólya's theorem (proposition 1 of section 2), that there is a conic C' in the pencil F having a double intersection with Γ at a point $x \in (x_i, x_{i+1})$. Since the intersections of C' with Γ have no interior point of accumulation, we may set:

1. $T_i C = C'$;
2. $T_i P_i = P'_i = (x, f(x))$;
3. $T_i P_{i+1} = P'_{i+1} = (x, f(x))$;
4. For $j \geq i + 2, P'_j$ is the first intersection (x, y) of C' with Γ having $x \geq x_{j-1}$, and clearly satisfies $x'_j \leq x_j$;
5. For $j \leq i - 1, P'_j$ is the first intersection (x, y) of C' with Γ having $x \leq x_{j+1}$, and clearly satisfies $x'_j \geq x_j$.

We now let T be the operation

$$T = T_5 T_4 T_3 T_2 T_1,$$

(that is, T_1 followed by T_2, T_3, \dots , up to T_5).

Let d be the infimum of $x_6 - x_1$ over all 7-tuples (C, P_i) consisting of conics and intersections of multiplicity 6.

Consider a sequence

$$(C^{(n)}, P_i^{(n)}), \quad n = 0, 1, \dots$$

This is a sequence in the compact space

$$\mathbb{R}P^5 \times I^6,$$

and hence has a cluster point (C', P'_i) . Clearly, (C', P'_i) is also an intersection of multiplicity 6. Hence the infimum d is attained by some (C^*, P_i^*) .

I claim that (C^*, P_i^*) is a 6-fold intersection at a single point. For suppose that, to the contrary, for some j we have

$$P_j^* \neq P_{j+1}^* = P_{j+2}^* = \dots = P_6^*.$$

Applying $T_j \dots T_1$ yields a new conic C^{**} and points P_i^{**} with

$$x_{j+1}^{**} < x_{j+1}^*.$$

If we have

$$x_6^{**} < x_6^*,$$

we have a contradiction, for the application of $T_5 \dots T_{j+1}$ cannot increase x_6^{**} , so that x_6^* could not have been part of the infimum. So we must have $P_6^{**} = P_6^*$, and hence for some $k \geq j + 1$ we now have

$$P_k^{**} \neq P_{k+1}^{**} = \dots = P_6^{**}.$$

We now apply $T_k \dots T_{j+1}$ and repeat the argument. After at most 5 repetitions, we are led to a contradiction, and this completes the proof of the theorem. \square

4. Main Lemma

Let I be a closed bounded interval and let Γ be the arc $y = f(x)$, where $f \in C^k(I)$. For dilation factors $H, K \geq 1$ let $\Gamma(H, K)$ be the graph of $y = Kf(x/H), x \in HI$. We now consider points of $\Gamma(H, K)$ defined over a number field. Let L be a finite extension of the rational numbers and S the set of embeddings of L into \mathbf{C} . An element ξ of L will be said to have *denominator* $\leq N$ if there is an integer r with $|r| \leq N$ such that $r\xi$ is an algebraic integer, and will be said to have *size* $\leq T$ if $|\xi^\sigma| \leq T$ for every $\sigma \in S$.

Let $P_1 = (x_1, y_1), \dots, P_s = (x_s, y_s)$ be the points of Γ for which Hx_i, Ky_i are elements of L of denominator $\leq N$ and size $\leq T$, arranged in order of increasing abscissae. Choose integers s_i, t_i such that $|s_i|, |t_i| \leq N$ and $s_i Hx_i, t_i Ky_i$ are algebraic integers. Set

$$\|f\|_k = \max_{\substack{\kappa \leq k \\ x \in I}} \frac{|f^{(\kappa)}(x)|}{\kappa!} .$$

We remark that if $I \subseteq [-B, B]$ then $\|x\|_k \leq B$ for all k .

Proposition 2. *Suppose $f_1, \dots, f_m \in C^k(I)$. Then*

$$\|f_1 \cdots f_m\|_k \leq ((k+1))^{m-1} \|f_1\|_k \cdots \|f_m\|_k .$$

In particular, if $I \subseteq [-B, B]$ then $\|x^p f^q\|_k \leq ((k+1))^{p+q-1} B^p \|f\|_k^q$, for positive integers p, q .

Proof. This is essentially Proposition 1 of [1]. \square

Proposition 3. *Suppose that $x_1, \dots, x_n \in I$ are distinct points and that $f_1, \dots, f_n \in C^{n-1}(I)$. Then*

$$|\det(f_j(x_i))| \leq |V(x_1, \dots, x_n)| n! \|f_1\|_{n-1} \cdots \|f_n\|_{n-1} ,$$

where $V(x_1, \dots, x_n)$ denotes the van der Monde determinant.

Proof. This follows from Proposition 2 of [1] and the subsequent discussion. \square

Let d be a positive integer, and define a finite sequence n_ℓ of integers as follows.

- (i) $n_0 = 1$
- (ii) Suppose $n_{\ell-1}$ has been defined. Then n_ℓ is the unique integer such that the points P_i for $n_{\ell-1} \leq i < n_\ell$ lie on some real algebraic curve of degree $\leq d$, but the points P_i for $n_{\ell-1} \leq i \leq n_\ell$ do not, if such an integer n_ℓ exists. Otherwise, the sequence terminates with $n_{\ell-1}$.

Let J_d denote the set of pairs $j = (j_1, j_2)$ with $0 \leq j_1, j_2 \leq j_1 + j_2 \leq d$. So $|J_d| = D = \frac{1}{2}(d+1)(d+2)$. If P is a point with coordinates (x, y) we write

$$P^j = P^{(j_1, j_2)} = x^{j_1} y^{j_2} .$$

Lemma 9. *For the sequence n_0, \dots, n_m associated to the curve $\Gamma : y = f(x)$, $x \in I$, $f \in C^{D-1}(I)$, $I \subseteq [-B, B]$ and any positive integer d we have*

$$|x_{n_{\ell+1}} - x_{n_\ell}| \geq (D^2 B \|f\|_{D-1})^{-\frac{4}{3(d+3)}} (D!)^{-\frac{2(n-1)}{D(D-1)}} (H K N^{6n} T^{2(n-1)})^{-\frac{4}{3(d+3)}} .$$

Proof. Since the points $P_{n_\ell}, \dots, P_{n_{\ell+1}}$ do not lie on any algebraic curve of degree $\leq d$, it follows from Lemma 1 of [1] that there is a subset $I \subset \{n_\ell, \dots, n_{\ell+1}\}$ of cardinality D such that

$$\Delta = \det \left(P_i^j \right)_{\substack{i \in I \\ j \in J_d}} \neq 0 .$$

Now, $\Lambda = \prod s_i^d \prod t_i^d (HK)^{\frac{dD}{3}} \Delta$ is a non-zero algebraic integer. Hence $\prod \Lambda^\sigma$ over $\sigma \in S$ is a non-zero rational integer, and so

$$\left| \prod_{\sigma \in S} \left(\prod_{i \in I} s_i^d \prod_{i \in I} t_i^d (HK)^{\frac{dD}{3}} \Delta \right)^\sigma \right| \geq 1 .$$

By our assumptions on the sizes and denominators, for any $\sigma \in S$,

$$\left| \left(\prod_{i \in I} s_i^d \prod_{i \in I} t_i^d (HK)^{\frac{dD}{3}} \Delta \right)^\sigma \right| \leq D! N^{2dD} T^{\frac{2dD}{3}}.$$

We now apply Proposition 2 with $n = D$, the points x_i with $i \in I$ and f_j the functions $x^{j_1} f(x)^{j_2}$ for $(j_1, j_2) \in J_d$ to obtain an upper bound for Δ . Since

$$|V(x_i; i \in I)| \leq |x_{n_{\ell+1}} - x_{n_\ell}|^{\frac{D(D-1)}{2}},$$

we conclude that (applying also Proposition 1):

$$\begin{aligned} |\Delta| &\leq |x_{n_{\ell+1}} - x_{n_\ell}|^{\frac{D(D-1)}{2}} D! \prod_{j \in J_d} \|x^{j_1} f(x)^{j_2}\|_{D-1} \\ &\leq |x_{n_{\ell+1}} - x_{n_\ell}|^{\frac{D(D-1)}{2}} D^D \prod_{j \in J_d} D^{j_1+j_2-1} B^{j_1} \|f\|_{D-1}^{j_2} \\ &\leq |x_{n_{\ell+1}} - x_{n_\ell}|^{\frac{D(D-1)}{2}} (D^2 B \|f\|_{D-1})^{\frac{dD}{3}}. \end{aligned}$$

Combining these estimates, we find that

$$1 \leq |x_{n_{\ell+1}} - x_{n_\ell}|^{\frac{D(D-1)}{2}} (D^2 B \|f\|_{D-1})^{\frac{dD}{3}} (HK)^{\frac{dD}{3}} N^{2Ddn} (D!)^{n-1} T^{\frac{2dD(n-1)}{3}},$$

from which the conclusion of the lemma follows by rearrangement. \square

As the length of the interval I is at most $2B$, we now conclude (using $D^{\frac{8}{3(d+3)}} < 3$ and $(D!)^{\frac{2}{D(D-1)}} \leq 3$ for every d):

Main Lemma. *Let $d \geq 1$, $D = \frac{1}{2}(d+1)(d+2)$ and $f \in C^{D-1}(I)$, $I \subseteq [-B, B]$. Let L be a finite extension of the rational numbers of degree n . Then the points $P = (x, y)$ of $\Gamma(H, K)$ with $x, y \in L$ having denominator $\leq N$ and size $\leq T$ lie on the union of not more than*

$$2 \cdot 3^n B \left(\|f\|_{D-1} B H K N^{6n} T^{2(n-1)} \right)^{\frac{4}{3(d+3)}} + 1$$

real algebraic curves of degree $\leq d$. \square

5. Integral Points

The condition $\Pi W(f, d) \neq 0$ for d -aversity can be used to give some refinements of the results of [1]. We do not appeal here to the new Main Lemma: the results below follow immediately by the methods of [1]. All we supply is an analytical statement of the hypothesis of d -aversity.

It was proved in [1] that if $f(x)$ is a C^∞ strictly convex function on a closed bounded interval I with graph Γ , and $\varepsilon > 0$, then there is a constant $c(f, \varepsilon)$ such that for all $t \geq 1$,

$$|t\Gamma \cap \mathbb{Z}^2| \leq c(f, \varepsilon)t^{\frac{1}{2}+\varepsilon},$$

where $t\Gamma$ is the homothetic dilation of Γ . The constant $c(f, \varepsilon)$ depends on the norms of the derivatives of $f(x)$ up to order roughly $1/\varepsilon^2$.

We can state theorems in which a $t^{\frac{1}{2}+\varepsilon}$ (or better) estimate is obtained assuming enhanced convexity, but only finite differentiability of $f(x)$.

Theorem 3. *Let $f(x)$ be a C^{104} function on a closed bounded interval I with graph Γ . Suppose that*

$$W(f, 2) = f'' \begin{vmatrix} f''' & 3f'' & 0 \\ f^{iv} & 4f''' & 6f'' \\ f^v & 5f^{iv} & 20f''' \end{vmatrix}$$

is non-vanishing throughout I . Then for any $\varepsilon > 0$, there is a constant $c(f, \varepsilon)$ such that for all $t \geq 1$,

$$|t\Gamma \cap \mathbb{Z}^2| \leq c(f, \varepsilon)t^{\frac{1}{2}+\varepsilon}.$$

Proof. We apply the Main Lemma of [1] with $d = 13$ to conclude that the integral points of $t\Gamma$ lie on the union of at most

$$c(f)t^{\frac{1}{8}}$$

real algebraic curves of degree ≤ 13 . We consider the irreducible components of these. For $\varepsilon > 0$, the components of degree ≥ 3 contain at most

$$c(f, \varepsilon)t^{\frac{1}{3}+\varepsilon}$$

integral points, by theorem 5 of [1] applied to a square of side $t \max(|I|, \text{osc}(f))$. By the non-vanishing hypothesis, the components of degree 1 and 2 intersect Γ in at most 2 and 5 points, respectively. \square

More generally, we have by a similar argument (and noting that, if $b \leq d$, then $\text{IIW}(f, b)$ divides $\text{IIW}(f, d)$, so that the non-vanishing of the latter implies the b -aversity of $f(x)$ for all $b \leq d$) :

Theorem 4. *Suppose $d \geq 1$, and let $f(x)$ be a C^{D-1} function on a closed bounded interval I with graph Γ . Suppose that $\text{IIW}(f, d)$ is non-vanishing on I . Then for $t \geq 1$,*

$$|t\Gamma \cap \mathbb{Z}^2| \leq c(f)t^{\frac{8}{3(d+3)}}.$$

Theorem 4*. *Suppose $d \geq 1$, and let $f(x)$ be a C^{D-1} function on a closed bounded interval I with graph Γ . Suppose that $\text{IIW}(f, b)$ is non-vanishing on I for some $1 \leq b < d$.*

Then for $\varepsilon > 0$ and $t \geq 1$,

$$|t\Gamma \cap \mathbb{Z}^2| \leq c(f, \varepsilon)t^{\frac{8}{3(d+3)} + \frac{1}{b+1} + \varepsilon}.$$

Proof of 4 and 4*. Apply the Main Lemma of [1] to conclude that the integral points of Γ lie on the union of at most

$$c(f)t^{\frac{8}{3(d+3)}}$$

real algebraic curves of degree $\leq d$. Under the hypotheses of Theorem 4, each component of each of these curves intersects Γ in at most $D - 1$ points. Under the hypotheses of Theorem 4*, the components of degree $\leq b$ intersect Γ in $\leq \frac{1}{2}b(b + 3)$ points, while those of degree $\geq b + 1$ contain at most

$$c(f, \varepsilon)t^{\frac{1}{b+1} + \varepsilon}$$

integral points in the appropriate square. \square

The dependence of the estimates on the norms of $f(x)$ can be eliminated, as in [1], by controlling the number of zeros of f^D in I . In applications where one has control over the number of zeros of $\Pi W(f, d)$ it is reasonable to assume control also over the zeros of f^D . One then gets the following results following the proof of Theorem 8 of [1].

Theorem 5. *Suppose $d \geq 4$, $N \geq 1$, and let $f(x)$ be a C^D function on a closed subinterval of $[0, N]$ with $|f'| \leq 1$ and graph Γ . Suppose that $f^{(D)} \Pi W(f, d)$ is non-vanishing on I . Then*

$$|\Gamma \cap \mathbb{Z}^2| \leq c(d)N^{\frac{8}{3(d+3)}} . \square$$

Theorem 5*. *Suppose $d \geq 4$, $N \geq 1$, and let $f(x)$ be a C^D function on a closed subinterval of $[0, N]$ with $|f'| \leq 1$ and graph Γ . Suppose that $f^{(D)} \Pi W(f, b)$ is non-vanishing on I for some $1 \leq b < d$. Let $\varepsilon > 0$. Then there is a constant $c(d, \varepsilon)$ such that:*

$$|\Gamma \cap \mathbb{Z}^2| \leq c(d, \varepsilon)N^{\frac{8}{3(d+3)} + \frac{1}{b+1} + \varepsilon} . \square$$

The $b = 2$ cases of this theorem can be improved by using the condition for 2-aversity of section 3 rather than $\Pi W(f, 2)$. We state only the $d = 13$ version.

Theorem 6. *Let $f(x)$ be a C^{105} function on a closed subinterval of $[0, N]$ with $|f'| \leq 1$ and graph Γ . Suppose that*

$$f'' f^{(105)} \begin{vmatrix} f''' & 3f'' & 0 \\ f^{iv} & 4f''' & 6f'' \\ f^v & 5f^{iv} & 20f''' \end{vmatrix}$$

is non-vanishing on I . Let $\varepsilon > 0$. Then there is a constant $c(\varepsilon)$ such that:

$$|\Gamma \cap \mathbb{Z}^2| \leq c(\varepsilon)N^{\frac{1}{2}+\varepsilon} . \square$$

This should be compared with the conjecture of Schmidt [6] that the estimate above obtains for $f \in C^3[0, N]$ with $|f| \leq N$ and $f''' \neq 0$.

6. Rational Points

Here we appeal to the Main Lemma of section 4 to give bounds for points on Γ with coordinates in a number field of bounded size and denominator. Theorem 7 generalizes Theorem 4 of the previous section. We cannot similarly generalize the other Theorems of section 5 because the recurrence argument used in [1] to eliminate dependence on the norms of functions is appealed to in their proofs. (In Theorems 3 and 4* this is hidden in the appeal to the uniform bounds of [1] for the integral points on an irreducible algebraic curve inside a square.) As mentioned in the introduction, this argument is not available in the present context. We give also our result for transcendental analytic functions, and conclude with some remarks on the application of the method of section 4 to algebraic functions.

Theorem 7. *Suppose $d \geq 1$, and let $f(x)$ be a C^{D-1} function on a closed bounded interval I with graph Γ . Suppose that $f(x)$ is b -averse for all $b \leq d$. Let n be a positive*

integer. There is a constant $c(f, n)$ such that for any finite extension L of the rational numbers of degree n , and any $N, T, H, K \geq 1$ the number of points (x, y) on $\Gamma(H, K)$ with $x, y \in L$ having denominator $\leq N$ and size $\leq T$ is bounded by

$$c(f, n) \left(H K N^{6n} T^{2(n-1)} \right)^{\frac{4}{3(d+3)}}.$$

Proof. Immediate from the Main Lemma. \square

Proposition 4. *Let $f(x)$ be an analytic function on an interval I , and M a linear space of real algebraic curves. If $W(f, M) = 0$ identically in I then $f \in M$.*

Proof. Consider n functions f_1, \dots, f_n possessing $n - 1$ derivatives on an interval I , and let $W(f_1, \dots, f_n)$ denote the Wronskian determinant. It is well known (see Pólya and Szegő [4, volume II, part 5, problem 60]) that if $W(f_1, \dots, f_n) = 0$ identically, while $W(f_1, \dots, f_{n-1})$ never vanishes then f_n is linearly dependent on f_1, \dots, f_{n-1} . Since our f is assumed analytic, it is enough for us if the smaller Wronskian is non-vanishing at a single point. So the conclusion follows by induction. \square

Corollary. *Let $f(x)$ be analytic on a closed bounded interval I with graph Γ and suppose that $f(x)$ is not algebraic. Let M be a linear space of real algebraic curves of dimension $D - 1$. Then $\Pi W(f, M)$ has only a finite number $\mu(f, M)$ of zeros in I . Thus Γ intersects any curve in M in at most $\gamma(f, M) = (\mu(f, M) + 1) (D - 1)$ distinct points. \square*

Combining the Corollary with the Main Lemma we conclude:

Theorem 8. *Let $f(x)$ be a real analytic function on a closed bounded interval I and suppose that $f(x)$ is not algebraic. Let n be a positive integer. Let $\varepsilon > 0$. There is a constant $c(f, n, \varepsilon)$ such that for any finite extension L of the rational numbers of degree*

n , and any $N, T, H, K \geq 1$ the number of points (x, y) on $\Gamma(H, K)$ with $x, y \in L$ having denominator $\leq N$ and size $\leq T$ is bounded by

$$c(f, n, \varepsilon) (H K N T)^\varepsilon . \square$$

Thus while it is possible (see van der Poorten [5]) to construct entire transcendental functions that (even together with their derivatives to all orders) map every algebraic number field into itself, the above shows that the sizes or denominators of the ordinates of these points must grow faster than any power of the sizes and denominators of the abscissae. Related constructions are discussed in Mahler [2].

For the special case of rational points on Γ itself one obtains the result quoted in the introduction:

Theorem 9. *Let $f(x)$ be a real analytic function on a closed bounded interval I and suppose that $f(x)$ is not algebraic. Let Γ be the graph of $f(x)$. Let $\varepsilon > 0$. There is a constant $c(f, \varepsilon)$ such that for any positive integer N , the number of rational points on Γ of height $\leq N$ is bounded by $c(f, \varepsilon)N^\varepsilon$. \square*

One can give a version of the Generalized Main Lemma of [1] after the manner of section 4 with a view to consider rational points on algebraic curves. However, since one is unable to eliminate norms and obtain uniform estimates one gets results for an individual algebraic function with weaker exponents than are obtainable by other means, but with constants of quite a different nature.

Thus the bound of Theorem 9 holds *a fortiori* if f is an algebraic function, unless the curve $y = f(x)$ admits a parametrization by rationally defined rational functions (in which case it is false), however the constants depend on the rank of the Mordell–Weil group of

the Jacobian, or on quantitative forms of the Mordell conjecture. Here we take the height of a rational point $P = (\frac{a}{c}, \frac{b}{c})$ to be $\max(|a|, |b|, |c|)$ where $a, b, c \in \mathbb{Z}$, and $(a, b, c) = 1$.

We can show that if f is a C^∞ function on a closed bounded interval I , satisfying an irreducible algebraic relation of degree d then the number of rational points on Γ of height $\leq N$ is bounded by $c(f, \varepsilon)N^{\frac{2}{d}+\varepsilon}$, where the constant depends only on the norms of derivatives of f up to a finite order depending on ε . The example $y = x^d$ shows that the exponent $2/d$ is best possible.

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