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Entire functions sharing arguments of integrality, II

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Abstract

This paper gives a slight strengthening of a special case of the six exponentials theorem and some related results.

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1. Introduction

This paper, like its predecessor [4], is devoted to a certain aspect of the following general question. Suppose that $X \subset \mathbb{C}$ is an infinite set with no finite points of accumulation, and that $f_1, f_2, \ldots, f_n$ are entire functions that take integer values on all $x \in X$. What conditions on the growth of the functions (and on X) are sufficient to conclude that they are algebraically dependent (over $\mathbb{Z}$), at least when restricted to X? The paradigm result of this type is due to Pólya [5, 2, 6], or see [4] for a statement (in a weakened form) and a generalization.

In this paper we consider a problem of the above general type related to the four exponentials conjecture in transcendental number theory.

The *four exponentials conjecture* is the following statement. *Let $\alpha, \beta \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$; let likewise $a, b \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then at least one of the four exponentials $\exp(a\alpha), \exp(a\beta), \exp(b\alpha), \exp(b\beta)$ is transcendental.* The *six exponentials theorem* (due to Lang and Ramachandra) asserts that *if one has α, β as above and $a, b, c \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then at least one of the six exponentials is transcendental.* (See [10] for references and further discussion.)

Suppose that α, β, a, b are a counterexample to the four exponentials conjecture, with moreover $\alpha, \beta, a, b \in \mathbb{R}_{>0}$ and $\exp(a\alpha), \exp(a\beta), \exp(b\alpha), \exp(b\beta) \in \mathbb{Z}$. (Note that this special case of the conjecture is still open.) Then the entire functions $f_1(z) = \exp(az)$ and $f_2(z) = \exp(bz)$ take integer values on the set $X_{\alpha,\beta} = \{i\alpha + j\beta : i, j \in \mathbb{N}\}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. The six exponentials theorem implies in particular that another entire function $g(z)$ of the form $\exp(cz)$ that takes integer values on $X_{\alpha,\beta}$ is algebraically dependent on f_1 and f_2.

Indeed the same conclusion holds for general entire functions g of somewhat faster growth. For an entire function h, denote by $M(h, r)$ the maximum modulus of an entire function h at radius r, and say that h is of *(strict) order* $\le \rho$ if there is a constant C such that

$$M(f, r) \le C^{r^\rho}.$$

A result of Waldschmidt [9, Theorem 2.2.1] that generalizes results of Lang [3, II, §2, Theorem 2] and Ramachandra [7, Theorem 1] (which in turn generalize results of Schneider) on algebraic values of meromorphic functions, implies that, under the above hypotheses, an entire function g that takes integer values on $X_{\alpha,\beta}$ and is of order $\le \rho$ for some $\rho < 2$ must be algebraically dependent on f_1 and f_2. (Indeed this is true for meromorphic functions, and without our special assumptions on the form of the counterexample to four exponentials.)

We extend this slightly, in the special case, showing that the same conclusion holds for certain entire functions of order ≤ 2 that may not be of order $\le \rho$ for any $\rho < 2$.

1.1. Theorem. *Let a, b, α, β be a counterexample to the four exponentials conjecture with moreover $\alpha, \beta, a, b \in \mathbb{R}_{>0}$ and $\exp(a\alpha), \exp(a\beta), \exp(b\alpha), \exp(b\beta) \in \mathbb{Z}$. Suppose that $g(z)$ is an entire function that takes integer values on $X_{\alpha,\beta}$ and that*

$$\lim_{r \to +\infty} \frac{\log M(g, r)}{r^2} \le \frac{1}{832\, ab\alpha^2\beta^2}.$$

Then the functions $f_1(z) = \exp(az), f_2(z) = \exp(bz), g(z)$ are algebraically dependent over $\mathbb{Z}$.

We have not attempted to optimize the numerical value 1/832. In this paper we will prove a generalization of Theorem 1.1 which we now proceed to formulate.

1.2. Definition. Let $X = \{x_0, x_1, \ldots\}$ be a strictly increasing sequence of non-negative real numbers.

(1) Call X a *scale* if, for any integer $t \geq 2$ and positive ϵ,

$$\lim_{n\to\infty} n^\epsilon \log\left(\frac{x_{tn}}{x_n}\right) = \infty.$$

(2) Define, for an integer $t \geq 2$,

$$\chi(X,t) = \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \frac{x_{tn} - x_j}{x_n - x_j}.$$

Note that if, for some positive integer t and some positive ϵ,

$$\limsup_{n\to\infty} n^\epsilon \log\left(\frac{x_{tn}}{x_n}\right) < \infty$$

then X is bounded (see 2.2); thus the scale condition may be seen as a mild regularity assumption when X has $x_j \to \infty$ as $j \to \infty$.

We will measure the growth of an entire function relative to X by considering the quantities

$$\omega_X(f,\sigma) = \limsup_{n\to\infty} \frac{\log M(f,x_n)}{n^\sigma}$$

where $0 < \sigma < 1$. There is at most one σ for which $0 < \omega_X(f,\sigma) < \infty$.

For $X_{\alpha,\beta}$ of Theorem 1.1 considered as an increasing sequence $\{x_0, x_1, \ldots\}$ we have

$$x_n \sim (2n\alpha\beta)^{1/2}$$

as $n \to \infty$ (see Proposition 2.6). Thus for f_1, f_2 of Theorem 1.1 we have

$$\omega_X(f_1, 1/2) = a\sqrt{2\alpha\beta}, \quad \omega_X(f_2, 1/2) = b\sqrt{2\alpha\beta}.$$

The following result is essentially a reformulation, in our special situation, of the aforementioned theorem of Waldschmidt, though our functions are not required to be of finite order.

1.3. Theorem. *Let X be a scale. Let $f_1, f_2, \ldots, f_k$ be entire functions that are integer valued on X, and suppose that $\omega_X(f_i,\sigma_i) < \infty$ for $i = 1, \ldots, k$ where $\sum_i (1-\sigma_i) > 1$. Then $f_1, \ldots, f_k$ are algebraically dependent over $\mathbb{Z}$ on X.*

By algebraic dependence of $f_1, f_2, \ldots, f_k$ over $\mathbb{Z}$ on X we mean that there is a polynomial $h \in \mathbb{Z}[t_1, t_2, \ldots, t_k]$, not identically zero, such that $h(f_1(x), \ldots, f_k(x)) = 0$ for all $x \in X$. We give a proof of Theorem 1.3 in Section 3.

Sequences of the form $X_{\alpha,\beta}$ are indeed scales (see 2.6). Under the hypotheses of Theorem 1.1 we have independent functions f_1, f_2, integer valued on $X = X_{\alpha,\beta}$, with $\omega_X(f_1,\sigma_1), \omega_X(f_2,\sigma_2) < \infty$, where $\sigma_1 = \sigma_2 = 1/2$ so that $\sum(1-\sigma_i) = 1$, the maximum possible according to 1.3, which thus already implies the six exponentials theorem (in the special case under consideration) as any additional function f_3 with $\omega_X(f_3,\sigma) < \infty$ for any $\sigma < 1$ would lead to algebraic dependence.

In our generalization of Theorem 1.1 we consider a scale X together with a finite set of entire functions $f_1, \ldots, f_k$ that are maximal in the sense of Theorem 1.3. It is convenient to measure the growth of g by reference to the scale X, so our growth hypothesis is stated in terms of $M(g, x_n)$.

1.4. Theorem. *Let X be a scale. Let $f_1, f_2, \ldots, f_k$ be entire functions that are integer valued on X, and that satisfy $\omega_X(f_i,\sigma_i) < \infty$ where $0 < \sigma_i < 1$ and $\sum_i (1-\sigma_i) = 1$. Suppose that $T < \chi(X,2)$, and that g is an entire function that is integer valued on X with*

$$\limsup_{n\to\infty} \frac{\log M(g,x_n)}{n} \leq \frac{T^{k+1}}{2(k+1)!3^{k+1}} \frac{1}{\prod \omega_X(f_i,\sigma_i)}.$$

Then $\{f_1, \ldots, f_k, g\}$ are algebraically dependent over $\mathbb{Z}$ on X.

For general X it is unclear whether one can conclude further (in 1.4 and 1.3) that $f_1, \ldots, f_k, g$ are algebraically dependent entire functions. But for the situation of 1.1 one can deduce this (using Jensen's formula). There is some further discussion of this issue in [4].

The proofs of all the theorems will be by Schneider's method from transcendental number theory, that is, by construction of an auxiliary function using Siegel's Lemma. This is also the method used in the results of Lang, Ramachandra, Waldschmidt mentioned above.

Our main motivation for the these results is the conjectural non-example afforded by the four exponentials conjecture, as hypothesized in Theorem 1.1. However, we are not able to exhibit examples satisfying the hypotheses of 1.4 either.

1.5. Question. Are there any examples of a scale X and entire functions $f_1, \ldots, f_k$, algebraically independent and integer valued on X, with growth rates as in the hypothesis of 1.4?

In Section 4 we establish a more general version of Theorem 1.4 allowing several additional functions g_i. One could give a specific formulation of this theorem for the situation of Theorem 1.1. This raises the possibility of proving the four exponentials conjecture by constructing some integer valued entire functions on $X_{\alpha,\beta}$ of suitable growth. Of course it is not clear how to do this. Some further results are presented in Section 4.

2. Preliminaries

An estimate following from Cauchy's theorem

Our proofs follow a standard method of transcendence theory. We use Siegel's Lemma to construct an entire function that vanishes at certain prescribed points, and then we show it is small (and so must vanish) at further points. The following is the result we use to effect this last step. It is a simple consequence of Cauchy's integral theorem.

Let $X = \{x_0, x_1, \ldots\}$ be a strictly increasing sequence of non-negative real numbers. For $n, m \in \mathbb{N}, m > n$, set

$$Q_X(n,m) = \frac{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}{(x_m - x_0)(x_m - x_1)\ldots(x_m - x_{n-1})}, \qquad R_X(n,m) = \frac{x_m}{x_m - x_n}.$$

2.1. Proposition. *Suppose g is an entire function vanishing at $x_0, x_1, \ldots, x_{n-1}$. Let $m > n$. Then*

$$|g(x_n)| \le \; Q_X(n,m)\, R_X(n,m)\, M(g, x_m).$$

Proof. This is Corollary 2.2 of [4]. □

Scales

We first verify the assertion after Definition 1.2 about the scale condition.

2.2. Proposition. *Suppose that $X = \{x_0, x_1, \ldots\}$ is a strictly increasing sequence of non-negative real numbers, and $t \ge 2$ an integer. Let $\epsilon > 0$. Suppose*

$$\limsup_{n\to\infty} n^\epsilon \log\left(\frac{x_{tn}}{x_n}\right) < \infty.$$

Then X is bounded.

Proof. Suppose that, for all $n \ge A$,

$$n^\epsilon \log\left(\frac{x_{tn}}{x_n}\right) \le B.$$

Then, for any positive integer m,

$$\log\left(\frac{x_{t^m A}}{x_A}\right) \le \frac{B}{A^\epsilon}\Big(1 + \frac{1}{t^\epsilon} + \frac{1}{(t^\epsilon)^2} + \ldots\Big) \le \frac{B}{A(1 - t^{-\epsilon})}. \; \square$$

The next two propositions show that the quantity $R_X(n,tn)$ of 2.1 is innocuous for a scale.

2.3. Proposition. *Let $t>0$. Then* $-\log(1-\exp(-t)) \;\le\; t^{-1}$.

Proof. We have $-\log(1-\exp(-t)) = \sum_{k\ge1}\exp(-tk)/k \le \sum_{k\ge1}\exp(-tk) = (\exp(t)-1)^{-1} \le \frac{1}{t}$. □

2.4. Proposition. *Let $X=\{x_0,x_1,\ldots\}$ be a scale, $t\ge2$ an integer, and $\epsilon>0$. Then*

$$\lim_{n\to\infty}\frac{\log R_X(n,tn)}{n^\epsilon}=0.$$

Proof. Applying 2.3,

$$\frac{\log R_X(n,tn)}{n^\epsilon}=\frac{-\log(1-\exp(-\log(x_{tn}/x_n)))}{n^\epsilon}\le(n^\epsilon\log(x_{tn}/x_n))^{-1}$$

and the conclusion follows from the condition that X is a scale. □

Estimation of $\chi(X,t)$ for certain sequences X

2.5. Proposition. *Let $\beta_1,\ldots,\beta_k$ be positive real numbers. Let $L=L(\beta_1,\ldots,\beta_k)$ be the region of $\mathbb{R}^k$ defined by*

$$L=\{(u_1,\ldots,u_k)\in\mathbb{R}^k : 0\le u_i, i=1,\ldots,k, \sum_{i=1}^k\frac{u_i}{\beta_i}\le1\}.$$

Then the number $\#L\cap\mathbb{Z}^k$ of integral lattice points in L satisfies

$$\mathrm{vol}(L)=\frac{1}{k!}\prod_{i=1}^k\beta_i\le\#L\cap\mathbb{Z}^k\le\left(1+\sum_{i=1}^k\frac{1}{\beta_j}\right)^k\mathrm{vol}(L).$$

Proof. For a point $u=(u_1,\ldots,u_k)$ of $\mathbb{R}^k$ let B_u denote the closed k-cube with bottom corner at u, namely $B_u=\{y=(y_1,\ldots,y_k)\in\mathbb{R}^k : u_i\le y_i\le u_i+1, i=1,\ldots,k\}$. Then the union of boxes B_u over $u\in L\cap\mathbb{Z}^k$ includes all L. Hence the lower estimate for $\#L\cap\mathbb{Z}^k$. On the other hand, the same union is contained in the region $\{(u_1,\ldots,u_k) : 0\le u_i, i=1,\ldots,k, \sum u_j/\beta_i\le(1+\sum1/\beta_i)$. □

For $\alpha_1,\ldots,\alpha_k\in\mathbb{R}$ positive and linearly independent over $\mathbb{Q}$ let

$$X_{\alpha_1,\ldots,\alpha_k}=\{\sum_{j=1}^k i_j\alpha_j : i_j\in\mathbb{N}, j=1,\ldots,k\}$$

considered as an increasing sequence.

2.6. Proposition. *Let $\alpha_1,\ldots,\alpha_k\in\mathbb{R}$ be positive and linearly independent over $\mathbb{Q}$. Let $X=X_{\alpha_1,\ldots,\alpha_k}=\{x_0,x_1,\ldots\}$ as above. Then*

$$x_n\sim\big(nk!\prod_{i=1}^k\alpha_i\big)^{1/k}$$

as $n\to\infty$ so that $X_{\alpha_1,\ldots,\alpha_k}$ is a scale. Further,

$$\chi(X_{\alpha_1,\ldots,\alpha_k},t)\ge\chi_k(t)=\int_0^1\log\left(\frac{t^{1/k}-v^{1/k}}{1-v^{1/k}}\right)dv.$$

Proof. The range of sums and products throughout the proof is $i=1,\ldots,k$. For $B\ge0$ set

$$L_B=\{(u_1,\ldots,u_k)\in\mathbb{R}^k, u_i\ge0, \sum\alpha_iu_i\le B\}.$$

According to 2.5 it holds that

$$\frac{B^k}{k!\prod\alpha_i}\le\#L_B\cap\mathbb{Z}^k\le(1+\frac{1}{B}\sum\alpha_i)^k\,\frac{B^k}{k!\prod\alpha_i}.$$

If $B = x_n$ then $\#L_B \cap \mathbb{Z}^k = n+1$. Therefore

$$\Big(k!\prod \alpha_i\Big)^{1/k}(n+1)^{1/k} - \sum \alpha_i \le x_n \le \Big(k!\prod \alpha_i\Big)^{1/k}(n+1)^{1/k}$$

and it follows that

$$x_n \sim \big(nk!\prod_{i=1}^{k} \alpha_i\big)^{1/k}$$

as $n \to \infty$, whence $x_{tn}/x_n \to t^{1/k}$ as $n \to \infty$ and $X_{\alpha_1,\ldots,\alpha_k}$ is a scale.

The function $\chi(X_{\alpha_1,\ldots,\alpha_k}, t)$ may be estimated by comparison with suitable integrals. Fixing n, set $B = x_n, A = x_{tn}$. The function

$$\log\big(\frac{A - \sum \alpha_i u_i}{B - \sum \alpha_i u_i}\big)$$

is increasing in each variable u_i. Therefore its value at a point $(u_1, \ldots, u_k)$ exceeds the integral of the function over the cube $\{(\xi_1, \ldots, \xi_k) : u_i - 1 < \xi_i \le u_i, i = 1, \ldots, k\}$.

Thus if $0 < \delta < 1$ then, once n is sufficiently large (depending on δ),

$$\sum_{j=0}^{n-1} \log\left(\frac{x_{tn} - x_j}{x_n - x_j}\right) \ge \int_{L_{\delta B}} \log\left(\frac{x_{tn} - \sum \alpha_i u_i}{x_n - \sum \alpha_i u_i}\right) du$$

and so, for any such δ,

$$\chi(X, t) \ge \liminf_{n\to\infty} \frac{1}{n}\int_{L_{\delta B}} \log\left(\frac{A - \sum \alpha_i u_i}{B - \sum \alpha_i u_i}\right) du.$$

For n sufficiently large we will also have $A = x_{tn} \ge t^{1/k}B\delta$. Further, $\mathrm{vol}(L_B) \sim n$ as n or B go to infinity. Therefore

$$\chi(X, t) \ge \liminf_{B\to\infty} \frac{1}{\mathrm{vol}(L_B)}\int_{L_{\delta B}} \log\left(\frac{t^{1/k}B\delta - \sum \alpha_i u_i}{B - \sum \alpha_i u_i}\right) du.$$

Let $C = \delta B$. Then taking the lim inf as $C \to \infty$, changing the variable of integration using $w = \sum \alpha_i u_i$ and $v^{1/k} = w/B$,

$$\chi(X, t) \ge \liminf_{C\to\infty} \frac{1}{\mathrm{vol}(L_{C/\delta})}\int_{L_C} \log\left(\frac{t^{1/k}C - \sum \alpha_i u_i}{C/\delta - \sum \alpha_i u_i}\right) du$$

$$= \liminf_{C\to\infty} \frac{1}{\mathrm{vol}(L_{C/\delta})}\int_0^C \log\left(\frac{t^{1/k}C - w}{C/\delta - w}\right) \frac{w^{k-1}\,dw}{(k-1)!\prod \alpha_i} = \liminf_{C\to\infty} \frac{\mathrm{vol}(L_C)}{\mathrm{vol}(L_{C/\delta})}\int_0^1 \log\left(\frac{t^{1/k} - v^{1/k}}{1/\delta - v^{1/k}}\right) dv$$

$$= \delta^k \int_0^1 \log\left(\frac{t^{1/k} - v^{1/k}}{1/\delta - v^{1/k}}\right) dv.$$

We may now let $\delta \to 1$ by dominated convergence. □

One would expect $\chi(X_{\alpha_1,\ldots,\alpha_k}, t) = \chi_k(t)$, which would seem to require some weak control on x_n, x_{n-1} being extremely close together, giving a large contribution to the sum defining $\chi(X_{\alpha_1,\ldots,\alpha_k}, t)$. The lower bound obtained above suffices for our purposes.

Basic integer valued polynomials

Let $\phi_n(z), n \in \mathbb{N}$ denote the basic integer valued polynomials:

$$\phi_0(z) = 1, \quad \phi_1(z) = z, \ldots, \phi_n(z) = \frac{z(z-1)\ldots(z-n+1)}{n!}, \ldots.$$

2.7. Proposition. *Let $n \in \mathbb{N}, C \ge 1, E \ge n$. Then $M(\phi_n, CE) \le e^{2C}E^n$.*

Proof. Since $CE \ge n$, $M(\phi_n, CE) \le (2EC)^n/n! \le e^{2C}E^n$. □

A consequence of Jensen's formula

2.8. Proposition. *Let $f(z)$ be analytic for $|z| < R$ and suppose $f(0) = 1$. Let $x_1, \dots x_n$ be a subset of the zeros of f (allowing multiplicity, i.e. x_i may be repeated so long as $f(z)/\prod(z - x_i)$ remains entire). Let $r > \max\{|x_i|\}$. Then*

$$\frac{r^n}{|x_1|\,|x_2| \dots |x_n|} \le M(f, r).$$

Proof. Let $r_1, r_2, \dots$ be the moduli of the zeros of $f(z)$, arranged in non-decreasing order and taken with multiplicity. Let m be the largest index for which $r_m < r$; thus r_{m+1}, if it exists, satisfies $r \le r_{m+1}$. It follows from Jensen's formula (see for e.g. [8, §3.61]) that

$$\frac{r^m}{|r_1|\,|r_2| \dots |r_m|} \le M(f, r).$$

Now $|x_1|, |x_2|, \dots, |x_n|$ occur, with multiplicity, among $r_1, r_2, \dots r_m$. Therefore

$$\frac{r^n}{|x_1|\,|x_2| \dots |x_n|} \le \frac{r^m}{|r_1|\,|r_2| \dots |r_m|}. \ \square$$

2.9. Proposition. *Let $X = X_{\alpha_1, \dots, \alpha_k}$ where α_i are positive and linearly independent over $\mathbb{Q}$. Let f be an entire function, vanishing on X, with*

$$\limsup_{n\to\infty} \frac{\log M(f, x_n)}{n} < \frac{1}{k}.$$

Then f vanishes identically.

Proof. In view of 2.8 it suffices to show that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{x_n^n}{x_1 x_2 \dots x_n} = \frac{1}{k},$$

for if f is not identically zero it may be divided by a suitable finite power of z and a suitable constant to meet the hypotheses of 2.8 yielding a contradiction.

According to the proof of 2.6 we have, for some suitable constants a, b, c,

$$c(n + a)^{1/k} - b \le x_n \le c(n + a)^{1/k} + b.$$

It is elementary to establish the above limit under these conditions, noting immediately that replacing x_n by x_n/c we may assume $c = 1$. $\square$

Siegel's Lemma

2.10. Lemma. ([1, Lemma 2.9.1]) *Let $a_{ij} \in \mathbb{Z}$ for $i = 1, \dots, M, j = 1, \dots, N$, not all zero. Suppose $|a_{ij}| \le B$, and $N > M$. Then the homogeneous linear system*

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{iN}x_N = 0, \quad i = 1, \dots, M$$

has a solution $x_1, x_2, \dots, x_N$ in integers, not all 0, with

$$\max |x_j| \le (NB)^{M/(N-M)}. \ \square$$

2.11. Corollary. *Let $Y = \{y_1, y_1, \dots, y_M\}$ be a set of distinct complex numbers. Let $\psi_1, \psi_2, \dots, \psi_N$, where $N > M$, be entire functions (not necessarily distinct!) with $\psi_j(y_i) \in \mathbb{Z}$ and $|\psi_j(y_i)| \le B$, where $B \ge 1$. There exist integers $t_1, \dots, t_M$, not all zero, with $|t_j| \le (NB)^{M/(N-M)}$ such that the function*

$$h = \sum_{j=1}^{M} t_j \psi_j$$

vanishes at $y_1, \dots, y_N$.

Proof. We require a non-trivial solution in integers to the homogeneous system of equations

$$\sum_{j=1}^{M} t_j \psi_j(y_i) = 0, \quad i = 1, \ldots, M.$$

If not all $\psi_j(y_i) = 0$, a solution satisfying the required bound is afforded by Lemma 2.10. If all $\psi_j(y_i) = 0$ we can again find non-trivial solutions since $B \geq 1$. □

3. Proof of Theorem 1.3

3.1. Proposition. *Let $X = \{x_0, x_1, \ldots\}$ be a scale, $t \geq 2$ an integer and $\epsilon > 0$. Then*

$$\lim_{n\to\infty} \frac{n^\epsilon \log Q_X(n, tn)}{n} = -\infty.$$

Proof. The conclusion follows directly from the estimate

$$\frac{\log Q_X(n, tn)}{n} \leq -\log\left(\frac{x_{tn}}{x_n}\right).$$

and the hypothesis that X is a scale. □

Proof of Theorem 1.3. Choose $C \geq 1, w_j, j = 1, \ldots, k$, such that, for each j and all n,

$$M(f_j, x_n) \leq C \exp\left(w_j n^{\sigma_j}\right).$$

Set

$$\delta = \frac{1}{2k+1}\left(\sum_{j=1}^{k}(1-\sigma_j) - 1\right).$$

Let n be a positive integer, and set

$$I_n = \{i = (i_1, \ldots, i_k) \in \mathbb{N}^k : 0 \leq i_j \leq n^{(1-\sigma_j-2\delta)}\}$$

and for $i \in I_n$ set

$$\psi_i(z) = f_1^{i_1}(z) \ldots f_k^{i_k}(z).$$

Then, throwing away elements of I_n if necessary,

$$n^{1+\delta} \leq \#I_n \leq (2n)^{1+\delta},$$

and, for $i \in I_n$ and any $m \in \mathbb{N}$ with $m \geq n$ we have, putting $B = \log C + \sum w_i$,

$$M(\psi_i, x_m) \leq \exp(Bm^{1-2\delta}).$$

Apply Siegel's Lemma (Lemma 2.11) to build a non-trivial integral linear combination $h(z) = h_n(z)$ of the functions $\psi_i, i \in I_n$ that vanishes at $x_0, \ldots, x_{n-1}$ using integer coefficients of absolute value not exceeding

$$\left((2n)^{1+\delta}\exp(Bn^{1-2\delta})\right)^{n/(n^{1+\delta}-n)} \leq \exp(B'n^{1-\delta})$$

for suitable B' and all sufficiently large n. Thus, for suitable B'' and any $m \geq n$,

$$M(h, x_m) \leq \exp(B''m^{1-\delta}).$$

Suppose that h vanishes at $x_0, \ldots, x_{m-1}, m \geq n$. Setting $r = x_{2m}$ and applying the estimate of 2.1 shows that, upon taking logs and dividing by $m^{1-\epsilon}$ where $0 < \epsilon < \delta$,

$$\frac{\log|h(x_m)|}{m^{1-\epsilon}} \leq \frac{\log Q_X(m, 2m)}{m^{1-\epsilon}} + \frac{\log R_X(m, 2m)}{m^{1-\epsilon}} + \frac{\log M(h, x_{2m})}{m^{1-\epsilon}}.$$

The second (by 2.4) and third terms on the right hand side $\to 0$, while the first term $\to -\infty$ by 3.1, whence

$$\frac{\log|h(x_m)|}{m^{1-\delta}} \to -\infty$$

for $m \geq n \to \infty$. Thus $\log|h(x_m)| < 0$ for all $m \geq n$ once n is sufficiently large, so that, h vanishes identically on X. □

4. Proof of Theorem 1.4 and further results

We prove a generalization of 1.4 allowing several faster growing entire function g_i. We also observe that we get a bound on the degree in g_i of the polynomial h giving algebraic dependence on the scale X. This is crucial in deducing the algebraic dependence of the functions in 1.1, so we carry the degree bound through our proof of 1.4, although this was not part of the statement of 1.4, and then deduce 1.1 from 1.4. For a real number a we denote by $[a]$ the integer part of a, so that $a-1<[a]\le a$.

4.1. Theorem. *Let X be a scale. Let $f_1, f_2, \dots, f_k$ be entire functions that are integer valued on X, and that satisfy $\omega_i = \omega_X(f_i, \sigma_i) < \infty$ where $0 < \sigma_i < 1$ and $\sum_i (1-\sigma_i) = 1$. Suppose that $g_1, \dots, g_q$ are entire functions, integer valued on X, and $\lambda_1, \dots, \lambda_q$ are non-negative real numbers with*

$$\limsup_{n\to\infty} \frac{M(g_i, x_n)}{n} \le \lambda_i$$

for $i = 1, \dots, q$. Put $\lambda = (\lambda_1, \dots, \lambda_q)$ and, for $j = (j_1, \dots, j_q) \in \mathbb{N}^q$, put $j \cdot \lambda = j_1\lambda_1 + \dots + j_q\lambda_q$. Suppose that $t \ge 2$ is an integer, $s > 1$ and $A > 0$ are such that

$$A < \frac{\chi(X,t)}{t + (s-1)^{-1}}, \quad \textit{and} \quad \sum_{j\in J} (A - j\cdot\lambda)^k > s\, k!\, \omega_1 \dots \omega_k,$$

where $J = \{\, j \in \mathbb{N}^q : j \cdot \lambda \le A\}$. Then $f_1, \dots, f_k, g_1, \dots, g_q$ are algebraically dependent over $\mathbb{Z}$ on X.

Moreover, there is a non-zero $h \in \mathbb{Z}[t_1, \dots, t_k, s_1, \dots, s_q]$ whose degree in each s_i is at most $[A/\lambda_i]$ such that $h(f_1, \dots, f_k, g_1, \dots, g_q)$ vanishes identically on X.

Proof. Under the hypotheses it is possible to choose positive real numbers $w_i, i = 1, \dots, k, \ell_i, i = 1, \dots, q, C, B$ with the following properties:

$$\omega_i \le w_i, \text{ and } M(f_i, x_n) \le C \exp(w_i n^{\sigma_i})$$

for all i and n,

$$\lambda_i \le \ell_i, \text{ and } M(g_i, x_n) \le C \exp(\ell_i n)$$

for all i and n,

$$A \le B < \frac{\chi(X,t)}{t+(s-1)^{-1}}$$

such that

$$[B/\ell_i] \le [A/\lambda_i]$$

for all i, and, finally, setting $W = \prod_{i=1}^k w_i$, $\ell = (\ell_1, \dots, \ell_q)$,

$$\sum_{j\in K} (B - j\cdot\ell)^k > W k!$$

where

$$K = \{j \in \mathbb{N}^q : j\cdot\ell \le B\}.$$

Let $n \in \mathbb{N}$ be so large that $\exp(w_j n^{\sigma_j}) \ge n$ for each j (for applicability of 2.7). Set

$$I_n = \{(i,j) = (i_1, \dots, i_k, j_1, \dots, j_q) \in \mathbb{N}^{k+q} : \sum_{a=1}^k i_a w_a n^{\sigma_a - 1} + j\cdot\ell \le B\},$$

and for $(i,j) \in I_n$ put

$$\psi_{i,j} = \phi_{i_1}(f_1(z)) \dots \phi_{i_k}(f_k(z))\, g_1(z)^{j_1} \dots g_q(z)^{j_q}.$$

The functions $\psi_{i,j}$ take integer values on X.

We have

$$\#I_n = \sum_{j\in K} \#L_j \cap \mathbb{N}^k$$

where

$$L_j = \{(u_1,\ldots,u_k):\ 0\le u_i,\ i=1,\ldots,k, \sum_{i=1}^{k} u_i\, \frac{w_i n^{\sigma_i-1}}{B-j\cdot\lambda} \le 1\}.$$

By Proposition 2.5

$$\#L_j \cap \mathbb{N}^k \ge \frac{(B-j\cdot\lambda)^k\, n}{W\, k!}$$

and so, by the assumptions, $s\,n \le \#I_n$. By throwing away some elements of I_n if needed it may be assumed that, for each n, $sn \le \#I_n \le sn+1$ (we cannot insist on $sn = \#I_n$ as s is only assumed to be real > 1, though later in proving 1.4 and 1.1 we will take $s = 2$).

Apply Siegel's Lemma (2.11) to construct a non-trivial integral linear combination $h = h_n$ of the functions $\psi_{i,j}, (i,j) \in I_n$ vanishing at $x_0,\ldots,x_{n-1}$. Suppose $(i,j)\in I_n$ and $m\in\mathbb{N}$. Then

$$M(\psi_{i,j},x_m) \le \prod_{a=1}^{k} M(\phi_{i_a}, M(f_a,x_m)) \prod_{b=1}^{q} M(g_b,x_m)^{j_b} \le e^{2kC} C^{j_1+\ldots+j_q} \exp\Big(\sum_a w_a i_a m^{\sigma_a} + \sum_b u_b j_b m\Big).$$

If $m\le n$ we have

$$\sum_a w_a i_a m^{\sigma_a} + \sum_b u_b j_b m \le \sum_a w_a i_a n^{\sigma_a} + \sum_b u_b j_b n \le \Big(\sum_a w_a i_a n^{\sigma_a-1} + \sum_b u_b j_b\Big) n \le Bn$$

while if $m>n$, as $\sigma_a<1$,

$$\Big(\sum_a w_a i_a m^{\sigma_a-1} + \sum_b u_b j_b\Big) m \le \Big(\sum_a w_a i_a n^{\sigma_a-1} + \sum_b u_b j_b\Big) m \le Bm.$$

Since $j_i < B/\ell_i$, we have $j_1+\ldots+j_q \le Q$ where $Q = B\sum 1/\ell_i$, and thus for all n,m we have

$$M(\psi_{i,j},x_m) \le e^{2kC} C^Q \exp(B\max(m,n)).$$

Accordingly the function h may be constructed using integers of absolute value at most

$$((sn+1)e^{2kC}C^Q\exp(Bn))^{1/(s-1)},$$

and for $m\ge n$ we have

$$M(h,x_m) \le (sn+1)((sn+1)e^{2kC}C^Q\exp(Bn))^{1/(s-1)} e^{2kC}C^Q\exp(B\max(m,n)).$$

Now suppose that h vanishes at $x_0,\ldots,x_{m-1}$ where $m\ge n$. Then, by 2.1,

$$|h(x_m)| \le (sn+1) Q_X(m,tm)\ R_X(m,tm)\ ((sn+1)e^{2kC}C^Q\exp(Bn))^{1/(s-1)}\ e^{2kC}C^Q\exp(Btm).$$

But then for n sufficiently large and $m\ge n$, by definition of $\chi(X,t)$ and Proposition 2.4,

$$\limsup_{n\to\infty} \frac{\log|h(x_m)|}{m} \le B((s-1)^{-1}+t) - \chi(X,t) < 0.$$

Thus h also vanishes at x_m and hence h vanishes at x_m for all m.

The construction shows that the degree of h in g_j is at most $[B/\ell_j] \le [A/\lambda_j]$. □

4.2. Proof of 1.4. Apply 4.1 with $q=1$, $g=g_1$ and $\lambda=\lambda_1$. Since the function $(A-j\lambda)^k$ is positive and decreasing as a function of λ, $J=\{j: 0\le j\le A/\lambda\}$ and

$$\sum_{j\in J}(A-j\lambda)^k > \int_0^{A/\lambda}(A-x\lambda)^k dx = \frac{1}{\lambda}\int_0^A y^k dy.$$

Thus $f_1,\ldots,f_k,g$ are algebraically dependent over $\mathbb{Z}$ on X provided

$$A < \frac{\chi(X,t)}{t+(s-1)^{-1}} \quad\text{and}\quad \lambda \le \frac{A^{k+1}}{s\,(k+1)!\,\omega_1\ldots\omega_k}.$$

The statement of 1.4 follows upon taking $s=t=2$ and $A=T/3$. We note further that the constructed polynomial h giving algebraic dependence has degree at most $[A/\lambda]$ in g. □

4.3. Proof of 1.1. By Proposition 2.6 we have $\chi(X,2) \geq \chi_2(2) = \int_0^1 \log((2^{1/2} - v^{1/2})/(1-v^{1/2})dv = 1 - \sqrt{2} + \log 2 - \log(\sqrt{2}-1)$. A numerical computation gives $\chi_2(2) = 1.16031\ldots$. Apply 1.4 with the hypotheses of 1.1 and $T = 1.16$, noting that

$$\frac{1}{2(k+1)!}\left(\frac{T}{3}\right)^3 \frac{1}{\omega_X(f_1,1/2)\omega_X(f_2,1/2)} \geq \frac{1}{24ab\alpha\beta}\left(\frac{1.16}{3}\right)^3 \geq \frac{1}{416\,ab\alpha\beta}.$$

Thus 1.4 gives dependence of the functions on X provided

$$\limsup_{n\to\infty} \frac{\log M(g,x_n)}{n} \leq \frac{1}{416\,ab\alpha\beta}.$$

The function h constructed in 4.1 is a polynomial in f_1, f_2, g of degree $\leq 1.16/(3\lambda)$ in g. Therefore

$$\limsup_{n\to\infty} \log \frac{M(h,x_n)}{n} \leq \frac{1.16}{3} < 1/2$$

and Jensen's formula (2.9) shows that h vanishes identically given that it vanishes on $X_{\alpha,\beta}$. Thus the functions are algebraically dependent over $\mathbb{Z}$. Since $x_n \sim \sqrt{2n\alpha\beta}$, putting $r = x_n$, $n = r^2/(2\alpha\beta)$ gives the statement of 1.1. □

Theorem 4.1 may also be applied when there are no functions g_i (one still has the zero vector in the set J), in which case it asserts that the growth rates of the functions f_i cannot be too small relative to the sequence X. This gives some quantitative improvement in Theorem 1.3.

4.4. Corollary of 4.1. *Let X be a scale. Let $f_1, \ldots, f_k$ be entire functions that are integer valued on X, and that satisfy $\omega_i = \omega_X(f_i, \sigma_i) < \infty$ where $\sum_i (1-\sigma_i) = 1$. Suppose that $t \geq 2$ is an integer and $s > 1$ are such that*

$$sk!\omega_1 \ldots \omega_k < \left(\frac{\chi(X,t)}{t + (s-1)^{-1}}\right)^k.$$

Then $f_1, \ldots, f_k$ are algebraically dependent over $\mathbb{Z}$ on X. □

Such an application yields another variant strengthening of the six exponentials theorem. While Theorem 1.1 views the six exponentials theorem as a result about *three* functions that are integer valued on a semigroup of points generated by *two* real numbers, it may alternatively be viewed as a theorem about *two* functions that are integer valued on a semigroup of points $X = X_{\alpha,\beta,\gamma}$ generated by *three* real numbers.

Let $\alpha = \log a, \beta = \log b, \gamma = \log c$, where a, b, c are multiplicatively independent positive integers, with the principal (real) value of the logarithm. Then $\exp(z)$ takes integer values on X. We have $x_n \sim (6n\alpha\beta\gamma)^{1/3}$, so that

$$\omega_X(e^z, 1/3) = (6\alpha\beta\gamma)^{1/3}.$$

We have $\chi(X,t) \geq \chi_3(t)$ from 2.6 so we get the following instance of 4.4.

4.5. Proposition. *With $X = X_{\alpha,\beta,\gamma}$ as above, if f is an entire function, integer-valued on $X_{\alpha,\beta,\gamma}$ with*

$$\omega_X(f, 2/3) = \limsup_{n\to\infty} \frac{\log M(f,x_n)}{n^{2/3}} < \frac{\chi_3(2)^2}{36\,(6\alpha\beta\gamma)^{1/3}},$$

or, equivalently,

$$\limsup_{r\to\infty} \frac{\log M(f,r)}{r^2} < \frac{\chi_3(2)^2}{216\alpha\beta\gamma},$$

then f, e^z are algebraically dependent over $\mathbb{Z}$ on X. □

More generally, suppose $\alpha_1 = \log a_1, \ldots, \alpha_\ell = \log a_\ell$ where $a_1, \ldots, a_\ell$ are multiplicatively independent positive integers, and f is entire and integer valued on $X = X_{\alpha_1,\ldots,\alpha_\ell}$. Then

$$\omega_X(e^z, 1/\ell) = \Big(\ell! \prod \alpha_i\Big)^{1/\ell},$$

and one finds that f, e^z must be algebraically dependent over $\mathbb{Z}$ on X if

$$\omega_X(f, (\ell-1)/\ell) < \frac{\chi_\ell(2)^2}{36(\ell! \prod \alpha_i)^{1/\ell}}.$$

or, equivalently,

$$\limsup_{r\to\infty} \frac{\log M(f,r)}{r^{\ell-1}} < \frac{\chi_\ell(2)^2}{36\ell!\alpha_1 \ldots \alpha_\ell}.$$

If f is entire and takes integer values at all points of $X = \{\log n \in \mathbb{R}, n = 1, 2, 3\}$ then Theorem 1.3 of [4] applies, and if

$$\limsup_{r\to\infty} \frac{\log M(f,r)}{e^r} < 0.005,$$

then f, e^z are algebraically dependent over $\mathbb{Z}$ on X.

4.6. Remark. In defining $\chi(X,t)$ we assumed $t \in \mathbb{N}$ in 1.2. To optimize the numerical constants obtained one would allow $t \in \mathbb{Q}$ by restricting the calculation of $\chi(X,t)$ to suitable subsequences. Since the contants obtained would presumably not be optimal, the present choice was preferred for simplicity.

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