Functional transcendence via o-minimality

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These notes are an edited version of notes prepared for a series of five lectures delivered during the LMS-EPSRC Short Course on “O-minimality and Diophantine Geometry”, held at the University of Manchester, 8-12 July 2013. Not everything here was covered in the lectures, in particular many details were skipped.

Synopsis. We describe Schanuel's conjecture, the differential version (“Ax-Schanuel”), and their modular analogues. We sketch proofs of the “Ax-Lindemann” parts in both settings using o-minimality, and describe connections with the Zilber-Pink conjecture.

1. Algebraic independence

1.1. Definition. Let L be a field and $K \subset L$ a subfield.

1. An element $x \in L$ is called *algebraic over* K if there exists a polynomial $p \in K[X]$, non-zero, such that $p(x) = 0$.

2. Elements $x_1, \ldots, x_n \in L$ are called *algebraically dependent over* K if there is a polynomial $p \in K[X_1, \ldots, X_n]$, non-zero, such that $p(x_1, \ldots, x_n) = 0$. Otherwise, they are called *algebraically independent over* K.

1.2. Proposition. *Let L be a field, $K \subset L$ a subfield, and $x \in L$. The following assertions are equivalent:*

1. x is algebraic over K.

2. There exists a finite dimensional K-vector space $V \subset L$ such that $xV \subset V$.

Proof. Exercise. □

1.3. Corollary. *The collection of $x \in L$ which are algebraic over K form a subfield of L (containing K).* **Exercise.** □

An element that is not algebraic (over K) will be called *transcendental (over K)*. If K is not specified, we take it to be $\mathbb{Q}$; if L also not specified we take it to be $\mathbb{C}$. We denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers in $\mathbb{C}$, i.e. the elements of $\mathbb{C}$ that are algebraic over $\mathbb{Q}$.

1.4. Definition. Let L be a field and $K \subset L$ a subfield.

1. A *transcendence basis* for L over K is a maximal algebraically independent (over K) subset.

[Transcendence bases exists; if T is a transcendence basis for L over K then every element of L is algebraic over $K(T)$; any two transcendence bases of L over K have the same cardinality. (**Exercises**: Like vector space dimension. Hint: use the

1.5. Steinitz Exchange Principle. Let $K \subset L$ be fields and $u, v, w_1, \ldots, w_k, y \in L$. Say that v *depends on* u *over* $w_1, \ldots, w_k$ if v is algebraic over $K(w, u)$ but not over $K(w)$ Suppose that v depends on u over $w_1, \ldots, w_k$. Then u depends on v over $w_1, \ldots, w_k$ and if y is algebraic over $K(w, u)$ then it is algebraic over $K(w, v)$.)]

2. The *transcendence degree* of L over K is the cardinality of a transcendence basis; it is denoted $\text{tr.d.}_K L$ or $\text{tr.d.}(L/K)$. If S is a set we will also write $\text{tr.d.}_K S$ for $\text{tr.d.}_K K(S)$, and also $\text{tr.d.}(S)$ for $\text{tr.d.}_{\mathbb{Q}} S$.

1.6. Examples. $\text{tr.d.}(\overline{\mathbb{Q}}) = 0$; $\text{tr.d.}(\mathbb{C}) = 2^{\aleph_0}$; $\text{tr.d.}_{\mathbb{C}}\big(\mathbb{C}(X_1, \ldots, X_n)\big) = n$, for independent indeterminates X_i; $\text{tr.d.}_{\mathbb{Q}}(K) = \text{tr.d.}_{\overline{\mathbb{Q}}}(K)$.

2. Transcendental numbers

References for this section are Baker [5], Lang [31], Nesterenko [43], Waldschmidt [65]. The existence of transcendental numbers (in $\mathbb{C}$ over $\mathbb{Q}$) may be established by a counting argument: the set of algebraic numbers is countable, but the set of complex numbers is uncountable. Before Cantor's theory of transfinite sets was formulated, Liouville showed in 1844 that certain fast-converging series are transcendental, because irrational algebraic numbers do not admit "very good" approximations by rationals; e.g. $\sum 10^{-n!}$ ([5]).

But the first "naturally occurring" number to be shown transcendental was e, the base of the natural logarithm, by Hermite in 1873. The transcendence of π was proved a little later by Lindemann in 1882. Lindemann stated the following generalisation of his results, a full proof being given by Weierstrass (1885).

2.1. Theorem. (Lindemann/Lindemann-Weierstrass) *Let $x_1, \ldots, x_n$ be algebraic numbers which are linearly independent over $\mathbb{Q}$. Then*

$$e^{x_1}, \ldots, e^{x_n}$$

are algebraically independent over $\mathbb{Q}$. $\square$

The transcendence of e follows ($n = 1, x_1 = 1$); the transcendence of any non-zero logarithm of an algebraic number also follows, in particular the transcendence of (πi and hence of) π.

Observe that the result is sharp: if $x_1, \ldots, x_n$ are linearly dependent over $\mathbb{Q}$ then $e^{x_1}, \ldots, e^{x_n}$ are algebraically dependent over $\mathbb{Q}$. (Just exponentiate the linear relation and use the functional property of exp.)

Euler had expressed the view that a ratio of logarithms, if irrational, must be transcendental, e.g. $\log 3/\log 2$. This became Hilbert's 7th problem and was later solved independently in the 30's by Gelfond and Schneider.

The logarithm function is multivalued. One can define principal values which are single valued, but in general for a nonzero complex number a we will use $\log a$ to denote any complex number c with $e^c = a$.

The non-zero elements of a field K will be denoted $K^\times$.

2.2. Theorem. (Gelfond-Schneider) *Let* $a, b \in \overline{\mathbb{Q}}^\times$. *Then* $\log b/\log a$ *is either rational or transcendental.* □

Otherwise put: if $a \in \overline{\mathbb{Q}}$ is algebraic and not equal to 0 or 1 and $r \in \overline{\mathbb{Q}} - \mathbb{Q}$ then $b = a^r$ is transcendental (else $r = \log b/\log a$ contradicts the theorem). E.g. $2^{\sqrt{2}}$ is transcendental. Also $e^\pi = (-1)^{-i}$. This theorem was generalised by Baker.

2.3. Theorem. (Baker, 1966) *Suppose* $a_1, \ldots, a_n \in \overline{\mathbb{Q}}^\times$ *and* $\log a_1, \ldots, \log a_n$ *are linearly independent over* $\mathbb{Q}$. *Then*

$$1, \log a_1, \ldots, \log a_n$$

are linearly independent over $\overline{\mathbb{Q}}$. □

The methods of proof of these theorems will not be relevant to our discussion of the *functional* versions, and we won't discuss them.

The functional relation $\exp(x + y) = \exp(x).\exp(y)$ forces e.g. $\log 6, \log 2, \log 3$ to be algebraically dependent. On the general principle that "numbers defined using exponentiation should be as algebraically independent as permitted by the functional relation" one expects things like:

2.4. Conjecture on algebraic independence of logarithms. If $a_1, \ldots, a_n \in \overline{\mathbb{Q}}^\times$ are *multiplicatively independent* (i.e. there are no nontrivial multiplicative relations $\prod a_i^{k_i} = 1, k_i \in \mathbb{Z}$) then any determination of the $\log a_i$ are algebraically independent. However, it is not known that $\mathrm{tr.d.}(\log \overline{\mathbb{Q}}) > 1$. Baker's Theorem (above) is the strongest result known here.

2.5. Gelfond's conjecture. Suppose $a \in \overline{\mathbb{Q}} - \{0, 1\}$ and b is algebraic of degree d. Then the numbers $a^b, a^{b^2}, \ldots, a^{b^{d-1}}$ are algebraically independent; one knows this for $d = 2, 3$ (Gelfond); in general it is known that $\mathrm{tr.d.}(a^b, a^{b^2}, \ldots, a^{b^{d-1}}) \geq \left[\frac{d+1}{2}\right]$ (Brownawell, Waldschmidt, Philippon, Nestrenko, Diaz; see [43]).

2.6. Various conjectures. e^e is transcendental; $e, e^e, \ldots$ are algebraically independent; $e + \pi$ is transcendental and e and π are algebraically independent. A result of Brownawell/ Waldschmidt (1974/3) implies (see Baker): *Either* e^e *or* e^{e^2} *is transcendental.* A result of Nesterenko (1996) implies: π *and* e^π *are algebraically independent* (see [43]).

2.7. The "Four exponentials" conjecture. Suppose $x_1, x_2 \in \mathbb{C}$ are l.i./$\mathbb{Q}$, and that $y_1, y_2 \in \mathbb{C}$ are l.i./$\mathbb{Q}$. Then at least one of the four exponentials $\exp(x_i y_j)$ is transcendental. E.g. suppose t is irrational (so $1, t$ are l.i./$\mathbb{Q}$). Since $\log 2, \log 3$ are l.i./$\mathbb{Q}$, at least one of $2, 3, 2^t, 3^t$ should be transcendental. This is not known: indeed it is not known that if $t \in \mathbb{R}$ and $2^t, 3^t$ are both integers then $t \in \mathbb{N}$. If one has three x_i then the conclusion is known ("six exponentials"; Siegel, Lang, Ramachandra; see [31]).

3. Schanuel's conjecture

Schanuel, in the 1960's, came up with a conjecture that implies all the theorems and conjectures above, succinctly summarising all the expected transcendence properties of the exponential function. It is stated in Lang's book [31, p30].

3.1. Schanuel's conjecture. (SC) Let $x_1, \ldots, x_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\text{tr.d.}_{\mathbb{Q}}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n.$$

For example, when all the x_i are algebraic we recover Lindemann's theorem, and when all the e^{x_i} are algebraic we recover the conjecture on algebraic independence of logarithms.

Exercise. Deduce the other statements above from SC (the ones involving π and e take a bit of work).

References: Kirby [26], Lang [31], Waldschmidt [65, 66], Zilber [71, 72] and especially [73].

4. Differential fields

According to Ax [3], Schanuel made the same conjecture for power series and (more generally) for differential fields (i.e. fields with derivations). As a reference, see Lang [32].

4.1. Definition. A *differential field* is a pair (K, D) where K is a field and $D : K \to K$ is a *derivation*: an additive function satisfying the Leibniz rule: $D(xy) = xDy + yDx$.

In a differential field, the kernel of D is a field (**Exercise**) called the *field of constants*. It always contains the prime field. More generally we will deal with fields with several (commuting) derivations. The derivations D of a field K form a vector space over K: $(zD)(x) = z(Dx)$. Let L be finitely generated over K, of transcendence degree r. Denote by $\mathcal{D}$ the vector space of derivations of L over K (i.e. trivial on K). We have a pairing $(\mathcal{D}, L) \to L$ given by

$$(D, x) \mapsto Dx.$$

Thus each $x \in L$ gives an element dx of the dual space of $\mathcal{D}$, and we have $d(yz) = ydz + zdy, d(y+z) = dy + dz$. These form a subspace of the dual space of $\mathcal{D}$ if we define ydz by $(D, ydz) = yDz$.

4.2. Proposition. (see Lang [32, VIII, 5.5]) *Let L be a separably generated and finitely generated extension of a field K, of transcendence degree r. Then the vector space $\mathcal{D}$ (over L) of derivations of L which are trivial on K has dimension r. Elements $t_1, \dots, t_r \in L$ form a separating transcendence basis of L over K iff $dt_1, \dots dt_r$ form a basis for the dual of $\mathcal{D}$ over L.* □

4.3. Examples. The paradigm examples are fields of functions, where the derivation is induced by differentiation.

1. In algebraic geometry one considers an (affine) algebraic set $V \subset \mathbb{C}^n$ defined as the locus of common zeros of some set of polynomials in $\mathbb{C}[X_1, \dots, X_n]$, and hence of the ideal I generated by them. The *coordinate ring*

$$\mathbb{C}[V] = \mathbb{C}[X_1, \dots, X_n]/I$$

is the ring of functions induced on V by $\mathbb{C}[X_1, \dots, X_n]$. This ring is a domain just if I is prime, and then V is called irreducible (over $\mathbb{C}$), or an *(affine) variety* (though this word is used with a lot of flexibility), and then it has a quotient field $\mathbb{C}(V)$, which is called an algebraic function field.

Such fields have derivations: on $\mathbb{C}(X)$ one has the derivative with respect to X, which extends non-trivially to $\mathbb{C}(V)$ if X is non-constant on V. If $\dim V = k$ it has k independent derivations D_i (over $\mathbb{C}$) given by extending the derivations corresponding to k independent coordinates X_i (as $\mathbb{C}(V)$ is finitely separably generated over $\mathbb{C}$). E.g. if $V : F(X,Y) = 0$ the derivation D with $DX = 1$ extends to $\mathbb{C}(V)$ with $DY = -F_X/F_Y$.

2. The same in an analytic context: let V be a complex analytic variety. Then the field K of meromorphic functions on V has $\dim V$ independent (over K, trivial on $\mathbb{C}$) derivations coming from differentiation with respect to suitably chosen coordinate functions (say $V \subset \mathbb{C}^N$).

In both examples, one can add exponentials of any finite number of elements to these fields, perhaps restricting to a neighbourhood of some point of V, in such a way that if $y = \exp(x)$ then $Dy = yDx$ for the derivations mentioned.

5. Ax-Schanuel

We follow Ax's terminology. We consider a tower of fields $\mathbb{Q} \subset C \subset K$ and a set of derivations $\mathcal{D} = \{D_1, \dots, D_m\}$ on K with $C = \bigcap_j \ker D_j$. By "rank" below we mean rank over K.

5.1. Definition. Elements $x_1, \dots, x_n \in K$ are called *linearly independent over $\mathbb{Q}$ modulo C*, which we write "l.i./$\mathbb{Q}$ mod C", if there is no nontrivial relation

$$\sum_{i=1}^{n} q_i x_i = c, \quad q_i \in \mathbb{Q}, c \in C$$

where nontrivial means not all q_i, c are zero.

5.2. Definition. Elements $y_1, \dots, y_n \in K^\times$ are called *multiplicatively independent modulo* C if there is no nontrivial relation

$$\prod_{i=1}^{n} y_i^{k_i} = c, \quad k_i \in \mathbb{Z}, c \in C$$

where nontrivial means that not all $k_i = 0$.

5.3. Theorem. (**"Ax-Schanuel"**; Ax [3], 1971) *Let* $x_i, y_i \in K^\times, i = 1, \dots, n$ *with*

(a) $D_j y_i = y_i D_j x_i$ *for all* j, i

(b) the x_i *are l.i. over* $\mathbb{Q}$ *modulo* C *[or (b′) the* y_i *are mult. indpt. over* C*]*

Then

$$\text{tr.d.}_C C(x_1, \dots, x_n, y_1, \dots, y_n) \geq n + \text{rank}(D_j x_i)_{i=1,\dots,n, j=1,\dots,m}. \ \square$$

The proof (like the setting) is differential algebra. See also Ax [4], Kirby [25], Brownawell-Kubota, Bertrand-Pillay [9] for generalisations, including to the semi-abelian setting.

We now consider this statement in a complex setting. We take

$$\pi : \mathbb{C}^n \to (\mathbb{C}^\times)^n$$

given by

$$\pi(z_1, \dots, z_n) = (\exp z_1, \dots, \exp z_n).$$

Let $A \subset U$ be a complex analytic subvariety of some open set $U \subset \mathbb{C}^n$, so that locally the coordinate functions $z_1, \dots, z_n$ and $\exp(z_1), \dots, \exp(z_n)$ are meromorphic on A, and we have derivations $\{D_j\}$ with $\text{rank}(D_j z_i) = \dim A$, the rank being over the field of meromorphic functions, and with $D_j e^{z_i} = e^{z_i} D z_j$ for all i, j. (I.e. we take the D_j to be differentiation with respect to some choice of $\dim A$ independent coordinates on A.

5.4. "Complex Ax-Schanuel" Conjecture. In the above setting, if the z_i are linearly independent over $\mathbb{Q}$ modulo $\mathbb{C}$, then

$$\text{tr.d.}_{\mathbb{C}} \mathbb{C}(z_1, \dots, z_n, \exp(z_1), \dots, \exp(z_n)) \geq n + \dim A.$$

This clearly implies a weaker "two-sorted" version where the transcendence degree of the z_i and $\exp(z_i)$ are computed separately: with the same setting and hypotheses,

$$\text{tr.d.}_{\mathbb{C}} \mathbb{C}(z_1, \dots, z_n) + \text{tr.d.}_{\mathbb{C}} \mathbb{C}(\exp(z_1), \dots, \exp(z_n)) \geq n + \dim A.$$

5.5. Definition. A subvariety $W \subset \mathbb{C}^n$ will be called *geodesic* if it is defined by (any number ℓ of) equations of the form

$$\sum_{i=1}^{n} q_{ij} z_i = c_j, \quad j = 1, \dots, \ell,$$

where $q_{ij} \in \mathbb{Q}, c_j \in \mathbb{C}$.

5.6. Definition. By a *component* we mean a complex-analytically irreducible component of $W \cap \pi^{-1}(V)$ where $W \subset \mathbb{C}^n$ and $V \subset (\mathbb{C}^\times)^n$ are algebraic subvarieties.

Let A be a component of $W \cap \pi^{-1}(V)$. We can consider the coordinate functions z_i and their exponentials as elements of the field of meromorphic functions (at least locally) on A, and we can endow this field with $\dim A$ derivations $\{D_j\}$ as above with $\mathrm{rank}(Dz_i) = \dim A$. Then (with Zcl denoting Zariski closure)

$$\dim W \geq \dim \mathrm{Zcl}(A) = \mathrm{tr.d.}_{\mathbb{C}}(z_i), \quad \dim V \geq \dim \mathrm{Zcl}(\exp(A)) = \mathrm{tr.d.}_{\mathbb{C}}(\exp z_i)$$

and the "two-sorted" Ax-Schanuel conclusion becomes

$$\dim W + \dim V \geq \dim X + \dim A$$

provided that the functions z_i are l.i. over $\mathbb{Q}$ mod $\mathbb{C}$.

This last condition is equivalent to A not being contained in a proper geodesic subvariety. Let us take U' to be the smallest geodesic subvariety of $\mathbb{C}^n$ containing A. Let $X' = \exp U'$, which is a coset of an algebraic subtorus of $(\mathbb{C}^\times)^n$, and put $W' = W \cap U'$, $V' = V \cap X'$. We can choose coordinates $z_i, i = 1, \ldots, \dim A$ which are l.i. over $\mathbb{Q}$ mod $\mathbb{C}$ and derivations as previously with $\mathrm{rank}(Dz_i) = \dim A$. We then get the following variant of Ax-Schanuel in this setting.

5.7. Formulation A. *Let U' be a geodesic subvariety of $\mathbb{C}^n$. Put $X' = \exp U'$ and let A be a component of $W \cap \pi^{-1}(V)$, where $W \subset U'$ and $V \subset X'$ are algebraic subvarieties. If A is not contained in any proper geodesic subvariety of U' then*

$$\dim A \leq \dim V + \dim W - \dim X'.$$

I.e. (and as observed still more generally by Ax [4]), the components of the intersection of W and $\pi^{-1}(V)$ never have "atypically large" dimension, except when A is contained in a proper geodesic subvariety. It is convenient to give an equivalent formulation.

5.8. Definition. Fix $V \subset (\mathbb{C}^\times)^n$.

1. A *component with respect to* V is a component of $W \cap \pi^{-1}(V)$ for some $W \subset \mathbb{C}^n$.
2. If A is a component we define its *defect* by $\delta(A) = \dim \mathrm{Zcl}(A) - \dim A$.
3. A component A with respect to V is called *optimal* for V if there is no strictly larger component B w.r.t. V with $\delta(B) \leq \delta(A)$.
4. A component A w.r.t. V is called *geodesic* if it is a component of $W \cap \pi^{-1}(V)$ for some geodesic subvariety W, with $W = \mathrm{Zcl}(A)$.

5.9. Formulation B. *Let $V \subset \mathbb{C}^n$. An optimal component for V is geodesic.*

We show that these two formulations are equivalent using only formal properties of weakly special subvarieties, so that the equivalence will hold in more general settings we will consider.

Proof that A implies B. We assume Formulation A and suppose that the component A of $W \cap \pi^{-1}(V)$ is optimal, where $W = \mathrm{Zcl}(A)$. Suppose that U' is the smallest geodesic subvariety containing A, and let $X' = \pi(U')$. Then $W \subset U'$. Let $V' = V \cap X'$. Then A is optimal for V' in U', otherwise it would fail to be optimal for V in $\mathbb{C}^n$. Since A is not contained in any proper geodesic subvariety of U' we must have

$$\dim A \leq \dim W + \dim V' - \dim X'.$$

Let B be the component of $\pi^{-1}(V')$ containing A. Then B is also not contained in any proper geodesic subvariety of U', so, by Formulation A,

$$\dim B \leq \dim V' + \dim \mathrm{Zcl}(B) - \dim X'.$$

But $\dim B = \dim V'$, whence $\dim \mathrm{Zcl}(B) = \dim X'$, and so $\mathrm{Zcl}(B) = X'$, and B is a geodesic component. Now

$$\delta(A) = \dim W - \dim A \geq \dim X' - \dim V' = \delta(B)$$

whence, by optimality, $A = B$. □

Proof that B implies A. We assume Formulation B. Let U' be a geodesic subvariety of $\mathbb{C}^n$, put $X' = \pi(U)$. Suppose $V \subset X', W \subset U'$ are algebraic subvarieties and A is a component of $W' \cap \pi^{-1}(V')$ not contained in any proper geodesic subvariety of U'. There is some optimal component B containing A, and B is geodesic, but since A is not contained in any proper geodesic, B must be a component of $\pi^{-1}(V')$ with $\mathrm{Zcl}(B) = U'$ and we have

$$\dim W - \dim A \geq \delta(A) \geq \delta(B) = \dim X' - \dim V$$

which rearranges to what we want. □

6. "Ax-Lindemann"

We retain the setting $\pi : \mathbb{C}^n \to (\mathbb{C}^\times)^n$ and terminology from the previous section. A component of defect zero with respect to $V \subset X$ is then just an algebraic subvariety $W \subset \pi^{-1}(V)$. We thus have by Formulation B:

6.1. Ax-Lindemann, Form 1. *A maximal algebraic subvariety $W \subset \exp^{-1}(V)$ is geodesic.*

Let us explicate Form 1. If $W \subset \pi^{-1}(V)$ then we may consider W to be a component w.r.t. V. If W is not contained in any proper geodesic subvariety we find

$$\dim W \leq \dim V + \dim W - \dim X$$

so that $\dim X \leq \dim V$, i.e. $\pi(W)$ is Zariski-dense in X. Let us consider then a subvariety $W \subset U$ with $z_1, \ldots, z_n$ denoting the elements of $\mathbb{C}(W)$ induced by the coordinate functions. We get the following.

6.2. Ax-Lindemann, Form 2. *If $z_1, \ldots, z_n \in \mathbb{C}(W)$ are l.i./$\mathbb{Q}$ mod $\mathbb{C}$ then the functions*

$$e^{z_1}, \ldots, e^{z_n}$$

are algebraically independent over $\mathbb{C}$.

In this form it should be clear that this is an analogue of Lindemann's theorem for algebraic functions (i.e. elements of the algebraic function field $\mathbb{C}(W)$), hence the neologism "Ax-Lindemann" to denote (retrospectively) this part of Ax-Schanuel.

However it is Form 1 that is important in the applications: for it essentially characterises the "algebraic part" of $\pi^{-1}(V)$. The algebraic part is defined in terms of real semi-algebraic subsets of $\pi^{-1}(V)$ (connected and of positive dimension). Because $\pi^{-1}(V)$ is complex analytic, it turns out that $\pi^{-1}(V)^{\text{lag}}$ is in fact a union of complex algebraic varieties. By "Ax-Lindemann" it is a union of geodesic subvarieties.

We give direct proofs of the equivalence of these two forms, which are formal (and therefore will hold in more general settings).

Proof that 1 implies 2. Suppose $e^{z_1}, \ldots, e^{z_n}$ as in Form 2 are not algebraically independent over $\mathbb{C}$. So $\exp(W) \subset V$ for some proper algebraic subvariety $V \subset (\mathbb{C}^*)^n$. By Form 1, there is a geodesic W' with $W \subset W' \subset \exp^{-1}(V)$. Since V is a proper subvariety, so is W' and so there is a non-trivial equation $\sum q_i z_i = c$ that holds on W. Hence the coordinate functions $z_1, \ldots, z_n$ are linearly dependent over $\mathbb{Q}$ modulo $\mathbb{C}$. □

Proof that 2 implies 1. We consider V as in the statement of Form 1, and $W \subset \exp^{-1}(V)$ maximal. Choose a maximal subset $z_i, i \in I \subset \{1, \ldots, n\}$ such that e^{z_i} are algebraically independent over $\mathbb{C}$. So all the other z_j are "geodesically dependent" on these, i.e. there is an equation $z_j = c_j + \sum_{i \in I} q_{ij}, c_j \in \mathbb{C}, q_{ij} \in \mathbb{Q}$. Since the $\exp z_i, i \in I$ are algebraically independent, we see that the geodesic subvariety T defined by the above equations for each $j \notin I$ is contained in $\pi^{-1}(V)$. By maximality $W = T$. □

7. The modular function

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ denote the complex upper half-plane. The *(elliptic) modular function* or *modular invariant* or *j-function* is a holomorphic function

$$j : \mathbb{H} \to \mathbb{C}$$

with remarkable arithmetic properties. We will describe some of its properties before briefly indicating the role this function plays in the arithmetic of elliptic curves.

In the following, various 2×2 real matrices with positive determinant act on $\mathbb{H}$ as *Mobius transformations* as follows:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{acts by} \quad z \mapsto gz = \frac{az + b}{cz + d}.$$

The condition $\det g > 0$ ensures $g(\mathbb{H}) = \mathbb{H}$.

Firstly, j is invariant under the action by $\mathrm{SL}_2(\mathbb{Z})$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ has a classical fundamental domain

$$F = \{z \in \mathbb{H} : |\mathrm{Re}(z)| \leq 1/2, |z| \geq 1\}.$$

More precisely this is the closure of a true fundamental domain F^* (i.e. where each $\mathrm{SL}_2(\mathbb{Z})$ orbit is represented just once), as the transformation $z \mapsto z+1$ identifies the vertical strips $\mathrm{Re}(z) = \pm 1/2$ and the transformation $z \mapsto -1/z$ identifies two segments of the circular boundary. A proof that F is a fundamental domain can be found in Serre [57]. Examining it yields a quantitative statement that will be important for us.

7.1. Proposition. ([48]) *Let $z \in \mathbb{H}$. The (unique) $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma z \in F^*$ has entries bounded by a polynomial (of degree ≤ 7) in* $\max(|z|, (\mathrm{Im}(z))^{-1})$. □

More generally, we consider the action by $\mathrm{GL}_2^+(\mathbb{Q})$, where the $^+$ denotes positive determinant. Scaling a matrix does not change its action, so we could reduce everything to actions by elements of $\mathrm{SL}_2(\mathbb{R})$. However this does not preserve rationality of the entries, so it is convenient to work with $\mathrm{GL}_2^+(\mathbb{Q})$.

For each $g \in \mathrm{GL}_2^+(\mathbb{Q})$ we may scale the matrix until its entries are in $\mathbb{Z}$ but relatively prime. The determinant of this matrix we denote $N = N(g)$.

For each $N \geq 1$ there is a *modular polynomial*

$$\Phi_N \in \mathbb{Z}[X, Y],$$

symmetric for $N \geq 2$ ($\Phi_1 = X - Y$), such that, if $N(g) = N$,

$$\Phi_N\big(j(z), j(gz)\big) = 0,$$

i.e. the two functions $j(z), j(gz)$ are algebraically dependent (over $\mathbb{Q}$). For example, $2z = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} z$, and $X = j(z), Y = j(2z)$ are related by $\Phi_2(X, Y) = 0$ where

$$\Phi_2 = -X^2Y^2 + X^3 + 1488(X^2Y + XY^2) + Y^3 - 162.10^3(X^2 + Y^2) + 40773375XY$$

$$+8748.10^3(X + Y) - 157464.10^9.$$

More details and examples can be found e.g. in Zagier [67], Diamond-Sherman [17].

Since j is invariant under $z \mapsto z + 1$ it has a Fourier expansion, known as the *q-expansion*

$$j(z) = q^{-1} + 744 + \sum_{m=1}^{\infty} c_m q^m, \quad q = e^{2\pi i z}$$

where (it turns out) $c_i \in \mathbb{Z}$ (e.g. $c_1 = 196884$). Thus, as one goes vertically to infinity, say along $z = it$, $j(z)$ grows like $e^{2\pi t}$ and has an essential singularity at ∞, and likewise at every rational point on the real line ($\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{Q} \cup \{\infty\}$).

If $z \in \mathbb{H}$ with $[\mathbb{Q}(z) : \mathbb{Q}] = 2$ then $j(z)$ is algebraic, indeed it is an algebraic integer with rich arithmetical properties described by the theory of *complex multiplication of elliptic curves*. For now we mention that, if z satisfies the quadratic equation

$$az^2 + bz + c = 0$$

where $a, b, c \in \mathbb{Z}, a > 0, (a, b, c) = 1$, letting $D(z) = b^2 - 4ac < 0$ be its *discriminant* then

$$[\mathbb{Q}\big(j(z)\big) : \mathbb{Q}] = h(D)$$

where $h(D)$ is the *class number* of the corresponding quadratic order; in particular if D is square-free then the corresponding order is the ring of integers in $\mathbb{Q}(z)$, and $h(D)$ is the order of its *class group*, the (finite) group of ideal classes under composition.

By a result of Schneider (1937), there are no other $z \in \mathbb{H}$ for which z and $j(z)$ are simultaneously algebraic ("Modular Hermite-Lindemann").

The modular function satisfies a third order algebraic differential equation, but none of any smaller order (Mahler [34]). Indeed (see e.g. Bertrand-Zudilin [10])

$$j''' \in \mathbb{Q}(j, j', j''),$$

more precisely (Masser [35])

$$Sj + \frac{j^2 - 1968j + 2654208}{2j^2(j-1728)^2}(j')^2 = 0$$

where

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2$$

is the *Schwarzian derivative.* We have $Sf = 0$ iff $f \in \mathrm{SL}_2(\mathbb{C})$.

Let us now say a little about how $j(z)$ arises in the theory of elliptic and curves. If $\Lambda \subset \mathbb{C}$ is a lattice (discrete $\mathbb{Z}$ module of rank 2), one can create doubly periodic meromorphic functions. By scaling one can always consider such lattices to be of the form

$$\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau, \quad \tau \in \mathbb{H}.$$

Then one forms the Weierstarsss $\wp$-function $\wp_\tau(z)$ by summing a suitable simple expression over the lattice being careful that it converges. It is doubly periodic and has double poles at the lattice points.

Its derivative is also Λ_τ periodic, and by taking suitable combinations one can eliminate the pole. The resulting function must vanish and one finds a relation of the form

$$\wp'^2 = 4\wp^2 - g_2(\tau)\wp(z) - g_3(\tau).$$

for suitable $g_2(\tau), g_3(\tau) \in \mathbb{C}$. Thus the map

$$z \mapsto (\wp, \wp')$$

maps $\mathbb{C}/\Lambda_\tau$ to a complex algebraic curve (including the point at ∞, giving a smooth projective curve). This is an *elliptic curve*, which we denote E_τ, and inherits a group law from the additive structure on $\mathbb{C}/\Lambda_\tau$. On E_τ the group law is given by some rational functions.

An elliptic curve is determined up to isomorphism over $\mathbb{C}$ by its j-invariant

$$j(\tau) = 1728\frac{g_2^3(\tau)}{g_2^3(\tau) - 27g_3^2(\tau)}.$$

The isomorphism can also be read in the lattices: τ_1, τ_2 give isomorphic curves if they are equivalent under $\mathrm{SL}_2(\mathbb{Z})$ which amounts to change of basis and rescaling. Thus the $\mathrm{SL}_2(\mathbb{Z})$ invariance of j.

Actions by $g \in \mathrm{GL}_2^+(\mathbb{Q})$ correspond to *isogenies* (homomorphisms with finite kernel) between elliptic curves. This can be seen as taking the quotient of a given curve by some (cyclic) subgroup, and so the j-invariant of the quotient has some algebraic relation to the given curve.

It is an elaborate and beautiful theory. See Diamond and Shurman [17], Zagier [67].

8. Modular Schanuel Conjecture

8.1. Definition. A point $z \in \mathbb{H}$ is called *special* if $[\mathbb{Q}(z) : \mathbb{Q}] = 2$.

Our principle now is that "numbers defined using the j-function should be as algebraically independent as permitted by the modular relations and the special values". We introduce a suitable "independence" notion.

8.2. Definition. Elements $z_1, \ldots, z_n \in \mathbb{H}$ are called $\mathrm{GL}_2^+(\mathbb{Q})$-*independent* if the z_i are not special and there are no relations

$$z_i = gz_j, \quad i \neq j, \quad g \in \mathrm{GL}_2^+(\mathbb{Q}).$$

In fact the special points are fixed points, so one could rephrase this as "no non-trivial $\mathrm{GL}_2^+(\mathbb{Q})$ relations between the z_i", the trivial ones being $z_i = 1z_i$. Note also that the relations are pairwise: if a set of n elements is dependent then one or two of them are already dependent. This is the hallmark of a "trivial pregeometry".

A first formulation might be the following.

8.3. Conjecture. Suppose $z_1, \ldots, z_n$ are $\mathrm{GL}_2^+(\mathbb{Q})$- independent. Then

$$\mathrm{tr.d.}\big(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)\big) \geq n.$$

Schneider's result gives this for $n = 1$; for $n \geq 2$ it is not known. Even the "Lindemann" statement (i.e. with z_i algebraic) is open. Many things are known beyond Schneider's result, for which I refer to Diaz [18] and Nesterenko [43].

The above conjecture does not take into account the derivatives of j. These fit into a much bigger conjectural picture which includes elliptic functions as well as modular ones (and the higher dimensional analogues), namely the *Generalised period conjecture* of Grothendieck-André: see André [2], Bertolin [7]. The following may be deduced from an explication of this conjecture in the case of one-dimensional "motives" in [7].

8.4. Modular Schanuel Conjecture. Suppose $z_1, \ldots, z_n$ are $\mathrm{GL}_2^+(\mathbb{Q})$- independent. Then

$$\text{tr.d.}\big(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n), j'(z_1), \ldots, j'(z_n), j''(z_1), \ldots, j''(z_n)\big) \geq 3n.$$

This does not reflect some transcendence properties of the derivatives at special points, but it is sufficient for our purposes here.

9. "Modular Ax-Schanuel"

We work in the complex setting rather than digressing on formulating a "Modular Ax-Schanuel" in a differential field. We consider

$$\pi : \mathbb{H}^n \to \mathbb{C}^n, \quad \pi(z_1, \ldots, z_n) = (j(z_1), \ldots, j(z_n)).$$

Let $A \subset U$ be a complex analytic subvariety of some open $U \subset \mathbb{H}^n$, with the coordinate functions $z_1, \ldots, z_n$ and $j(z_1), \ldots, j(z_n)$ meromorphic on A, and with derivations $\{D_k\}$ induced by differentiation w.r.t. z_k such that $\text{rank}(D_k z_\ell) = \dim A$, the rank being over the field of meromorphic functions on A.

9.1. Definition. The functions $z_1, \ldots, z_n$ on A are called *geodesically independent* if no z_i is constant and there are no relations $z_k = g z_\ell$ where $k \neq \ell$ and $g \in \mathrm{GL}_2^+(\mathbb{Q})$.

The following conjecture might be considered the analogue of "Ax-Schanuel" for the j-function.

9.2. "Modular Ax-Schanuel" Conjecture. In the above setting, suppose that the z_i are geodesically independent. Then

$$\text{tr.d.}_{\mathbb{C}}\mathbb{C}(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n)) \geq n + \dim A.$$

This conjecture is open beyond some special cases described below (including further below in §16); it is of intrinsic interest, but also very useful in addressing Zilber-Pink problems (see §15).

We pursue now geometric formulations analogous to those obtained earlier for the exponential function, and they will take exactly the same form. To frame these we need a definition of "geodesic subvariety", but we also need to pause on the meaning of an "algebraic subvariety" of $\mathbb{H}^n$. We can map $\mathbb{H}^n$ to the product Δ^n of open unit discs by an invertible algebraic map, whence one sees that there can be no positive dimensional algebraic varieties contained inside $\mathbb{H}^n$.

9.3. Definition. By a "subvariety" of $\mathbb{H}^n$ we mean an irreducible (in the complex analytic sense) subvariety of $W \cap \mathbb{H}^n$ for some algebraic subvariety $W \subset \mathbb{C}^n$.

9.4. Definition. A subvariety $W \subset \mathbb{H}^n$ is called *geodesic* if it is defined by some number of equations of the forms

$$z_i = c_i, \quad c_i \in \mathbb{C}; \quad z_k = g_{k\ell} z_\ell, \quad g \in \mathrm{GL}_2^+(\mathbb{Q}).$$

These are the "weakly special subvarieties" in the Shimura sense. The word "geodesic" is adopted from Moonen [41] who shows that, in a Shimura variety, the weakly special subvarieties are the "totally geodesic" ones. I wanted a word that gave a readable "suchly independent" phrase in analogy with "linearly independent" and "algebraically independent".

Since we have defined the "weakly special subvarieties" it is opportune to define the special ones.

9.5. Definition.

1. A *special point* in $\mathbb{H}^n$ is a tuple of special (i.e. quadratic) points.
2. A *special subvariety* in $\mathbb{H}^n$ is a weakly special subvariety containing a special point; equivalently, the fixed coordinates c_i above are all special.
3. The images under π of these are the *special subvarieties* in $\mathbb{C}^n$.

We now define components, their defects, and optimal components exactly as before and find that the conjecture above implies the following two formulations of a "Weak Modular Ax-Schanuel" conjecture, which are equivalent by exactly the same proofs given previously.

9.6. Formulation A. *Let U' be a geodesic subvariety of $\mathbb{H}^n$. Put $X' = \exp U'$ and let A be a component of $W \cap \pi^{-1}(V)$, where $W \subset U'$ and $V \subset X'$ are algebraic subvarieties. If A is not contained in any proper geodesic subvariety of U' then*

$$\dim A \leq \dim V + \dim W - \dim X'.$$

9.7. Formulation B. *Let $V \subset \mathbb{C}^n$. An optimal component for V is geodesic.*

Formulation B is the form that is needed to tackle Zilber-Pink problems using o-minimality and point-counting. However, a true "Modular Ax-Schanuel" should take into account the derivatives of j.

9.8. Conjecture (Modular Ax-Schanuel with derivatives). In the setting of "Modular Ax-Schanuel" above, if z_ℓ are geodesically independent then

$$\mathrm{tr.d.}_{\mathbb{C}}\mathbb{C}(z_1, \ldots, z_n, j(z_1), \ldots, j(z_n), j'(z_1), \ldots, j'(z_n), j''(z_1), \ldots, j''(z_n)) \geq 3n + \dim A.$$

The "geodesic independence" condition is evidently equivalent in the complex setting (in analogy with Ax [3]) to: the $j(z_i)$ are "modular independent", i.e. non-constant and no relation $\Phi_N(j(z_k), j(z_\ell)) = 0$ holds for $k \neq \ell, N \geq 1$.

10. “Modular Ax-Lindemann”

We retain the setting of the previous section, and consider $V \subset \mathbb{C}^n$.

10.1. “Modular Ax-Lindemann” Form 1. *A maximal algebraic subvariety $W \subset j^{-1}(V)$ is geodesic.*

10.2. “Modular Ax-Lindemann” Form 2. Let $W \subset \mathbb{C}^n$ with $W \cap \mathbb{H}^n \neq \emptyset$. Suppose that the coordinate functions $z_1, \ldots, z_n \in \mathbb{C}(W)$ are geodesically independent. Then the functions

$$j(z_1), \ldots, j(z_n),$$

defined locally on W, are algebraically independent over $\mathbb{C}$.

These two formulations are equivalent, by variants of the proofs for the exponential case (**Exercise**). It is Form 1 that arises in the o-minimal approach to the André-Oort conjecture for products of modular curves. It is proved in [48], and we sketch the proof later. A version “with derivatives” is established in [49].

11. The general setting

Both settings described above: $\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n, \quad j : \mathbb{H}^n \to \mathbb{C}^n$ fit into a bigger picture

$$\pi : U \to X$$

where X is a *Shimura* or *mixed Shimura variety* (see Pink [55, 56], or Daw [16] in this volume), and U is (essentially) its universal cover.

The prototypical Shimura varieties are modular varieties. For example $\mathbb{C}$, as the j-line, is the moduli space of elliptic curves up to isomorphism over $\mathbb{C}$. The higher dimensional analogues are the *Siegel moduli spaces* $\mathcal{A}_g$ which parameterise (principally polarised) abelian varieties of dimension g, i.e. g-dimension complex tori which admit an algebraic structure (when $g \geq 2$ not all do). The dimension of $\mathcal{A}_g$ is $g(g+1)/2$. One has

$$\pi_g : \mathbb{H}_g \to \mathcal{A}_g$$

where $\mathbb{H}_g$ is the *Siegel upper half space*, and the uniformisation (which is given by *Siegel modular forms*) is invariant under $\mathrm{Sp}_{2g}(\mathbb{Z})$. See e.g. van der Geer [19].

Each point $x \in \mathcal{A}_g$ parameterises an abelian variety A_x; the corresponding *mixed* Shimura variety consists of $\mathcal{A}_g$ fibered by the A_x. The simplest example is given by the Legendre family of elliptic curves, that is the elliptic surface

$$y^2 = x(x-1)(x-\lambda)$$

considered as a family of elliptic curves, one for each $\lambda \in \mathbb{C} - \{0, 1\}$ fibered over the λ-line.

Maybe here is the point to mention that $\mathbb{H}$ is not the universal cover of $\mathbb{C}$. The covering by j is ramified at two points whose pre-images are fixed by elements of $\mathrm{SL}_2(\mathbb{Z})$, namely $j(i)$ and $j(\rho)$. But if one takes a suitable finite index (congruence) subgroup one has no such points: the covering

$$\lambda : \mathbb{H} \to \mathbb{C} - \{0, 1\}$$

associated with the Legendre family is universal, and the corresponding congruence subgroup is isomorphic to the free group on two generators.

In general, mixed Shimura varieties arise as quotients of symmetric hermitian domains by suitable arithmetic subgroups of their group of biholomorphic self-maps. They have the structure of an algebraic variety. Each mixed Shimura variety X has a collection $\mathcal{S} = \mathcal{S}_X$ of "special subvarieties" and a larger collection $\mathcal{W} = \mathcal{W}_X$ of "weakly special subvarieties", which is what I have termed "geodesic".

Shimura varieties are the setting for an arithmetic conjecture called the "Andre-Oort conjecture". This fits into the much broader "Zilber-Pink" conjecture in the setting of mixed Shimura varieties, which concerns the interaction between subvarieties $V \subset X$ and the collection of "special subvarieties" (see §15).

In the approach to these conjectures via o-minimality, suitable functional transcendence statements are a key ingredient. In particular, to carry out this approach to prove AO for a Shimura variety X one requires:

11.1. "Ax-Lindemann" Conjecture for X. *Let $V \subset X$. A maximal algebraic subvariety $W \subset \pi^{-1}(V)$ is weakly special.*

Klingler-Ullmo-Yafaev have recently announced [30] a proof of this for all Shimura varieties: Tsimerman and I [52] proved it for $\mathcal{A}_g$, building in part on Ullmo-Yafaev's proof [64] for all *compact* Shimura varieties (when there are no cusps in the fundamental domain, the quotient is a compact, i.e. projective, variety). In proving this theorem (using o-minimality) they also established the definability of the uniformisation restricted to a fundamental domain, extending the work of Peterzil-Starchenko [47] who did it for $\mathcal{A}_g$ (indeed for the mixed Shimura variety associated with $\mathcal{A}_g$). The extension to mixed Shimuras may not be far away.

This means, by work of Ullmo [60], that a full proof of AO is now reduced to (1) a statement about Galois orbits of special points being "large", and (2) a statement that the height of a pre-image of a special point is "not too large". For $\mathcal{A}_g$, the latter was proved in [51]; for $g \leq 6$, the Galois lower bound is known due to Tsimerman ([58]; under GRH it is known for all g by Tsimerman and independently by Ullmo-Yafaev [62]). More generally one expects the following.

11.2. Weak Ax-Schanuel Conjecture. Let X be a (mixed) Shimura variety and $V \subset X$. An optimal component for $V \subset X$ is weakly special.

For a still more general setting see Zilber [73].

12. Exponential Ax-Lindemann via o-minimality

We give a proof of "Ax-Lindemann" using o-minimality and point-counting, to motivate the proof of the modular analogue which follows.

For a proof of the full Ax-Schanuel statement via o-minimality and point-counting see Tsimerman [59], in this volume. We consider

$$\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n, \quad V \subset (\mathbb{C}^*)^n.$$

The complex exponential is definable when restricted to a fundamental domain for the $2\pi i\mathbb{Z}$ action (by translation) on $\mathbb{C}$. We take say

$$F = \{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) < 2\pi i\}.$$

Then exp is definable on F^n, and we let

$$Z = \exp^{-1}(V) \cap F^n,$$

which is also definable.

12.1. Theorem. *A maximal algebraic subvariety $W \subset \exp^{-1}(V)$ is geodesic.*

12.2. Idea. 1. The action of $(2\pi i\mathbb{Z})^n$ on F^n divides $\mathbb{C}^n$ into fundamental domains γF^n, where $\gamma \in (2\pi i\mathbb{Z})^n$. We find that W is "present" in "many" of them. Then the suitable translation of these pieces back to F^n belongs to $\exp^{-1}(V)$.

2. The $\gamma \in (2\pi i\mathbb{Z})^n$ for which W is "present" in γF belong to a certain definable subset of $(2\pi i\mathbb{R})^n$ for which the corresponding translate of W is contained in $\exp^{-1}(V)$, and which thus contains "many" rational points. By the Counting Theorem, this set contains positive dimensional semi-algebraic families of translates of W.

3. Consider such a family of translations, say with a real parameter t. If the union over this family of translations is bigger than W, we could "complexify" the parameter and get a complex variety W' containing W but of bigger dimension. This contradicts our assumption that W is maximal. So these translations must translate W along itself. This forces W to be linear and even to be a coset of a rational subspace.

Proof. We suppose $W \subset \exp^{-1}(V)$ is maximal, say of dimension k. We can suppose that $z_1, \ldots, z_k$ are independent functions on W, and that the other variables depend algebraically on them

$$z_\ell = \psi_\ell(z_1, \ldots, z_k), \quad \ell = k+1, \ldots, n.$$

Of course these algebraic functions will have some branching, but locally at smooth points they are functions and can be analytically continued throughout $z_1, \ldots, z_k$-space avoiding some lower-dimensional branching locus.

We will write below z for the tuple of "free" variables $(z_1, \ldots, z_k)$, and ψ for the tuple of functions $(\psi_{k+1}, \ldots, \psi_n)$.

Fix some small product of discs $U \subset \mathbb{C}^k$ in the $z_1, \ldots, z_k$-variables such that the ψ_ℓ are all unbranched at points

$$(z + 2\pi i t) = (z_1 + 2\pi i t_1, z_2 \ldots, z_k)$$

for $(z_1, \ldots, z_k) \in U$ and all sufficiently large real t_1 (this is true generically). By the periodicity of exp, for any translation of W by integer multiples of $2\pi i$ on the coordinates is again inside $\exp^{-1}(V)$.

But we are going to use definability, so can only make use of exp on finitely many fundamental domains. We will just use F^n.

For any integer t, there exists a unique integer vector

$$m'(t) = (m_{k+1}, \ldots, m_n)$$

such that the graph on U of

$$\psi(z + 2\pi i t) - 2\pi i m'(t)$$

intersects Z in a set of real dimension $2k$ (which is its full real dimension).

For any $m' \in \times\mathbb{R}^{n-k}$ and $t \in \mathbb{R}$ we let

$$W(U, m', t)$$

denote the graph on U of the functions

$$\psi(z + 2\pi i t) - 2\pi i m'.$$

Fixing U we consider now the definable set

$$Y = \{(m', t) \in \mathbb{R}^{n-k} \times \mathbb{R} : \dim_{\mathbb{R}} \big(W(U, m', t) \cap Z\big) = 2k\}.$$

Since the functions ψ_ℓ have polynomial growth in t, the components of $m'(t)$ are bounded by some polynomial in t. Therefore Y contains "many" rational (in fact integer) points.

Therefore Y contains semi-algebraic curves which contain arbitrarily large finite numbers of integer points, which seems to give us a positive family of translates of W contained in $\exp^{-1}(V)$ if we "complexify" the parameter t locally.

But W is maximal, so we must be just translating W along itself, in particular we have for suitable integers $s_1 \neq t_1$ and integer vectors $m'(s_1), m'(t_1)$

$$(*) \quad \psi_\ell(z_1 + 2\pi i s_1, z_2, \ldots, z_k) - \psi_\ell(z_1 + 2\pi i t_1, z_2, \ldots, z_k) = 2\pi i m'(s_1) - 2\pi i m'(t_1)$$

holding for all ℓ *identically in z.*

Fix $(z_2, \ldots, z_k)$. Differentiating with respect to z_1, we see that

$$\psi'_\ell(z_1 + 2\pi i s_1, z_2, \ldots, z_k) - \psi'_\ell(z_1 + 2\pi i t_1, z_2, \ldots, z_k) = 0.$$

The algebraic function $\psi'_\ell(z_1, z_2, \ldots, z_k)$ (as a function of z_1, the other z_i being fixed) with a period must be constant. So we have

$$\psi_\ell(z_1, z_2, \ldots, z_k) = q(z_2, \ldots, z_k)z_1 + r(z_2, \ldots, z_k).$$

Since we have integer points, if we go back to $(*)$ the coefficient $q(z_2, \ldots, z_k)$ must be rational. But then $q(z_2, \ldots, z_k)$, which is an algebraic function, must be constant (could use definability here) and we have

$$\psi_\ell(z_1, z_2, \ldots, z_k) = q_{\ell 1} z_1 + r(z_2, \ldots, z_k), \quad q_{\ell 1} \in \mathbb{Q}.$$

We repeat the argument for all the free variables to find that

$$\psi_\ell = r_\ell + \sum q_{\ell i} z_i, \quad q_{\ell i} \in \mathbb{Q}, r_\ell \in \mathbb{C}, \quad \ell = k+1, \ldots, n$$

and so W is geodesic. □

13. Modular Ax-Lindemann via o-minimality

We sketch a proof of Modular Ax-Lindemann via o-minimality and point-counting, essentially following the argument in [48]. This follows the previous argument, with just a few minor additional technicalities due to the boundary of $\mathbb{H}^n$ and the more intricate group action. We now consider

$$j : \mathbb{H}^n \to \mathbb{C}^n, \quad V \subset \mathbb{C}^n.$$

13.1. Definability. Let F be the standard fundamental domain for the $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathbb{H}$, as described above. Then $j|_F : F \to \mathbb{C}$ is definable in $\mathbb{R}_{\text{an exp}}$. (Follows from the q-expansion; Peterzil-Starchenko [46] observed this while proving definability for the Weierstrass $\wp$ function as a function of both variables.)

13.2. Theorem. *A maximal algebraic subvariety $W \subset j^{-1}(V)$ is geodesic.*

13.3. Idea. The same.

Proof. We suppose $W \subset j^{-1}(V)$ is maximal, with $\dim W = k$, and that, locally on some region $D \subset \mathbb{C}^k$, we may take $z_1, \ldots, z_k$ as independent variables and parameterise W by

$$z_\ell = \phi_\ell(z_1, \ldots, z_k), \quad \ell = k+1, \ldots, n.$$

If $g \in \mathrm{SL}_2(\mathbb{Z})^n$ then gW is also a maximal algebraic subvariety. It is locally parameterised by

$$z_\ell = g_\ell \phi_\ell(g_1^{-1} z_1, \ldots g_k^{-1} z_k), \quad \ell = k+1, \ldots, n$$

on $(g_1, \ldots, g_k)D$.

We can then analytically continue these functions, perhaps with some branching, remaining inside $\mathbb{H}^n$ (and hence within $j^{-1}(V)$), until some free or dependent variable runs into its real line.

For example, keeping $z_2, \ldots, z_k$ in a small neighbourhood, we can analytically continue the functions in z_1 up to the real boundary unless some ϕ_ℓ becomes real. This

ϕ_ℓ then depends on z_1 over $z_2, \dots, z_k$, and we can exchange z_1 and z_ℓ, and now we have a parameterisation that goes up to the boundary of z_1.

This gives a subregion of $\mathbb{H}^n$ bounded by loci where some $z_1, \dots, z_k$ or some $\phi_\ell(z_1, \dots, z_k)$ becomes real, in particular including some "disc" U_1 where z_1 becomes real, the other free variables remain away from their real lines (so it is a product of a half-disc in z_1, and discs in the other free z_i). Some of the dependent z_ℓ may also be real on U_1, others not: we can move to a smaller "disc" so that such dependent variables are either real on all of U_1, or are away from their real lines - and contained in a single fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.

Again moving to a smaller "disc" if necessary we can assume that all the ϕ_ℓ are regular and non-branching.

We let Φ denote the tuple $(\phi_{k+1}, \dots, \phi_n)$ and put

$$W_1 = \{(u, \Phi(u)) : u \in U_1\} \subset W,$$

a definable (even semi-algebraic) set.

The point will be that in the z_1 half disc there are infinitely many fundamental domains but the variables away from their real lines will be confined to finitely many fundamental domains.

Fix a fundamental domain F_1 inside the z_1 half disc, and a rational point a/c with $(a, c) = 1$ on the boundary of this half disc. Take a matrix

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and write

$$g_0(t) = \begin{pmatrix} a & b+t \\ c & d+t \end{pmatrix}, \quad t \in \mathbb{R}.$$

For large real t and $z \in F_1$, $g_0(t)z$ is in the half disc. Let

$$G_0 = \{g \in \mathrm{SL}_2(\mathbb{R})^n : g_1 = g_0(t), \text{ some } t, g_i = 1, i \le 2 \le k\}$$

with no restriction on $g_{k+1}, \dots, g_\ell$. This set is clearly definable.

For any definable $G' \subset \mathrm{SL}_2(\mathbb{R})^n$, $W' \subset W$ (of full complex dimension k say) and $Z' \subset j^{-1}(V)$ the set

$$R(G', W', Z') = \{g \in G' : \dim_{\mathbb{R}}(gW' \cap Z) = 2k\}$$

is definable. Further, for any such g we have $gW \subset j^{-1}(V)$ by dimensional considerations and analytic continuation.

Consider the definable set

$$R(G_0, Y_1, Z).$$

For large t the action by $g_0(t)$ keeps part of the z_1 half disc within itself. For any such t we may find elements of $\mathrm{SL}_2(\mathbb{Z})$ to bring the relevant coordinates to F, for t a large integer this will give an element of $R(G_0, Y_1, Z)$, and the size of the group element are bounded by some polynomial in t (by Proposition in §7). So $R(G_0, Y_1, Z)$ has "many" integer points, and by the Counting Theorem there are semi-algebraic subsets with arbitrarily large finite numbers of integer points.

Now maybe all these sets have fixed t. Then we can find an integer t and a positive dimensional set of translations, and hence a smooth one-dimensional set of translations contained in $j^{-1}(V)$ containing an integer point. But the integer translation of W is maximal, and is contained in a larger family by complexifying the parameter.

So we have semi-algebraic sets with many integer points and variable t. By the maximality, these translates parameterise the same translate of W.

Now we observe that the dependent variables away from their boundaries did not need to move. Therefore these variables do not depend on z_1.

For the other variables, we get identities (using two integer points on the same algebraic set where the translate is constant) of the form

$$\phi(gz) = h\phi(z), \quad \phi(z) = \phi_\ell(z, z_2, \ldots, z_k), \quad g, h \in \mathrm{SL}_2(\mathbb{Z}).$$

We know that g is of the form

$$g_0(s)g_0(t)^{-1} = \begin{pmatrix} 1 - ac(s-t) & a^2(s-t) \\ -c^2(s-t) & 1 + ac(s-t) \end{pmatrix}$$

and so is parabolic with fixed point a/c, and we get such identities for every $a/c \in I$, the real boundary of the z_1 half disc.

Now there is an "end-game" to show that $\phi \in \mathrm{GL}_2^+(\mathbb{Q})$. This part gets more conceptual in the various generalisations [64, 52, 30] : using monodromy considerations one shows essentially that W is an orbit of the group that stabilises it. Here I use elementary arguments.

1. $\phi \in \mathrm{SL}_2(\mathbb{C})$

The following argument is different to the argument in [48] and to the alternative argument offered in [49].

We have $P(x, \phi(x)) = 0$ for some irreducible $P \in \mathbb{C}[X, Y]$. We have infinitely many parabolic g with distinct (real) fixed points for which we have an identity

$$\phi(gz) = h\phi(z).$$

This identity continues to hold wherever we may continue ϕ. If x_g is the fixed point of g then $y_g := \phi(x_g)$ is fixed by h, and there are infinitely many distinct y_g, even with ϕ pre-images distinct from branch points of ϕ.

If $\phi(x) = y_g$, then also $\phi(gx) = y_g$. Then x is pre-periodic under g, but since g is parabolic it has no pre-periodic points other than its unique fixed point. So for such y_g there is only one x_g (the fixed point) with $\phi(x_g) = y_g$.

Since this holds for infinitely many distinct y_g, P must be linear in X. Exchanging roles (Steinitz exchange), it is also linear in Y. So ϕ is a fractional linear transformation.

2. $\phi \in \mathrm{SL}_2(\mathbb{R})$

Because it preserves the real line (**Exercise**).

3. ϕ is independent of $z_2, \ldots, z_k$

Because there is no non-constant holomorphic function (here complex algebraic) to $\mathrm{SL}_2(\mathbb{R})$. E.g. the image of 0 must be a real function of a complex variable.

So the dependencies are all by fixed elements of $\mathrm{SL}_2(\mathbb{R})$, and extend throughout all $\mathbb{H}^k$. And so we get identities for all rational a/c (and so even all $r \in \mathbb{R}$).

4. $\phi \in \mathrm{GL}_2^+(\mathbb{Q})$

Some elementary work. Write $\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $AD - BC = 1$. We must show that the ratios of entries are all rational (so it is in the image of $\mathrm{GL}_2^+(\mathbb{Q})$).

We can take g with $a = 1, c = 0$. Write $u = (s - t)$. Then for some $\lambda \in \mathbb{R}$, $h \in \mathrm{GL}_2^+(\mathbb{Q})$ we have

$$\phi g \phi^{-1} = \begin{pmatrix} 1 - uAC & uA^2 \\ -uC^2 & 1 + uAC \end{pmatrix} = \lambda h$$

for suitable (many) integer choices of u. If $C = 0$ we see that $A^2 \in \mathbb{Q}$ and then $AD = 1$ implies $A/D \in \mathbb{Q}$. Similarly, $A = 0$ implies $B/C \in \mathbb{Q}$. Otherwise $(A, C \neq 0)$ we have $A^2/C^2 \in \mathbb{Q}$ and $(1 - uAC)/C^2 \in \mathbb{Q}$, for many different u, giving $A/C \in \mathbb{Q}$. Taking $a = 0, c = 1$ we get similarly

$$\begin{pmatrix} 1 - uBD & uB^2 \\ -uD^2 & 1 + uBD \end{pmatrix} = \lambda h.$$

Now $B = 0$ leads to $A/D \in \mathbb{Q}$, $D = 0$ leads to $B/C \in \mathbb{Q}$ and otherwise $(B, D \neq 0)$ we have $B/D \in \mathbb{Q}$.

Now suppose $C = 0$, so we have $A/D \in \mathbb{Q}$. If $B = 0$ we have the required form. We cannot have $D = 0$, so $B \neq 0$ gives $B/D \in \mathbb{Q}$ and we have again the right form. Similarly, if any of $A, B, C, D = 0$ we get the right form: If $B = 0$ we have $A/D \in \mathbb{Q}$. If $C = 0$ we are done. We can't have $A = 0$, so if $C \neq 0$ we get $A/C \in \mathbb{Q}$ and are done.

So we may assume all $A, B, C, D \neq 0$. We Have $A/C = q, B/D = r \in \mathbb{Q}$ and ϕ is up to scaling

$$\begin{pmatrix} 1 & \alpha \\ q & r\alpha \end{pmatrix}, \quad r \neq q, \alpha \in \mathbb{R}.$$

Then

$$\psi = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \phi = \begin{pmatrix} 1 & \alpha \\ 0 & (r - q)\alpha \end{pmatrix}$$

satisfies identities of the same kind (with the same g), but now there is a zero entry, so we are done! □

14. SC and CIT

Boris Zilber's work on the model theory of the exponential function led him to formulate ([70, 71]) an arithmetic conjecture which he called CIT: "Conjecture on Intersections with Tori".

In the language of exponential fields one cannot formulate SC in a first order way. One can list all subvarieties $V \subset \mathbb{C}^{2n}$ defined over $\mathbb{Q}$ and of dimension $\dim V < n$. Then

$$\text{tr.d.}(z, e^z) < n, \quad z = (z_1, \dots, z_n), \quad e^z = (e^{z_1}, \dots, e^{z_n})$$

just means that (z, e^z) lies on one of these V and one could aspire to go through them asserting: *"If* $(z, e^z) \in V$ *then..."*. However one cannot assert that the coordinates of z are l.i over $\mathbb{Q}$ in a first order way, as this requires a quantification over $\mathbb{Q}$. One could do this, however, if for each such V only finitely many such linear dependencies arise: one could then just write them out explicitly.

But one must be a bit careful: the assertion *"Let $V \subset \mathbb{C}^{2n}$ be defined over $\mathbb{Q}$. There exists finitely many non-trivial linear forms $L(z_1, \dots, z_n)$ with integer coefficients such that if*

$$(z, e^z) = (z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}) \in V$$

then

$$L(z_1, \dots, z_n) = 0.$$

for (at least) one of these forms" is simply false.

14.1. Example. Take $V \subset \mathbb{C}^3 \times \mathbb{C}^3$ defined by

$$z_1 z_2 = z_3^2, \quad w_1 = 1, w_2 = 1, w_3 = 1.$$

So $\dim V = 2$. If $k_1 k_2 = k_3^2$ and $z_\ell = 2\pi i k_\ell$ then

$$(z_1, z_2, z_3, e^{z_1}, e^{z_2}, e^{z_3}) \in V,$$

but these points are not contained in finitely many rational subspaces (they all lie in some proper rational subspace though!).

The right statement is a variant of this:

14.2. Uniform Schanuel Conjecture. (USC; [71]) Let $V \subset \mathbb{C}^{2n}$ be a closed algebraic set defined over $\mathbb{Q}$ with $\dim V < n$. There exists a finite set $\mu(V)$ of proper $\mathbb{Q}$-linear subspaces of $\mathbb{C}^n$ such that if

$$(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n}) \in V$$

then there is $M \in \mu(V)$ and $\overline{k} \in \mathbb{Z}^n$ and such that $(z_1 + 2\pi i k_1, \dots, z_n + 2\pi i k_n) \in M$. Moreover if M is codimension 1 (in $\mathbb{C}^n$) then $k = 0$.

For more on the model theory of exponentiation see [69, 28]. For the present purposes I want to work with a weaker version of USC:

14.3. Weak SC. If tr.d.$(z, e^z) < n$ then the coordinates of e^z are multiplicatively dependent.

14.4. Uniform Weak SC. Let $V \subset \mathbb{C}^{2n}$ defined over $\mathbb{Q}$ with $\dim V < n$. There is a finite set $K = K(V)$ of non-trivial integer tuples $k \in \mathbb{Z}^n$ such that such that if $(z, e^z) \in V$ then $\prod \exp(z_i k_i) = 1$ for some $k \in K$.

Now we formulate "CIT". We consider algebraic subgroups of $X = (\mathbb{C}^*)^n$. These are subvarieties defined by some number of equations of the form

$$\prod_{i=1}^{n} x_i^{k_i} = 1$$

for vectors $k = (k_1, \ldots, k_n)$ of integers. These can be reducible (e.g. $x_1^2 = 1$ in $\mathbb{C}^*$), and they decompose into finitely many irreducible subvarieties which are called *tori* if they are subgroups, or cosets of tori by torsion points, called *torus cosets*, generally.

We will also call torus cosets *special subvarieties*, and denote the collection of them by $\mathcal{S} = \mathcal{S}(X)$. This is a countable collection.

Now two algebraic subvarieties $V, W \subset X$ generically intersect in an algebraic set whose components have dimension

$$\dim V + \dim W - \dim X$$

by simple "counting conditions" (i.e. codim V conditions are required to be on V, codim W conditions to be on W). It is a basic fact that such components can never have smaller dimension than this (see e.g. Mumford [42, 3.28]); but it can be bigger.

14.5. Definition. Let $V \subset X = (\mathbb{C}^*)^n$.

1. A component $A \subset V \cap T$, where $T \in \mathcal{S}$, is called *atypical* if

$$\dim A > \dim V + \dim T - \dim X.$$

2. Denote by

$$\mathrm{Atyp}(V) = \bigcup A$$

the union of all atypical components of $V \cap T$ over all $T \in \mathcal{S}$.

Thus Atyp(V) is potentially a countable union.

14.6. Conjecture. (CIT) For $V \subset (\mathbb{C}^*)^n$, Atyp$(V)$ is a finite union.

Otherwise put: V contains only finitely many *maximal* atypical components.

14.7. Remarks.

1. Zilber [70, 71] stated the conjecture for semi-abelian ambient varieties, and for V defined over $\mathbb{Q}$, which is what is needed for the SC application.

2. Zilber showed that CIT for (semi-)abelian varieties implies the "Mordell-Lang conjecture" (a theorem of Faltings, Raynaud, Vojta, Faltings, McQuillan), including the Manin-Mumford conjecture (Raynaud), and exponential CIT implies the multiplicative versions (Mann-Lang-Liardet-Laurent).

3. The same conjecture in the exponential setting (in a different formulation) was stated by Bombieri-Masser-Zannier ([13], for $V/\mathbb{C}$). They earlier proved partial results for curves [12]. They later proved [15] that the various formulations were equivalent in the exponential case, and that CIT/$\mathbb{Q}$ implies CIT/$\mathbb{C}$.

4. Exponential CIT is open. Various partial results are known: including a complete result for curves; see Bombieri-Masser-Zannier [12, 13, 14], Maurin [39, 40], Habegger [20, 21]; and [11].

5. The same kind of conjecture was formulated (again independently) by Pink [55, 56] in the setting of "mixed Shimura varieties". Apparently his object was to find a unifying statement including the Mordell-Lang on the "semiabelian side" and André-Oort [1, 44, 29, 61] on the Shimura side. See Zannier's book [68].

Zilber [70, 71] proves the following theorem.

14.8. Theorem. *SC + CIT implies USC.* □

I will prove this for the weak uniform version adapting the proof in [71].

14.9. Theorem. *SC + CIT implies UWSC.*

Proof. We consider some $V \subset \mathbb{C}^{2n}$ defined over $\mathbb{Q}$ and of dimension $\dim V < n$. We let W be the projection of V onto the second $\mathbb{C}^n$ factor, d the dimension of the generic fibre of this projection, and $V' \subset V$ the proper subvariety where the fibre dimension exceeds d.

According to CIT, there is a finite set of torus cosets $S_1, \ldots, S_\ell$ whose atypical components with W contain all atypical components.

Now suppose $(z, e^z) \in V$. According to SC, z lies in some proper rational subspace $T \subset \mathbb{C}^n$, whose dimension we may take to be l.d.(z). The image of $\exp(T)$ is a subtorus $S \subset (\mathbb{C}^*)^n$, of the same dimension.

14.10. Claim. Suppose $(z, e^z) \in V - V'$. Then e^z lies in an atypical component of $W \cap S$.

Pf. We estimate tr.d.(z, e^z) below by SC and above by the intersection of V with the π pre image of $W \cap S$. Let A be the component of $W \cap T$ containing e^z. We find

$$\dim T \leq \text{tr.d.}(z, e^z) \leq d + \dim A = \dim V - \dim W + \dim A < n - \dim W + \dim A.$$

Rearranging we see that

$$\dim A > \dim T + \dim W - n$$

and this proves the claim.

So if (z, e^z) lies in $V - V'$ then e^z satisfies one of finitely many multiplicative relation. Otherwise $(z, e^z) \in V'$, and we repeat the argument with its components V'_i, putting $W'_i = \pi V'_i$ with generic fibre dimension d'_i outside $V''_i \subset V'$ etc. □

15. Zilber-Pink

Let X be a mixed Shimura variety, and $\mathcal{S}$ its collection of "special subvarieties". The following is essentially Zilber's formulation in Pink's setting. We define atypical components and $\mathrm{Atyp}(V)$ for $V \subset X$ exactly as previously.

15.1. Zilber-Pink conjecture. Let $V \subset X$. Then $\mathrm{Atyp}(V)$ is a finite union.

It is natural to formulate this conjecture for $V/\mathbb{C}$, but it is again the version for $V/\mathbb{Q}$ that connects a Schanuel conjecture with its uniform version.

We consider the modular setting ($X = \mathbb{C}^n$).

15.2. Modular SC. (MSC) Let $z \in \mathbb{H}^n$ and let T be the smallest special subvariety of $\mathbb{H}^n$ containing z. Then

$$\mathrm{tr.d.}(z, j(z)) \geq \dim T.$$

15.3. UMSC. Let $V \subset \mathbb{C}^{2n}$ defined over $\mathbb{Q}$ and of dimension $\dim V < n$. There exists finitely many proper special subvarieties $T_1, \ldots, T_\ell$ such that if

$$(z, j(z)) \in V$$

then there exists T_i and $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$ such that $z \in T_i$. Equivalently, there are finitely many special subvarieties $S_i = j(T_i) \subset \mathbb{C}^n$ such that $j(z) \in S_i$ for some i.

15.4. Theorem. *MSC + MZP implies UMSC.*

Proof. This is just the same as the proof of 14.9. Let V be given and define W, d, V' as before. Suppose $(z, j(z)) \in V$. Then $z \in T$ for some proper special T. Let $S = j(T)$, so $\dim S = \dim T$. Suppose $(z, j(z)) \in V - V'$. Let A be the component of $W \cap S$ containing $j(z)$. Then

$$\dim T \leq \mathrm{tr.d.}(z, j(z)) \leq d + \dim A = \dim V - \dim W + \dim A < n - \dim W + \dim A.$$

So A is atypical and by MZP it is contained in an atypical component of one of finitely many proper specials S_i. Repeat for V'. □

16. Zilber-Pink and Ax-Schanuel

Other special cases of ZP have been successfully proved via o-minimality and point-counting (e.g. Masser-Zannier [36, 37, 38], Habegger-Pila [23], Bertrand-Masser-Pillay-Zannier [8], Orr [45], Bays-Habegger [6]) and in several of these one needs a suitable "Ax" type statement. In [23] it the modular analogue of "Ax-Logarithms", the algebraic independence of logarithms of algebraic functions.

16.1. Theorem (Ax-Logarithms). (Ax) *Suppose $C \subset (\mathbb{C}^*)^n$ is a curve. If $\log C$, locally on some disc, is contained in an algebraic hypersurface then C is contained in a proper weakly special subvariety.* □

16.2. Theorem (Modular Ax-Logarithms). ([23]) *Suppose $C \subset \mathbb{C}^n$ is a curve. If $j^{-1}(C)$, locally on some disc, is contained in an algebraic hypersurface then C is contained in a proper weakly special subvariety.* $\square$

This is proved using monodromy (not o-minimality). The required statements in Masser-Zannier are also proved using monodromy arguments.

In work in progress [24], Habegger-Pila show that "Weak Modular Ax-Schanuel" (as in §9) together with a suitable arithmetic statement that Galois orbits of certain atypical intersections are "large" implies, via o-minimality and point-counting, the full Zilber-Pink conjecture for $\mathbb{C}^n$.

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References

1. Y. André, *G-functions and geometry,* Aspects of Mathematics E13, Vieweg, Braunschweig, 1989.
2. Y. André, *Une Introduction aux Motifs,* Panoramas et Sythèses, **17**, SMF, Paris 2004.
3. J. Ax, On Schanuel's conjectures, *Annals* **93** (1971), 252–268.
4. J. Ax, Some topics in differential algebraic geometry I: Analytic subgroups of algebraic groups, *Amer. J. Math.* **94** (1972), 1195–1204.
5. A. Baker, *Transcendental Number Theory,* CUP, 1974, 1979.
6. M. Bays and P. Habegger, A note on divisible points on curves, arXiv:1301.5674 and *Trans. A. M. S.,* to appear.
7. C. Bertolin, Périodes de 1-motifs et transcendance, *J. Number Th.* **97** (2002), 204–221.
8. D. Bertrand, D. Masser, A. Pillay, U. Zannier, Relative Manin-Mumford for semi-abelian surfaces, manuscript, 2011/2013, arXiv:1307.1008.
9. D. Bertrand, A. Pillay, A Lindemann-Weierstrass theorem for semi-abelian varieties over function fields, *J. Amer. Math. Soc.* **23** (2010), 491–533.
10. D. Bertrand and W. Zudilin, Derivatives of Siegel modular forms and exponential functions, *Izvestiya Math.* **65** (2001), 659–671.
11. E. Bombieri, P. Habegger, D. Masser, and U. Zannier, A note on Maurin's theorem, *Rend. Lincei. Mat. Appl.* **21** (2010), 251–260.
12. E. Bombieri, D. Masser and U. Zannier, Intersecting a curve with algebraic subgroups of multiplicative groups, *IMRN* **20** (1999), 1119–1140.

13. E. Bombieri, D. Masser and U. Zannier, Anomalous subvarieties – structure theorems and applications, *IMRN* **19** (2007), 33 pages.
14. E. Bombieri, D. Masser and U. Zannier, Intersecting a plane with algebraic subgroups of multiplicative groups, *Ann. Scuola Norm. Pisa Cl. Sci (5)* **7** (2008), 51–80.
15. E. Bombieri, D. Masser, and U. Zannier, On unlikely intersections of complex varieties with tori, *Acta Arithmetica* **133** (2008), 309–323.
16. C. Daw, this volume.
17. F. Diamond and J. Shurman, *A First Course in Modular Forms,* Graduate Texts in Mathematics **228**, Springer, New York, 2005.
18. G. Diaz, Transcendence et indépendance algébriques: liens entre les points de vue elliptique et modulaire, *Ramanujan J.* **4** (2000), 157–199.
19. G. van der Geer, Siegel modular forms and their applications, in *The 1-2-3 of Modular Forms,* Universitext, Springer, Berlin, 2008.
20. P. Habegger, Intersecting subvarieties of $\mathbf{G}_m^n$ with algebraic subgroups, *Math. Annalen* **342** (2008), 449–466.
21. P. Habegger, On the bounded height conjecture, *IMRN* **2009**, 860–886.
22. P. Habegger, Torsion points on elliptic curves in Weierstrass form, *Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5)* **12** (2013), 687–715.
23. P. Habegger and J. Pila, Some unlikely intersections beyond André–Oort, *Compositio* **148** (2012), 1–27.
24. P. Habegger and J. Pila, O-minimality and certain atypical intersections, arXiv:1409.0771.
25. J. Kirby, The theory of the exponential differential equations of semiabelian varieties *Selecta Math.* **15** (2009), 445–486.
26. J. Kirby, Variants of Schanuel's conjecture, manuscript, 2007.
27. J. Kirby and B. Zilber, The uniform Schanuel conjecture over the real numbers, *Bulletin London Math Soc.* **38** (2006), 568–570.
28. J. Kirby and B. Zilber, Exponentially closed fields and the Conjecture on Intersections with Tori, *Ann. Pure Appl. Logic* **165** (2014), 1680–1706.
29. B. Klingler and A. Yafaev, The André-Oort conjecture, *Annals* **180** (2014), 867–925.
30. B. Klingler, E. Ullmo, and A. Yafaev, The hyperbolic Ax-Lindemann-Weierstrass conjecture, 2013 manuscript.
31. S. Lang, *Introduction to transcendental numbers,* Addison-Wesley, Reading, 1966.
32. S. Lang, *Algebra,* Revised Third Edition, Graduate Texts in Mathematics **211**, Springer, New York, 2002. (Or earlier editions published by Addison-Wesley.)
33. S. Lang, *Elliptic functions,* second edition, Graduate Texts in Mathematics **112**, Springer, New York, 1987.
34. K. Mahler, On algebraic differential equations satisfied by automorphic functions, *J. Austral. Math. Soc.* **10** (1969), 445–450.

35. D. Masser, Heights, Transcendence, and Linear Independence on Commutative Group Varieties, Lecture Notes in Mathematics **1819**, Amoroso and Zannier, eds, 1-51, Springer-Verlag, Berlin, 2003.
36. D. Masser and U. Zannier, Torsion anomalous points and families of elliptic curves, *C. R. Math. Acad. Sci. Paris* **346** (2008), 491–494.
37. D. Masser and U. Zannier, Torsion anomalous points and families of elliptic curves, *Amer. J. Math.* **132** (2010), 1677–1691.
38. D. Masser and U. Zannier, Torsion points on families of squares of elliptic curves, *Math. Ann.* **352** (2012), 453–484.
39. G. Maurin, Courbes algébriques et équations multiplicatives, *Math. Annalen* **341** (2008), 789-824.
40. G. Maurin, Équations multiplicatives sur les sous-variétés des tores, *IMRN* **2011**.
41. B. J. J. Moonen, Linearity properties of Shimura varieties, I, *J. Alg. Geom.* **7** (1998), 539–567.
42. D. Mumford, *Algebraic Geometry I: Complex Projective Varieties,* Grundlehren der math. Wiss. **221**, Corrected second printing, Springer, Berlin, 1976.
43. Yu. V. Nesterenko, *Algebraic Independence,* TIFR Mumbai/Narosa, 2009.
44. F. Oort, Canonical lifts and dense sets of CM points, *Arithmetic Geometry, Cortona, 1994,* 228–234, F. Catanese, editor, Symposia. Math., XXXVII, CUP, 1997.
45. M. Orr, Families of abelian varieties with many isogenous fibres, *Crelle,* to appear.
46. Y. Peterzil and S. Starchenko, Uniform definability of the Weierstrass $\wp$ functions and generalized tori of dimension one, *Selecta Math. N. S.* **10** (2004), 525–550.
47. Y. Peterzil and S. Starchenko, Definability of restricted theta functions and families of abelian varieties, *Duke Math. J.* **162** (2013), 731–765.
48. J. Pila, O-minimality and the André-Oort conjecture for $\mathbb{C}^n$, *Annals* **173** (2011), 1779–1840.
49. J. Pila, Modular Ax-Lindemann-Weierstrass with derivatives, *Notre Dame J. Formal Logic* **54** (2013), 553–565 (Oléron proceedings).
50. J. Pila, Special point problems with elliptic modular surfaces, *Mathematika* **60** (2014), 1-31.
51. J. Pila and J. Tsimerman, The André-Oort conjecture for the moduli space of abelian surfaces, *Compositio* **149** (2013), 204–216.
52. J. Pila and J. Tsimerman, Ax-Lindemann for $\mathcal{A}_g$, *Annals* **179** (2014), 659–681.
53. J. Pila and A. J. Wilkie, The rational points of a definable set, *DMJ* **133** (2006), 591–616.
54. J. Pila and U. Zannier, Rational points in periodic analytic sets and the Manin-Mumford conjecture, *Rend. Lincei Mat. Appl.* **19** (2008), 149–162.
55. R. Pink, A combination of the conjectures of Mordell-Lang and André-Oort, *Geometric methods in algebra and number theory,* F. Bogomolov, Y. Tschinkel, editors, pp 251–282, Prog. Math. **253**, Birkhauser, Boston MA, 2005.

56. R. Pink, A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang, 2005 preprint, available from the author's webpage.
57. J.-P. Serre, *A Course in Arithmetic,* Graduate Texts in Mathematics **7**, Springer, New York, 1973.
58. J. Tsimerman, Brauer-Siegel for arithmetic tori and lower bounds for Galois orbits of special points, *J. Amer. Math. Soc.* **25** (2012), 1091–1117.
59. J. Tsimerman, Ax-Schanuel and O-minimality, this volume.
60. E. Ullmo, Quelques applications du théorème de Ax-Lindemann hyperbolique, *Compositio* **150** (2014), 175–190.
61. E. Ullmo and A. Yafaev, Galois orbits and equidistribution of special subvarieties: towards the André-Ort conjecture, *Annals* **180** (2014), 823–865.
62. E. Ullmo and A. Yafaev, A characterisation of special subvarieties, *Mathematika* **57** (2011), 833–842.
63. E. Ullmo and A. Yafaev, Nombres de classes des tores de multiplication complexe et bornes inférieures pour orbites Galoisiennes de points spéciaux, arXiv:1209.0942.
64. E. Ullmo and A. Yafaev, Hyperbolic Ax-Lindemann in the Cocompact case, *Duke Math. J.*, **163** (2014), 433–463.
65. M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups,* Grundleheren der math. Wiss **326**, Springer, Berlin, 2000.
66. M. Waldschmidt, *Some Consequences of Schanuel's Conjecture,* Talk slides available from the author's website at www.math.jussieu.fr/ miw/
67. D. Zagier, Elliptic modular functions and their applications, in *The 1-2-3 of Modular Forms,* Universitext, Springer, Berlin, 2008.
68. U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry,* with appendices by D. Masser, *Annals of Mathematics Studies* **181**, Princeton University Press, 2012.
69. B. Zilber, Pseudo-exponentiation on algebraically closed fields of characteristic zero, *Ann. Pure Appl. Logic* **132** (2005), 67–95.
70. B. Zilber, Intersecting varieties with tori, 2001 preprint incorporated into [71].
71. B. Zilber, Exponential sums equations and the Schanuel conjecture, *J. London Math. Soc. (2)* **65** (2002), 27–44.
72. B. Zilber, *On transcendental number theory, classical analytic functions and Diophantine geometry,* Talk slides available from http://people.maths.ox.ac.uk/zilber/
73. B. Zilber, Model theory of special subvarieties and Schanuel-type conjectures, 2013 preprint available from http://people.maths.ox.ac.uk/zilber/

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