O-minimality and Diophantine geometry

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Abstract. This lecture is concerned with some recent applications of mathematical logic to Diophantine geometry. More precisely it concerns applications of o-minimality, a branch of model theory which treats tame structures in real geometry, to certain finiteness problems descending from the classical conjecture of Mordell.

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1. Introduction

This is a somewhat expanded version of my lecture at ICM 2014 in Seoul. It surveys some recent interactions between model theory and Diophantine geometry.

The Diophantine problems to be considered are of a type descending from the classical Mordell conjecture (theorem of Faltings). I will describe the passage from Mordell's conjecture to the far-reaching Zilber-Pink conjecture, which is very much open and the subject of lively study by a variety of methods on several fronts. The model theory is "o-minimality", which studies tame structures in real geometry, and offers powerful tools applicable to certain "definable" sets. In combination with an elementary analytic method for "counting rational points" it leads to a general result about the height distribution of rational points on definable sets. This result can be successfully applied to Zilber-Pink problems in the presence of certain functional transcendence and arithmetic ingredients which are known in many cases but seemingly quite difficult in general.

Both the methods and problems have connections with transcendental number theory. My further objective is to explain these connections and to bring out the pervasive presence of Schanuel's conjecture.

Though the broad family of Diophantine problems is the same, o-minimality is a rather different flavour of model theory to that employed in the Diophantine results of Hrushovski [62, 64] and the subsequent developments (e.g. [24, 129]; for more on "stability" and its applications see [63, 65]). However, both flavours involve fields with extra structure and hinge on suitable tame behaviour of the definable sets.

As there are excellent survey papers on these developments (e.g. [130, 131]), this exposition will stay at a broader level and keep technicalities to a minimum.

2. From Mordell to Zilber-Pink

The Mordell conjecture. Diophantine geometry deals in the first instance with the solution of systems of algebraic equations in integers and in rational numbers. It is a broad subject with a central place in number theory going back to antiquity. The problems we will consider are finiteness questions. One seeks to show that certain forms of Diophantine problems have only finitely many solutions, or a solution set that has a finite description in certain specific terms.

The ur-conjecture here is the Mordell conjecture asserting the finiteness of the number of rational points on curves of genus at least 2. For example, a non-singular plane quartic curve. This conjecture was proposed by Mordell [94] in 1922, and proved by Faltings [46] in 1983. In the meantime it evolved into the Mordell-Lang conjecture (ML; see Lang [77], I, 6.3) proved in the work of Faltings, Hindry, Laurent, McQuillan, Raynaud, Vojta, and others; see e.g. [90, 97, 18].

This was the first of three crucial steps in the evolution of the Mordell conjecture into what is known as the Zilber-Pink conjecture (ZP).

The Mordell-Lang conjecture. The first step, due to Lang (e.g. [76]), recasts the conjecture in terms of a subvariety (i.e. irreducible closed algebraic subset defined over $\mathbb{C}$; we identify varieties with their sets of complex points) V of a (semi-abelian) group variety X. The conjecture concerns the interaction of V with certain "special" subvarieties of X distinguished in terms of its group structure.

The simplest result of "Mordell-Lang" type concerns a curve $V \subset X = \mathbb{G}_{\mathrm{m}}^2$. Here $\mathbb{G}_{\mathrm{m}} = \mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathbb{C}^\times$ is the multiplicative group of non-zero complex numbers, so V is the set of solutions in $(\mathbb{C}^\times)^2$ of some irreducible (over $\mathbb{C}$) polynomial $F(x, y) = 0$. The result, which appears in Lang [76] is the following. *If there are infinitely many points $(\xi, \eta) \in V$ such that (ξ, η) is a torsion point of $(\mathbb{C}^\times)^2$, then F has either the form $x^n y^m = \zeta$ or $x^n = \zeta y^m$ for some non-negative integers n, m (not both zero) and root of unity ζ.* In the exceptional case V is a *torsion coset*: a coset of an irreducible algebraic subgroup (*subtorus*) of X by a torsion point, and, being positive-dimensional, contains infinitely many torsion points. Observe that a torsion point is a torsion coset (of the trivial group).

With a view to generalisations, torsion cosets of $X = \mathbb{G}_{\mathrm{m}}^n$ will be called "*special subvarieties*" and torsion points "*special points*". The (countable) collection of special subvarieties will be denoted $\mathcal{S} = \mathcal{S}_X$. For later use, general cosets of subtori will be called *"weakly special subvarieties"*. We observe that special points are Zariski dense in any special subvariety.

The Multiplicative Manin-Mumford conjecture, which is a special case of a theorem of Laurent [79] (for V defined over $\mathbb{Q}$ it may be deduced from results of Mann [82]; see Dvornicich-Zannier [42] for generalisations, see also an independent proof by Sarnak [126]), asserts the converse. Consider a subvariety $V \subset X$.

($*$) *If special points are Zariski-dense in V then V is a special subvariety.*

Since the Zariski-closure of any set of points consists of finitely many irreducible components ($*$) may be equivalently formulated as ($*'$) or ($*''$) as follows.

($*'$) *A component of the Zariski closure of a set of special points is special.*

($*''$) V *contains only finitely many maximal special subvarieties.*

If one replaces the group of torsion points by the division group Γ of a finitely generated subgroup of $\mathbb{G}_{\mathrm{m}}^n$, and takes special subvarieties to be cosets of subtori by elements of Γ, then ($*$) is the "Multiplicative Mordell-Lang conjecture", a theorem of Laurent [79].

The Manin-Mumford conjecture (MM; proved by Raynaud [122, 123]) is the statement ($*$) for a subvariety of an abelian variety with its torsion cosets as "special subvarieties" (the original formulations of Manin and Mumford concerned a curve of genus at least two embedded in its Jacobian); for the division group of a finitely generated subgroup it is ML. Note that ML, in both the multiplicative and abelian settings, is ineffective (one cannot bound the height of points, though one can bound their number).

While there is no explicit mention of rational points in the formulation of ML, implications for these, including the original Mordell conjecture, are recovered via the Mordell-Weil theorem: the group of rational points on an abelian variety over a number field is finitely generated ([78], I.4.1, [18]).

The André-Oort conjecture. André [1] and Oort [98] made conjectures analogous to the Manin-Mumford conjecture where the ambient variety X is a *Shimura variety* (the latter partially motivated by a conjecture of Coleman [33]). A combination of these has become known as the André-Oort conjecture (AO).

Shimura varieties have a central role in arithmetic geometry, in particular in the theory of automorphic forms see e.g. [91]. As the formal definition (see e.g. [91, 92]) is rather involved, I will just give some examples. The simplest examples are modular curves, for example the curve $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ whose set of complex points is just the affine line $\mathbb{C}$, parameterising isomorphism classes (over $\mathbb{C}$) of elliptic curves by their j-invariant (see e.g. [155]). However, the André-Oort conjecture is trivial for one-dimensional ambient varieties; the simplest non-trivial cases concern cartesian products of modular curves. The paradigm examples of Shimura varieties are the Siegel modular varieties $\mathcal{A}_g$ parameterising principally polarised abelian varieties of dimension g [17].

Associated with a Shimura variety X is a countable collection $\mathcal{S} = \mathcal{S}_X$ of *special subvarieties*, the zero-dimensional ones being called *special points*. For example, in $Y(1)^2$, a special subvariety of dimension 1 is: a "vertical line" $x = j_0$ or "horizontal line" $y = j_0$ where j_0 is the j-invariant of an elliptic curve with *complex multiplication* ("CM"; i.e. a "singular modulus" see e.g. [155], §6); or the zero set of a *modular polynomial* $\Phi_N(x, y)$ (see e.g. [155]). The other special subvarieties are $Y(1)^2$ itself and special points, being the points for which both coordinates are singular moduli. There is also a larger (uncountable) collection of *weakly special subvarieties* which includes, in addition, all vertical and horizontal lines. In $\mathcal{A}_g$ the special subvarieties become rather complicated to describe, but special points are again those $x \in \mathcal{A}_g$ for which the corresponding abelian variety A_x is CM (see e.g. [98]). Special points are Zariski dense in any special subvariety, and AO asserts the converse:

Let X be a Shimura variety and $V \subset X$ a subvariety. Then $(*)$ holds.

Equivalently, AO may be formulated as $(*')$, or $(*'')$ which we take as the "official" version.

Conjecture 2.1 (AO). *Let X be a Shimura variety and $V \subset X$. Then V contains only finitely many maximal special subvarieties.*

The simplest non-trivial case of AO, for $Y(1)^2$ (and more generally products of two modular curves), was established unconditionally by André [2]. AO is open in general, though it is known to be true under the Generalised Riemann Hypothesis for CM fields (by work of Edixhoven, Klingler, Ullmo, and Yafaev [43, 45, 71, 144]) and it is known unconditionally in several cases and under various additional hypotheses on the special points in question (see [152]). In particular, AO for arbitrary products of modular curves was affirmed using o-minimality and point-counting in [108]. We will describe this approach below as well as further results which have been established by the same methods. Though unconditional, theses results are ineffective in that they do not produce a bound on the height of the special points. The only effective result known is for products of two modular curves, due recently to Kühne [74] and Bilu-Masser-Zannier [16].

The broader class of *mixed Shimura varieties* (see e.g. [120]) includes for example the "mixed" variety $\mathcal{X}_g$ associated with $\mathcal{A}_g$, namely $\mathcal{A}_g$ fibered at each point by the abelian variety parameterised by that point, and analogous varieties with additional level structure (see e.g. [17]). These include elliptic modular surfaces. More exotic examples, like the Poincaré bi-extension ([13]), include copies of $\mathbb{G}_m$ as special subvarieties. The second step in the evolution of ZP is to enlarge the category of "ambient" varieties to that of mixed Shimura varieties, which also have a geometrically defined collection of "special subvarieties" [120].

This gives a class of varieties in which all the Diophantine problems so far considered can be comprehended. The "special point conjecture" $(*)$ in this setting was formulated by André [1]. It contains AO and MM for CM abelian varieties, but it does not include the full MM or ML statements.

The Zilber-Pink conjecture. One further extension, which significantly enlarges its scope and reach, gives the Zilber-Pink conjecture. The setting is again $V \subset X$ where X is a mixed Shimura variety, but instead of special subvarieties T *contained in* V, we consider (components of) intersections $V \cap T$, with T a special subvariety, which are *atypical in dimension* (see below).

This idea has three independent sources, expressed in different formulations. Zilber [159] formulated a version ("CIT") in the setting of semi-abelian varieties, motivated by his work on the model theory of complex exponentiation (see below). Bombieri-Masser-Zannier [20] proved a theorem and formulated a conjecture about curves in $\mathbb{G}_{\rm m}^n$ originating with a question of Schinzel [132], leading to a result for intersections of subvarieties with one-dimensional tori and the formulation of a general conjecture in [21]. Pink [121] formulated a conjecture encompassing MM, ML, and AO by the same device of "unlikely intersections", meaning intersections of a variety $V \subset X$ with special subvarieties of codimension exceeding $\dim V$.

The irreducible components of the intersection of two subvarieties $V, W \subset X$ typically have dimension

$$\dim V + \dim W - \dim X,$$

as one would expect by "counting conditions" (and never less if X is smooth [95]).

Definition 2.2. Let X be a mixed Shimura variety with collection $\mathcal{S}$ of special subvarieties, and let $V \subset X$. An irreducible component $A \subset V \cap T$, where $T \in \mathcal{S}$ is called an *atypical subvariety (of V in X)* if

$$\dim A > \dim V + \dim T - \dim X.$$

Conjecture 2.3 (ZP). *Let X be a mixed Shimura variety and $V \subset X$. Then V contains only finitely many maximal atypical subvarieties.*

This is essentially the formulation of Zilber and Bombieri-Masser-Zannier in Pink's setting. There are several alternative formulations; see [22] for a proof that they are equivalent in the multiplicative setting. As it is always atypical for a proper subvariety of X to contain a special subvariety, ZP for X and all its special subvarieties implies the assertion $(*'')$, the "special point" or "generalised André-Oort" conjecture for X, via an inductive argument.

There has been a lot of work on problems subsumed within ZP. Nevertheless it is open even in the multiplicative case. I will describe a theorem established in work of Bombieri, Masser, Zannier, and Maurin that affirms ZP for a curve in $\mathbb{G}_{\mathrm{m}}^n$.

An atypical subvariety of a curve in $\mathbb{G}_{\mathrm{m}}^n$ is either a point in its intersection with a subgroup of codimension at least 2, or the curve itself if it is contained in a subgroup of codimension 1. The following definition is convenient.

Definition 2.4. For a mixed Shimura variety X with its collection $\mathcal{S}$ of special subvarieties and a non-negative integer k we let $\mathcal{S}^{[k]}$ denote the (countable) union of all special subvarieties of X of codimension $\leq k$.

Theorem 2.5 ([20, 89, 22]). *Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be a curve defined over $\mathbb{C}$. If V is not contained in a proper special subvariety then $V \cap \mathcal{S}^{[2]}$ is a finite set.* $\square$.

An alternative proof (for V defined over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$) is given in [19] and, in conjunction with the "Bounded height theorem" of Habegger [52] leads to an effective result [53]. A proof of the main result of [20] using o-minimality and point-counting has been developed by Capuano [28].

ZP formally implies ML ([159, 121]), which may be seen in the multiplicative setting for curves as follows. Let $V \subset \mathbb{G}_{\mathrm{m}}^2$ be a curve and suppose that $c_1, \ldots, c_k \in \mathbb{C}^\times$ are multiplicatively independent (no nontrivial monomial on them gives unity). Define

$$V^* = \{(x, y, z_1, \ldots, z_k) \in \mathbb{G}_{\mathrm{m}}^{2+k} : (x, y) \in V, z_i = c_i, i = 1, \ldots, k\}.$$

Two multiplicative conditions on $(x, y, \overline{z}) \in V^*$ will in general mean that x and y belong to the division closure of the multiplicative group $\langle c_1, \ldots, c_k\rangle$ generated by $c_1, \ldots, c_k$. Thus ZP for all $\mathbb{G}_{\mathrm{m}}^n$ implies ML for all $\mathbb{G}_{\mathrm{m}}^n$.

I do not give a survey of results. The known results for abelian varieties are less complete than those for $\mathbb{G}_{\mathrm{m}}^n$, and in the Shimura setting less complete still. Below I will discuss various specific problems that have been tackled using o-minimality and point-counting. See Zannier [156] for further discussion and references on ZP as well as more general problems under the rubric of "unlikely intersections", and [157] for some specific problems and applications. See also Chambert-Loir [29]. For analogous results in other settings see [85, 30].

3. Transcendental Number Theory

Classical results. Transcendental number theory is concerned primarily with the algebraic nature of the values of special functions, especially the exponential function. I want to mention two famous results: Lindemann's theorem (also known as the Lindemann-Weierstrass theorem) and Baker's theorem (see e.g. [8]). Here $\log x$ means any determination of the logarithm of $x \in \mathbb{C}^\times$.

Theorem 3.1 (Lindemann (-Weierstrass)). *Let $x_1, \ldots, x_n \in \overline{\mathbb{Q}}$ be linearly independent over $\mathbb{Q}$. Then $e^{x_1}, \ldots, e^{x_n}$ are algebraically independent over $\mathbb{Q}$.* □

Theorem 3.2 (Baker). *Suppose that $x_1, \ldots, x_n \in \overline{\mathbb{Q}}$. If $\log x_1, \ldots, \log x_n$ are linearly independent over $\mathbb{Q}$ then they are linearly independent over $\overline{\mathbb{Q}}$.* □

Baker's theorem has been partially extended to elliptic and abelian functions in work of Baker, Bertrand, Masser, Philippon, Wüstholz and others (see e.g. [9]). These developments also impacted substantially on Diophantine problems, but I want to note in particular that the Masser-Wüstholz isogeny estimates led to a new proof [86] of the Mordell conjecture. More recently, Kühne [74] uses quantitative results for linear forms in (elliptic and classical) logarithms in his unconditional proof of AO for products of two modular curves.

So the methods of Diophantine geometry and transcendence theory are cognate; but the underlying conjectures are also cognate in the work of Zilber on the model theory of exponentiation described below.

Schanuel's conjecture. Schanuel's conjecture (SC; see Lang [77], p31) seems to encapsulate all reasonable transcendence properties of the exponential function.

Conjecture 3.3 (SC). *Let $z_1, \ldots, z_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then*

$$\text{tr. deg.}_{\mathbb{Q}}\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n.$$

The special case with all the z_i algebraic recovers Lindemann's theorem. The special case with all the $\exp z_i$ algebraic is open for $n \geq 2$, the best result known towards "algebraic independence of logarithms" is Baker's theorem.

"Ax-Schanuel". Ax [5] (see also [4]) established Schanuel's conjecture in the setting of a differential field (apparently also conjectured by Schanuel; see [5]); this theorem is known as "Ax-Schanuel".

Let $\mathbb{Q} \subset C \subset K$ be a tower of fields and $\{D_1, \ldots, D_m\}$ a set of commuting derivations of K with $C = \bigcap_\mu \ker D_\mu$. By "rank" below we mean rank over K.

Definition 3.4. Elements $x_1, \ldots, x_n \in K$ are called *linearly independent over* $\mathbb{Q}$ *modulo* C if there is no nontrivial relation $\sum_\nu q_\nu x_\nu = c$ where $q_\nu \in \mathbb{Q}$, $c \in C$.

Ax's theorem is then the following. Condition (a) encapsulates "$y_\nu = e^{x_\nu}$" in a general differential field. However, by the Seidenberg embedding theorem [133, 134], a finitely generated differential field may be embedded into a field of meromorphic functions.

Theorem 3.5 ("Ax-Schanuel"). *Let* $x_\nu, y_\nu \in K^\times, \nu = 1, \ldots, n$, *with*

(a) for all μ, ν, $D_\mu y_\nu = y_\nu D_\mu x_\nu$*:*

(b) the x_ν *are linearly independent over* $\mathbb{Q}$ *modulo* C *[or (b'), the* y_ν *are multiplicatively independent over* C*].*

Then

$$\text{tr. deg.}_C C(x_1, \ldots, x_n, y_1, \ldots, y_n) \geq n + \text{rank}\big(D_\mu x_\nu\big)_{\mu=1,\ldots,m,\nu=1,\ldots,n}.$$

This implies a (weaker) variant in the complex setting that will be important in the sequel. A statement along these lines was established by Ax [6] in the semiabelian setting.

We consider $\pi : \mathbb{C}^n \to \mathbb{G}_\mathrm{m}^n$ given by $z \mapsto e(z) = \exp(2\pi i z)$ on each coordinate. Fix $V \subset \mathbb{G}_\mathrm{m}^n$. Ax-Schanuel implies that the "best" intersections of $\pi^{-1}(V)$ with algebraic subvarieties $W \subset \mathbb{C}^n$ are achieved by weakly special W. We formulate a precise statement as follows.

Definition 3.6. 1. A *component with respect to* V is a complex analytically irreducible component A of $W \cap \pi^{-1}(V)$ for some irreducible algebraic $W \subset \mathbb{C}^n$.

2. If A is a component w.r.t. V we define its *defect* $\delta(A)$ to be $\dim \mathrm{Zcl}(A) - \dim A$ where $\mathrm{Zcl}(A)$ is the Zariski closure of A.

3. A component A w.r.t. $V$is called *optimal* for V if there is no component B w.r.t. V with $A \subset B$, $A \neq B$, and $\delta(B) \leq \delta(A)$. Note that if A is optimal it must be a component of $\mathrm{Zcl}(A) \cap \pi^{-1}(V)$.

4. A component A w.r.t. V is called *weakly special* if it is a component of $W \cap \pi^{-1}(V)$ for some weakly special $W = \mathrm{Zcl}(A)$.

Then Ax-Schanuel implies the following statement (see [111]).

Theorem 3.7. *An optimal component w.r.t.* $V \subset \mathbb{G}_\mathrm{m}^n$ *is weakly special.* □

In particular, we have $\delta(W) = 0$ just if $W \subset \pi^{-1}(V)$. An optimal component with defect zero is then a maximal irreducible algebraic subvariety contained in $\pi^{-1}(V)$.

Corollary 3.8. *A maximal algebraic subvariety* $W \subset \pi^{-1}(V)$ *is weakly special.* □

Another way to formulate the corollary is that if algebraic functions $z_1, \ldots, z_m$ (say elements of the function field $\mathbb{C}(W)$) are linearly independent modulo constants (i.e. the locus $z_1, \ldots, z_m$ is not contained in any weakly special subvariety) then the exponentials $\exp z_1, \ldots, \exp z_n$ are algebraically independent over $\mathbb{C}$, which is a functional analogue of Lindemann's theorem. Accordingly I call the assertion of the corollary and its various analogues "Ax-Lindemann"; see also [15].

Tsimerman [141] has recently given a new proof of Ax-Schanuel via o-minimality and point-counting.

4. Model Theory

Model theory of $\mathbb{C}$ and $\mathbb{R}$. See e.g. [160]. The first-order theory of the complex field $(\mathbb{C}, +, \times, 0, 1)$ is just the theory of algebraically closed fields of characteristic zero and is *categorical* (has a unique model up to isomorphism) in every uncountable power. This is a very strong property of a theory. Algebraically closed fields are also "strongly minimal": the definable subsets of $\mathbb{C}$ are either finite or cofinite. Indeed, by quantifier elimination, the definable (with parameters from $\mathbb{C}$) subsets of $\mathbb{C}^n$ are precisely the constructible sets: the Boolean algebra generated by the zero-sets of polynomials (with coefficients in $\mathbb{C}$) in $\mathbb{C}^n$. The theory is also decidable.

Strong minimality fails for the real field as the order is definable, whence intervals are definable; however (Tarski-Seidenberg theorem) the definable sets are just the semi-algebraic sets: finite boolean combinations of sets defined by finitely many polynomial equalities and inequalities. The theory is again decidable (Tarski [139]). A definable subset of $\mathbb{R}$ is still relatively simple, being a finite union of points and (possibly unbounded) intervals.

Model theory of complex exponentiation. The integers are definable in the complex numbers with exponentiation:

$$\mathbb{Z} = \{z \in \mathbb{C} : \forall w \in \mathbb{C}\big(\exp(w) = 1 \to \exp(zw) = 1\big)\}.$$

Therefore, by Gödel's Theorem, the first-order theory of $\mathbb{C}_{\exp} = (\mathbb{C}, +, \times, 0, 1, \exp)$ is undecidable and the definable sets can be "wild". The theory is very far from categorical. Nevertheless, Zilber showed that categoricity can be recovered if one works with a stronger infinitary logic. He used a Hrushovski-style construction in which Schanuel's conjecture plays a fundamental role to construct [158] a candidate "logically perfect" algebraically closed field of power continuum with a "standard" (cyclic kernel) exponentiation, $\mathbb{B}_{\exp}$, and conjectured that this field is isomorphic to $\mathbb{C}_{\exp}$ (entailing SC and more; see e.g. [34, 35]).

Considering the first-order theory of this structure led Zilber to his "CIT" conjecture [159] in the setting of $\mathbb{G}_{\mathrm{m}}^n$ and more generally semiabelian varieties: it is the "difference" between SC and a uniform version of SC that admits first-order axiomatization.

Conjecture 4.1 (Uniform Schanuel Conjecture; USC). *Let $V \subset \mathbb{C}^{2n}$ be a closed algebraic set defined over $\mathbb{Q}$ with $\dim V < n$. There exists a finite set $\mu(V)$ of proper $\mathbb{Q}$-linear subspaces of $\mathbb{C}^n$ such that if*

$$(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \in V$$

then there is $M \in \mu(V)$ and $\overline{k} \in \mathbb{Z}^n$ and such that $(z_1+2\pi ik_1, \ldots, z_n+2\pi ik_n) \in M$. Moreover if M is codimension 1 (in $\mathbb{C}^n$) then $k = 0$.

Part of this program has been carried out for the j-function by Harris [59] and more generally for Shimura curves [36] by Daw-Harris. A very general picture of "special subvarieties" and generalised Schanuel conjectures is set out in [161].

Model theory of real exponentiation. O-minimality grew out of the attempt to understand the model theory of the real field with exponentiation. The real exponential has no overt periodic behaviour, thus no obvious source of "Gödelian problems". Upon proving the decidability of the real field, Tarski [139] asked whether the theory of the real field with exponentiation, i.e. the structure $\mathbb{R}_{\exp} = (\mathbb{R}, +, \times, 0, 1, \exp)$, is decidable.

In studying this question, van den Dries [38] noted the key role played by the above mentioned finiteness property of semi-algebraic sets and formulated the condition "a definable subset of $\mathbb{R}$ is a finite union of points and intervals" that is the key defining property of a general theory of "o-minimal structures" subsequently undertaken by Pillay and Steinhorn [118] (see also [73, 119]). They prove the fundamental Cell Decomposition Theorem, from which the remarkable tameness and uniformity properties of o-minimal structures flow. For completeness I include a "model-theory free" definition of an o-minimal structure over the real field. Being a "structure" means that the sets in $\bigcup_n \mathbf{\Sigma}_n$ are precisely the definable sets (with parameters) in a suitable expansion of the real field.

Definition 4.2. 1. A *pre-structure* is a sequence $\mathbf{\Sigma} = (\mathbf{\Sigma}_n)_{n=1,2,\ldots}$ where each $\mathbf{\Sigma}_n$ is a collection of subsets of $\mathbb{R}^n$.

2. A pre-structure $\mathbf{\Sigma}$ is called a *structure (over the real field)* if, for all $n, m = 1, 2, \ldots$ with $m \geq n$, the following conditions are satisfied:
(i) $\mathbf{\Sigma}_n$ is a Boolean algebra
(ii) $\mathbf{\Sigma}_n$ contains every semi-algebraic subset of $\mathbb{R}^n$
(iii) if $A \in \mathbf{\Sigma}_n$ and $B \in \mathbf{\Sigma}_m$ then $A \times B \in \mathbf{\Sigma}_{n+m}$
(iv) if $A \in \mathbf{\Sigma}_n$ then $\pi(A) \in \mathbf{\Sigma}_m$ where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is a coordinate projection.
If $\mathbf{\Sigma}$ is a structure and $Z \subset \mathbb{R}^n$ we say that Z is *definable* in $\mathbf{\Sigma}$ if $Z \in \mathbf{\Sigma}_n$.

3. A structure $\mathbf{\Sigma}$ is called *o-minimal* if the boundary of each set in $\mathbf{\Sigma}_1$ is a finite set of points.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *definable* in a structure $\mathbf{\Sigma}$ if its graph is. If $A, \ldots, f, \ldots$ are sets or functions then $\mathbb{R}_{A,\ldots,f,\ldots}$ denotes the smallest structure containing $A, \ldots, f, \ldots$. By a *definable family* of sets we mean a definable subset $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ which we view as a family of fibres $Z_y \subset \mathbb{R}^n$ as y varies over the projection of Z onto $\mathbb{R}^m$ (which is definable, along with all the fibres Z_y). A family of functions is definable if the family of their graphs is.

In the sequel, a *definable set* will mean a definable set in some o-minimal structure over $\mathbb{R}$. The o-minimal condition has very strong consequences for definable sets and functions. For example, a definable function is continuous (and also differentiable) except at finitely many points. Moreover, in a definable family of functions, the number of points of discontinuity (or of non-differentiability) is bounded uniformly for all members of the family. Another example (relevant later): in a definable family, the set of parameters for which the fibre has a given dimension is definable. For these and other properties see van den Dries [39].

Of course the theory is only useful if there are non-trivial examples. van den Dries observed that the o-minimality of the structure $\mathbb{R}_{\mathrm{an}}$ generated by all *restricted analytic functions*, i.e. all $f : T \to \mathbb{R}$ where $T \subset \mathbb{R}^n$ is a compact box and f is analytic on an open neighbourhood of T, follows from a fundamental theorem of Gabrielov [49] on subanalytic functions.

This is a large and useful structure, but did not answer the question raised by van den Dries for the (unrestricted) exponential function. This was affirmed by Wilkie [148] (who in fact established the "model-completeness" of $\mathbb{R}_{\exp}$, giving its o-minimality in view of the results of Khovanskii [68]).

Theorem 4.3 (Wilkie [148]). *The structure* $\mathbb{R}_{\exp}$ *is o-minimal.* $\square$

The structure $\mathbb{R}_{\mathrm{an},\,\exp}$ generated by the union of $\mathbb{R}_{\exp}$ and $\mathbb{R}_{\mathrm{an}}$ is o-minimal ([41], see also [40]). Note that in general the structure generated by the union of o-minimal structures need not be o-minimal ([125]): there is no "largest" minimal structure over $\mathbb{R}$. Larger and stranger o-minimal structures followed ([138, 125]), but $\mathbb{R}_{\mathrm{an},\,\exp}$ suffices for all the applications we will consider ($\mathbb{R}_{\mathrm{an}}$ doesn't).

Macintyre and Wilkie [80] affirmed Tarski's original question assuming SC.

Theorem 4.4 ([80]). *Assuming SC, the theory of* $\mathbb{R}_{\exp}$ *is decidable.* $\square$

5. Counting Points

Counting rational points in algebraic varieties. Counting solutions to a Diophantine equation up to a given height T and probing the behaviour of their number $N(T)$ as $T \to \infty$ is a well-travelled path in Diophantine geometry, especially in connection with Waring's problem and, more recently, the Batyrev-Manin conjectures; see e.g. [61]. For example, it is believed (see [137]) that there is no positive integer n which can be written as a sum of two fifth powers in two essentially different ways. This amounts to saying that all solutions in non-negative integers to

$$X^5 + Y^5 = U^5 + V^5$$

are *trivial* in that $\{X, Y\} = \{U, V\}$. Hooley proved (see [60], improved in [26]) that there are at most $O_\epsilon(T^{5/3+\epsilon})$ non-trivial solutions with $0 \le X, Y, U, V \le T$, which are thus dominated by the $2T^2 + O(T)$ trivial ones. The conjectures of Bombieri and Lang (see e.g. [18], 14.3.7, [61], F.5.2) imply that all but finitely many rational points on a variety lie in the geometrically defined "special set".

Thus, conjecturally, general Diophantine problems, like the special ones of Mordell-Lang type, only have infinitely many solutions if there is a "reason".

Counting rational points in definable sets. Prompted by questions posed by Sarnak (motivated by his analytic proof of the multiplicative Manin-Mumford conjecture [126]; see also [127]), Bombieri-Pila [23] counted integer points up to a given height on plane curves in various categories (convex, transcendental real-analytic, algebraic) by an elementary real-variable method. The same idea was applied to rational points on a real analytic plane curve in [104]. Heath-Brown [60] introduced a variant p-adic "determinant method" applicable to rational points on algebraic varieties in any dimension, which prompted the idea of applying the "real" version to count rational points in higher-dimensional sets defined by analytic conditions.

We define the *height* of a rational number $x = a/b$ in lowest terms (i.e. $\gcd(a,b) = 1$) by $H(x) = \max(|a|, |b|)$, and the *height* of a tuple $x = (x_1, \ldots, x_n) \in \mathbb{Q}^n$ by $H(x) = \max(H(x_i), i = 1, \ldots, n)$. For a set $Z \subset \mathbb{R}^n$ we put

$$Z(\mathbb{Q}, T) = \{x \in Z : x \in \mathbb{Q}^n, H(x) \leq T\},$$

and define the *counting function* of Z by

$$N(Z, T) = \#Z(\mathbb{Q}, T).$$

We would like to have a result expressing that a "reasonable" set $Z \subset \mathbb{R}^n$ has "few" rational points unless there is a "reason". We will take "reasonable" to mean definable. If Z contains positive dimensional semi-algebraic subsets (e.g. a piece of a line or circle) then these may contain quite a lot of algebraic points; thus we will exclude such subsets from the counting.

Definition 5.1. Let $Z \subset \mathbb{R}^n$. We define the *algebraic part* Z^{alg} of Z to be the union of positive-dimensional connected semi-algebraic subsets of Z.

The algebraic part is a coarse analogue of the "special set". But one cannot expect finiteness of rational points outside the algebraic part in view of curves like $y = 2^x$. The following theorem provides a sense in which there are "few" rational points outside the algebraic part of a definable set.

Theorem 5.2 (Counting Theorem; Pila-Wilkie [116]). *Let $Z \subset \mathbb{R}^n$ be a definable set and $\epsilon > 0$. Then there is a constant $c(Z, \epsilon)$ such that, for all T,*

$$N(Z - Z^{\mathrm{alg}}, T) \leq c(Z, \epsilon) T^{\epsilon}. \quad \square$$

Suppose Z is the image of a map $\phi : (0,1)^k \to \mathbb{R}^n$. The underlying analytic idea of [23], extended to higher dimension in [105], is that $Z(\mathbb{Q}, T)$ is contained in the intersection $Z \cap V$ of Z with "few" hypersurfaces V of some suitable degree $d = d(Z, \epsilon)$, whose number depends on the maximum size of coordinate functions of ϕ and some number (depending on ϵ) of their partial derivatives. The key to proving the Counting Theorem is a parameterisation theorem ([116], Thm 2.3) by means of which the intersections $Z \cap V$ can be realised as images of finitely many maps whose derivatives up to a given order are bounded uniformly as V varies in the family of all hypersurfaces of given degree. This o-minimal version of the "Algebraic Lemma" of Yomdin-Gromov [154, 50] allows the analytic idea to be applied inductively; it yields a result which is uniform for definable families.

One can establish a bound of the same quality for algebraic points of bounded degree. For a definable set Z and $k \geq 1$ put

$$Z(k, T) = \{x \in Z : [\mathbb{Q}(x_i) : \mathbb{Q}] \leq k, H(x_i) \leq T, i = 1, \ldots, n\},$$

$$N_k(Z, T) = \#Z(k, T),$$

where $H(x)$ here is the multiplicative height (see [18], 1.5.7) of an algebraic number.

Then for definable Z, positive k and $\epsilon > 0$ we have

$$N_k(Z - Z^{\mathrm{alg}}, T) \leq c(Z, k, \epsilon)T^{\epsilon}.$$

The result is again uniform for Z in definable families.

A further refinement, necessitated by applications, makes the result look more like a generalised "special point" statement. Namely, one shows that $Z(k, T)$ is contained in "few" definable connected subsets which locally coincide with semialgebraic sets ("blocks"), and which come from finitely many (depending on Z, k, ϵ) definable families. For the precise statement I refer to [108].

To make the result effective for a particular o-minimal structure one would need an effective bound on the number of connected components of a definable set in that structure, as a function of the "complexity" of the formula which defines it. This is known in only special cases [11].

Non-archimedean analogues have been announced by Cluckers-Comte-Loeser [31]. For an earlier result about integer points on definable curves, including a much stronger bound for curves in $\mathbb{R}_{\mathrm{an}}$, see Wilkie [151]. For a still earlier application of Khovanskii theory to "unlikely intersections" see Cohen-Zannier [32].

Wilkie's conjecture. The Counting Theorem cannot be much improved in general (see [105]); in particular one cannot in general replace the $\ll_{\epsilon} T^{\epsilon}$ bound with a power of $\log T$. However, Wilkie ([116], 1.11) has conjectured:

Conjecture 5.3. *Let Z be definable in $\mathbb{R}_{\exp}$. Then there are constants $C(Z), c(Z)$ such that*

$$N(Z - Z^{\mathrm{alg}}, T) \leq C(\log T)^{c}.$$

Partial results are established in [27, 66, 67, 107], some for sets definable in the larger o-minimal structure $\mathbb{R}_{\mathrm{Pfaff}}$. It would be nice to go further and establish such results for the structures in which (the restrictions of) the uniformising maps of mixed Shimura varieties are definable.

6. O-minimality and "special point" problems

The setting. The basic strategy is due to Zannier, who proposed using the Counting Theorem to give a new proof of MM. This was implemented in [117]. He and Masser saw that the same strategy could be applied to certain "relative Manin-Mumford" problems posed by Masser ([87], further described below). These turn out to be also special cases of ZP.

The generalisation of the Counting Theorem to algebraic points [106], and the analogies between MM and AO (as highlighted e.g. in [153, 142]) prompted the idea of applying the same idea to the latter problem: for products of $Y(1)$, for example, the "special points" correspond to tuples of j-invariants of elliptic curves with complex multiplication. These are precisely the points $j(z)$ where $z \in \mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ is a quadratic irrationality, thus they correspond to algebraic points in $\mathbb{H}^n$ of bounded degree.

Indeed all the "special point" problems take a similar form. The ambient variety X has a transcendental uniformisation

$$\pi : U \to X$$

by a complex domain U with certain properties. Examples (set $e(u) = \exp(2\pi i u)$):

(i) $X = \mathbb{G}_{\mathrm{m}}^n$, $U = \mathbb{C}^n$, $\pi(u_1, \ldots, u_n) = (e(u_1), \ldots e(u_n))$;
(ii) X an abelian variety, $U = \mathbb{C}^{\dim X}$, π periodic under a suitable lattice Λ;
(iii) $X = Y(1)^n$, $U = \mathbb{H}^n$, and $\pi(u_1, \ldots, u_n) = (j(u_1), \ldots j(u_n))$;
(iv) $X = \mathcal{A}_g$, $U = \mathbb{H}_g$, Siegel upper half-space, π is $\mathrm{Sp}_{2g}(\mathbb{Z})$-invariant [17].

While a general abelian variety is not a mixed Shimura variety, it is a subvariety of one, and the "induced" ZP on its subvarieties is equivalent to the statement of ZP when X is endowed with its torsion cosets as "special subvarieties" [121].

This picture is essentially the same for any Shimura (or mixed Shimura) variety X, where U may be taken to be an open domain in some ambient complex affine space, π is invariant under a discrete arithmetic subgroup Γ of a real algebraic group G acting on U as biholomorphisms, and where U and the (graph of the) G action on it are semi-algebraic.

In each case, the pre-images of special points are algebraic points of bounded degree, when considered in suitable real coordinates on U (e.g. for an abelian variety we take a basis of Λ to define our real coordinates). Thus the search for special points in $V \subset X$ can be translated into a search for their pre-images (which we will also just call special points) in $\pi^{-1}(V)$. This set is in general far from algebraic.

Moreover, components of pre-images of special subvarieties are algebraic (in a sense described below) and appear in definable families. For example in $Y(1)^n$ they are the just the subvarieties of $\mathbb{H}^n$ defined by some collection of equations of the form $z_i = g_{ij} z_j$, where $(i, j) \in E \subset \{1, \ldots, n\}^2$ and $g_{ij} \in \mathrm{GL}_2^+(\mathbb{Q})$ acting by Mobius transformations. Here E is any set, possibly empty, and we allow $i = j$ in which case the fixed point z_i is quadratic. They sit in the family of subvarieties defined by relations from $\mathrm{SL}_2(\mathbb{R})$, and this family is definable (indeed semi-algebraic).

As the map π is invariant under the group Γ acting on U (in the examples: $\mathbb{Z}^n$, Λ, $\mathrm{SL}_2(\mathbb{Z})^n$, $\mathrm{Sp}_{2g}(\mathbb{Z})$), we may restrict our attention to a fundamental domain F for this action, which may also be taken to be semi-algebraic. We let

$$Z = \pi^{-1}(V) \cap F$$

and we are now interested in certain algebraic points of bounded degree in Z.

Definability. The map $e : \mathbb{C} \to \mathbb{C}^\times$ (i.e. its graph in $\mathbb{C} \times \mathbb{C}^\times$) is not definable in any o-minimal structure, due to the infinite discrete group acting, and the same holds for the map $\pi : U \to X$ for every mixed Shimura variety X of positive dimension. But in all the examples given so far, the restriction of π to a suitable fundamental domain F for Γ is definable in $\mathbb{R}_{\mathrm{an,\ exp}}$.

Theorem 6.1 (Peterzil-Starchenko [103]). *The restriction of* $\pi : \mathbb{H}_g \to \mathcal{A}_g$ *to the classical fundamental domain for the* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *action is definable.* □

Indeed the corresponding assertion holds for $\mathcal{X}_g$, generalising the earlier result by the same authors for Weierstrass $\wp$-functions [100] established in the course of a study of non-standard complex tori. The generalisation of this result to all Shimura varieties has been announced by Klingler-Ullmo-Yafaev [72], and to all mixed Shimura varieties by Gao [48]. With these results, o-minimal methods are available across the full breadth of the Zilber-Pink conjecture.

The strategy. The Counting Theorem tells us that $Z(k, T)$ is contained in "few" blocks contained in Z^{alg}. In the arithmetic settings, one then has essentially two tasks to turn this statement into the Diophantine conclusion:

(i) to characterise Z^{alg} as (essentially) coinciding with the exceptional locus in the Diophantine problem, i.e. weakly special subvarieties. This is a problem in *functional transcendence.*

(ii) to reduce "few" (i.e. $\ll_\epsilon T^\epsilon$) to finite. This is effected by playing off the upper bound against a *lower bound for the size of the Galois orbit of a special point.*

Characterising the algebraic part. We need to understand Z^{alg}, but it is more natural to consider first $\pi^{-1}(V)^{\mathrm{alg}}$, which turns out to be a union of complex algebraic subvarieties (intersected with U). These will generally not be fully contained in Z (or indeed in F).

I have not defined weakly special varieties except in the case of exponentiation, but they may be characterised by the following result of Ullmo-Yafaev [145]. By an "algebraic subvariety of U" we will mean a (complex analytically irreducible) component of $W \cap U$ where W is an algebraic subvariety of the ambient space (we always assume that the uniformising space U is semialgebraic).

Theorem 6.2 (Ullmo-Yafaev [145]). *Let* X *be a Shimura variety. A subvariety* $W \subset X$ *is weakly special if and only if the components of its pre-image in* U *are algebraic subvarieties.* □

Weakly special subvarieties are thus precisely the algebraic varieties preserved (as algebraic) by π. It turns out that the algebraic part of $\pi^{-1}(V)$ is equal to the union of weakly special subvarieties of positive dimension it contains:

Theorem 6.3 ("Ax-Lindemann"). *For X and $\pi : U \to X$ as in our examples, a maximal algebraic $W \subset \pi^{-1}(V)$ is weakly special.* □

Thus, the algebraic part of $\pi^{-1}(V)$, a coarse analogue of the special set, turns out to be a close relative. For the exponential function this follows, as already observed, from Ax-Schanuel; for abelian varieties it is likewise due to Ax [6]; see also [25, 69]. For $\mathcal{A}_g$ it is due to Pila-Tsimerman [114], building on [113, 147, 108]. Klingler-Ullmo-Yafaev [72] have announced the result for all Shimura varieties (and indeed a bit more generally), and a further generalisation to all mixed Shimura varieties has been announced by Gao [48]. A version for the modular function "with derivatives" is in [109]. While Ax's theorem is in the setting of differential fields and is proved by differential algebra, in all the Shimura variety settings mentioned "Ax-Lindemann" is proved directly in the complex setting using o-minimality and point-counting. This uses the fact that the group Γ gives rise to "many" integer points in suitable definable subsets of G. Mok has indicated how such results can be proved via complex differential geometry.

From "few" to finite. The definability of Z allows the Counting Theorem to be applied to the relevant algebraic points. This implies that there are "few" such points. How does one get from this to a finiteness statement?

The key here is that special points in X are algebraic and are (at least conjecturally) of high degree, while their pre-images have small height, relative to a suitable measure of their "complexity". For example, a root of unity ζ of order (precisely) T has degree

$$[\mathbb{Q}(\zeta) : \mathbb{Q}] = \phi(T) \gg_\epsilon T^{1-\epsilon}$$

for every positive ϵ (see e.g. [58], 18.4, Theorem 327), while its pre-image (under the map $e(z) = \exp(2\pi i z)$) in the fundamental domain $F = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}\, z < 1\}$ has height T. (The "complexity measure" here is the order T.)

Lower bounds for the size of Galois orbits of torsion points in abelian varieties are much studied, e.g. in connection with isogeny estimates and Serre's Open Image Theorem. Suitable results for [117] are due to Masser [83]. For products of elliptic modular surfaces (torsion points on CM curves) one has results of Silverberg [136].

Lower bounds for the size of Galois orbits of special points are essential in all current approaches to AO. The following was suggested by Edixhoven [44] for special points in $\mathcal{A}_g$, where an appropriate complexity measure for $x \in \mathcal{A}_g$ is afforded by the discriminant $\Delta(x)$ of the centre of the endomorphism ring of the corresponding abelian variety A_x.

Conjecture 6.4. *Let $g \geq 1$. There exist positive constants C_g, δ_g such that, for a CM point $x \in \mathcal{A}_g$,*

$$[\mathbb{Q}(x) : \mathbb{Q}] \geq C_g |\Delta(x)|^{\delta_g}.$$

For $g = 1$ the conjecture is affirmed by the theory of complex multiplication of elliptic curves and the (ineffective) Landau-Siegel lower bound for class numbers [75, 135]. It has been affirmed unconditionally for $g \le 6$, and for all g under GRH, by Tsimerman [140] (for the latter see also [146]).

A suitable upper bound for the height of the pre-image in a fundamental domain of a special point in $\mathcal{A}_g$ is established in [113] ; its generalisation to general Shimura varieties is expected. Then the upper and lower bounds for the number of points outside the algebraic part are incompatible for large height: there are only finitely many "isolated" special points.

Concluding the proof. Once the "Ax-Lindemann" result is established, a further property follows: that the maximal weakly special subvarieties contained in $\pi^{-1}(V)$ come from a finite number of "families" (because being "optimal" is a definable condition on a larger semi-algebraic collection of subvarieties of U containing all the weakly special subvarieties, while by Ax-Lindemann the weakly special families in which optimal subvarieties lie are characterised by rational data: a definable subset of $\mathbb{Q}$ must be finite). In the exponential case, this means that they are translates of finitely many rational linear spaces. In the other cases, one can also view the weakly special subvarieties in a given family as "'translates", parameterised by points in a suitable "quotient". The translate is special if and only if the corresponding parameter is a special point.

This finally enables the argument to be concluded by induction as follows. Given V, one has finitely many families $U_i, i = 1 \ldots, k$ of weakly special subvarieties, parameterised by points of some ambient varieties X_i of the same general type, but of lower dimension (except that points are weakly special and are parameterised by X itself). With each one has a subvariety $V_i \subset X_i$ consisting of those parameters for which the corresponding weakly special subvariety is contained in V. One may suppose by induction that V contains only finitely many special subvarieties of positive dimension. Then apply the Counting Theorem directly to see that a special point of large complexity has "many" Galois conjugates over V and all its positive dimensional special subvarieties, and leads to a contradiction. To conclude one observes that there are only finitely many special points whose complexity is below a given bound.

Theorem 6.5 ([37, 108, 113, 114, 143, 147]). *AO holds for $\mathcal{A}_g^n$, $n \ge 1$, $g \le 6$.* $\square$

The efficacy of the Counting Theorem in these applications lies firstly in that it may be applied even when the Galois lower bounds are far from optimal: the problem then devolves to understanding the algebraic part, which is a question in functional transcendence. Secondly, it can be applied to this latter problem due to the arithmetic nature of Γ. Ullmo [143] has shown that, for a Shimura variety X, these ingredients (definability of π on F, Ax-Lindemann, lower bound for Galois orbits, upper bound for the height of a pre-image of a special point in F, the last two in terms of a suitable "complexity") suffice to establish AO for X.

Special points results in mixed settings are obtained in [3, 48, 110]. A proof of semi-abelian MM along these lines is in [101].

7. O-minimality and atypical intersections

Torsion anomalous points. For $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$ we denote by E_λ the elliptic curve in Legendre form defined (in affine coordinates) by

$$y^2 = x(x-1)(x-\lambda).$$

We let $P_\lambda, Q_\lambda \in E_\lambda$ be the points

$$P_\lambda = \big(2, \sqrt{2(2-\lambda)}\big), \quad Q_\lambda = \big(3, \sqrt{6(3-\lambda)}\big)$$

(with some fixed determination of $\sqrt{\ }$; whether the point is torsion is independent of the choice). Masser and Zannier [87, 88] prove the following theorem.

Theorem 7.1. *There are only finitely many complex numbers $\lambda \neq 0, 1$ such that P_λ and Q_λ are both torsion point in E_λ.* $\square$

This is a "Relative Manin-Mumford" problem, in that it concerns a curve (the locus of (P_λ, Q_λ)) in a family of abelian varieties (the squares of the E_λ for $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$). A general "Relative Manin-Mumford" conjecture is framed by Pink [121] where it is shown to follow from his general conjecture (but note that it requires a slight correction: see Bertrand [13]).

The relative Manin-Mumford conjecture for a curve in the Poincaré bi-extension has been announced by Bertrand-Masser-Pillay-Zannier [14]. See Zannier [157] for further developments and applications.

Atypical modular intersections. It is natural then to apply a similar strategy to other problems of atypical intersections. Since special subvarieties are defined by rational (or bounded degree algebraic) data, and the dimension conditions characterising atypical intersections are detectable by definable sets, the methods are *prima facie* available once one has definability of π on F.

Habegger and Pila [56] establish a partial analogue of Theorem 2.5 concerning atypical intersections of a curve in $Y(1)^n$: i.e. points where the coordinates satisfy two independent "special" relationships (either the elliptic curves corresponding to two coordinates are isogenous, or the curve corresponding to one coordinate is CM). The result again depends on a functional transcendence statement (algebraic independence of "modular logarithms") and a suitable lower bound for Galois orbits. The lower bound is obtained only under an additional hypothesis.

Definition 7.2. For a curve $V \subset Y(1)^n$, define $\deg_i V$ to be the number of intersections of V with the hyperplane determined by a generic fixed value of the ith coordinate. The curve V is called *asymmetric* if, among the positive $\deg_i V$, there are no repetitions, save that one value may appear at most twice.

Theorem 7.3 ([56]). *Let $V \subset Y(1)^n$ be an asymmetric curve defined over $\overline{\mathbb{Q}}$. If V is not contained in a proper special subvariety then $V \cap \mathcal{S}^{[2]}$ is a finite set.* $\square$

Looking to atypical intersection problems more generally, it seems reasonable to conjecture the following "complex Ax-Schanuel" statement.

Conjecture 7.4 (Weak Complex Ax; WCA). *Let X be a mixed Shimura variety, with its uniformisation $\pi : U \to X$, and $V \subset X$. Then an optimal component for V is weakly special.*

Habegger and Pila [57] show that WCA for $Y(1)^n$ together with a conjecture on the size of Galois orbits of certain "optimal" atypical intersections in $Y(1)^n$ enable the point-counting strategy to be carried through to give ZP for $Y(1)^n$. (A proof of WCA for $Y(1)^n$ has been announced in [115].) The same ideas yield an unconditional result for curves in abelian varieties. We give some definitions in order to formulate this conjecture.

Let X be a mixed Shimura variety and $\mathcal{S}$ its collection of special subvarieties. Since $\mathcal{S}$ is closed under taking irreducible components of intersections, for any subvariety $A \subset X$ there is a smallest special subvariety containing A which we denote $\langle A \rangle$. We call $\partial(A) = \dim\langle A \rangle - \dim A$ the *defect* of A. Fix $V \subset X$. A subvariety $A \subset V$ is called *optimal (for V)* if there is no subvariety $B \subset V$ with $A \subset B, A \neq B$ and $\partial(B) \leq \partial(A)$.

Another formulation of ZP for X is then that for any $V \subset X$ there are only finitely many optimal subvarieties. (Apart from V itself, which is optimal for its defect, any optimal proper subvariety of V must be atypical.)

Definition 7.5. The *complexity* $\Delta(T)$ of $T \in \mathcal{S}_{Y(1)^n}$ is the maximum of the absolute values of the discriminant of any fixed (quadratic) coordinates and the heights of any $g \in \mathrm{GL}_2(\mathbb{Q})^+$ defining a pre-image of T in $\mathbb{H}^n$ (see [108]).

Conjecture 7.6 (Large Galois Orbits; LGO). *Let $X = Y(1)^n$ and $V \subset X$ defined over K, a field finitely generated over $\mathbb{Q}$. Then there are constants $C(V), \delta(V) > 0$ such that for any optimal isolated point component $\{x\}$ one has*

$$[K(x) : K] \geq C(V)\Delta(\langle\{x\}\rangle)^{\delta(V)}.$$

The following two results are announced in [57].

Theorem 7.7. *Assume WCA for $Y(1)^n$ and LGO. Then ZP holds for $Y(1)^n$.* □

The same blueprint works for abelian varieties (and I would expect a suitable formulation to apply to any mixed Shimura variety). WCA is known for abelian varieties (Ax [6]) while LGO may be affirmed unconditionally for curves when everything is defined over $\overline{\mathbb{Q}}$. This relies on a height inequality of Rémond [124].

Theorem 7.8. *Let X be an abelian variety, $V \subset X$ a curve, both defined over $\overline{\mathbb{Q}}$. If V is not contained in a proper special subvariety then $V \cap \mathcal{S}^{[2]}$ is a finite set.* □

Analogues of Mordell-Lang. Just as ZP for curves in $\mathbb{G}_{\mathrm{m}}^n$ entails ML for curves, ZP for curves in $Y(1)^n$ entails an analogue of ML. The same circle of ideas (o-minimality, point counting, and lower bounds for Galois orbits coming from isogeny estimates for elliptic curves [86]) enable a proof of "modular ML" for a general subvariety of $Y(1)^n$; see [56, 110]. The latter includes an extension to products of elliptic modular surfaces. Various partial results for subvarieties of $\mathcal{A}_g$ have been obtained by Orr, including the following full result for curves.

Theorem 7.9 (Orr [99]). *A curve $V \subset \mathcal{A}_g$ having infinitely many points for which the corresponding abelian varieties are isogneous is weakly special.* □

A further result in this general area (though not a special case of ZP) is an analogue of the Tate-Voloch conjecture for products of modular curves, proved by Habegger [55]. I will not state the result, but note that the proof makes use of the above mentioned modular Mordell-Lang, established via o-minimality, while results of Scanlon [128] in the original semi-abelian setting made use of the model theory of difference fields.

Some further questions. The section above contains many stated results but they are at the same time fragmentary. Functional transcendence questions and lower bounds for Galois orbits seem to pose significant (though fascinating) challenges. I would like to conclude with some further questions that seem to arise naturally from the considerations around the Ax-Schanuel theme.

Consider a mixed Shimura variety X, its uniformisation $\pi : U \to X$, and an algebraic subvariety $W \subset U$ in the sense defined earlier. When W is an orbit of a suitable kind, results from ergodic theory ("Ragunathan conjecture") govern when $\pi(W)$ is dense (in the usual analytic topology) in a weakly special subvariety. Ax-Lindemann says that the Zariski closure of $\pi(W)$ is always weakly special.

Question 7.10. *For $\pi : U \to X$ and an algebraic $W \subset U$ as above:*

1. Are there natural conditions under which $\pi(W)$ is dense in X?

2. Are there natural conditions under which $\pi(W)$ intersects every algebraic subvariety $V \subset X$ of complementary dimension (cf Ax [7])?

Ax-Schanuel is naturally stated (and proved) in the setting of a differential field. The function $j(z)$ satisfies a certain nonlinear third order algebraic differential equation, and none of lower order [81]. Specifically (see e.g. [84]),

$$J(j, j', j'', j''') = Sj + \frac{j^2 - 1968j + 2654208}{2j^2(j - 1728)^2}(j')^2 = 0,$$

where Sf denotes the *Schwarzian derivative* $Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ and $'$ indicates differentiation with respect to z. The full solution set is $\{j(gz) : g \in \mathrm{SL}_2(\mathbb{C})\}$.

Definition 7.11. Let $\mathbb{Q} \subset C \subset K$ be a tower of fields and $\{D_\mu\}$ a set of commuting derivations of K with $C = \bigcap_\mu \ker D_\mu$. Elements $j_1, \ldots, j_n \in K$ are called *modular-independent* if no $j_\nu \in C$ and no relation $\Phi_N(j_\nu, j_\mu) = 0$ holds with $N \geq 1$, $\nu \neq \mu$.

We can formulate a conjecture giving a modular analogue of "Ax-Schanuel" in a differential field setting. It implies WCA for $Y(1)^n$ as well as the result of [109]. (A modular analogue of Schanuel's conjecture may be deduced from the Grothendieck-André period conjecture [1], as explicated by Bertolin [12]; see [111].) Condition (a) below stipulates that $j'_\nu, j''_\nu, j'''_\nu$ are the derivatives of j_ν with respect to z_ν and that j_ν satisfies the j equation with respect to z_ν for each ν. The modular independence (b) implies that the quantities which appear in the denominator in J are non-zero. A corresponding "modular ZP with derivatives" is framed in [112].

Conjecture 7.12. *With K as above let $z_\nu, j_\nu, j'_\nu, j''_\nu, j'''_\nu \in K^\times, \nu = 1, \ldots, n$, with*

(a) for all ν, μ, $D_\mu j_\nu = j'_\nu D_\mu z_\nu$, $D_\mu j'_\nu = j''_\nu D_\mu z_\nu$, $D_\mu j''_\nu = j'''_\nu D_\mu z_\nu$ and

$$J\big(j_\nu, j'_\nu, j''_\nu, j'''_\nu\big) = 0;$$

(b) the j_ν are modular-independent.

Then

$$\text{tr. deg.}_C C(z_1, \ldots, z_n, j_1, \ldots, j_n, j'_1, \ldots, j'_n, j''_1, \ldots, j''_n) \geq 3n + \text{rank}\big(D_\mu z_\nu\big).$$

Freitag and Scanlon [47] have shown that the set defined by the differential equation satisfied by the j-function in a differentially closed field of characteristic zero is "strongly minimal" and "geometrically trivial". This uses the "modular Ax-Lindemann-Weierstrass with derivatives" result in [109]. For an introduction to differential fields in a model-theoretic setting, including definitions of the above terms (and related results on Painlevé transcendents) see Nagloo-Pillay [96]. One would like to generalise these results appropriately to the uniformising functions of mixed Shimura varieties.

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