

# Rational points in periodic analytic sets and the Manin-Mumford conjecture

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ABSTRACT. We present a new proof of the Manin-Mumford conjecture about torsion points on algebraic subvarieties of abelian varieties. Our principle, which admits other applications, is to view torsion points as rational points on a complex torus and then compare (i) upper bounds for the number of rational points on a transcendental analytic variety (Bombieri-Pila-Wilkie) and (ii) lower bounds for the degree of a torsion point (Masser), after taking conjugates. In order to be able to deal with (i), we discuss (Thm. 2.1) the semi-algebraic curves contained in an analytic variety supposed invariant for translations by a full lattice, which is a topic with some independent motivation.

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## §1. INTRODUCTION

The so-called *Manin-Mumford conjecture* was raised independently by Manin and Mumford and first proved by Raynaud [R] in 1983; its original form stated that *a curve  $C$  (over  $\mathbb{C}$ ) of genus  $\geq 2$ , embedded in its Jacobian  $J$ , can contain only finitely many torsion points* (relative of course to the Jacobian group-structure). Raynaud actually considered the more general case when  $C$  is embedded in any abelian variety. Soon afterwards, Raynaud [R2] produced a further significant generalization, replacing  $C$  and  $J$  respectively by a subvariety  $X$  in an abelian variety  $A$ ; in this situation he proved that *if  $X$  contains a Zariski-dense set of torsion points, then  $X$  is a translate of an abelian subvariety of  $A$  by a torsion point*. Other proofs (sometimes only for the case of curves) later appeared, due to Serre, to Coleman, to Hindry, to Buium, to Hrushovski (see [Py]), to Pink & Roessler [PR], to Baker & Ribet [BR]. We also remark that a less deep precedent of this problem was an analogous question for multiplicative algebraic groups, raised by Lang already in the '60s. (See [L]; Lang started the matter by asking to describe the plane curves  $f(x, y) = 0$  with infinitely many points  $(\zeta, \eta)$ , with  $\zeta, \eta$  roots of unity.)

In the meantime, the statement was put into a broader perspective, by viewing it as a special case of the general Mordell-Lang conjecture and also, from another viewpoint, of the Bogomolov conjecture on points of small canonical height on (semi)abelian varieties (we recall that torsion points are those of zero height). These conjectures have later been proved and unified (by Faltings, Vojta, Ullmo, Szpiro, Zhang, Poonen, David, Philippon,...) by means of different approaches providing, as a byproduct, several further proofs of the Manin-Mumford statement (we refer to the survey papers [Py] and [T] for a history of the topic and for references.) Recent work of Klingler, Ullmo and Yafaev proving (under GRH) the André-Oort conjecture, an analogue of the Manin-Mumford conjecture for Shimura varieties, has inspired another proof of Manin-Mumford due to Ratazzi & Ullmo [RU]. All of these approaches are rather sophisticated and depend on tools of various nature.

It is the purpose of this paper to present a completely different proof compared to the existing ones. Our approach too relies on certain auxiliary results, having however another nature (archimedean) with respect to the prerequisites of the previously known proofs; hence we believe that this treatment may be of some interest for a number of mathematicians. Also, the underlying principle has certainly other applications, as in work in progress [MZ]; we shall say a little more on this at the end.

In short, the basic strategy of our proof is as follows: view the torsion points as rational points on a real torus; estimate from above the number of rational points on a transcendental subvariety (Pila-Wilkie); estimate from below the number of torsion points by considering degree and taking conjugates (Masser); obtain a contradiction if the order of torsion is large. In more detail, we can proceed along the following steps, sticking for simplicity to the case of an algebraic curve  $X$ , of genus  $\geq 2$ , in the abelian variety  $A$ , both over a number field. (The general case of complex numbers can be dealt with in a similar way or reduced to this by specialization.)

(i) There is a complex analytic group-isomorphism  $\beta : \mathbb{C}^g/\Lambda \rightarrow A$ , where  $\Lambda$  is a certain lattice of rank  $2g$ , say with a  $\mathbb{Z}$ -basis  $\lambda_1, \dots, \lambda_{2g}$ ; so we can view a torsion point  $P \in A$  as the image  $P = \beta(x)$  where  $x$  is the class modulo  $\Lambda$  of a vector  $r_1\lambda_1 + \dots + r_{2g}\lambda_{2g} \in \mathbb{C}^g$ , where the  $r_i$  are rationals; if  $P$  has exact order  $T$ , the  $r_i$  will have exact common denominator  $T$ .

(ii) The algebraic curve  $X \subset A$  equals  $\beta(Y)$ , where  $Y = \beta^{-1}(X)$  is a complex analytic curve in  $\mathbb{C}^g/\Lambda$ ; in turn, if  $\pi : \mathbb{C}^g \rightarrow \mathbb{C}^g/\Lambda$  is the natural projection, we may write  $Y = \pi(Z)$ , where  $Z = \pi^{-1}(Y) \subset \mathbb{C}^g$  is an analytic curve which is invariant under translations in  $\Lambda$ .

(iii) We can use the basis  $\lambda_1, \dots, \lambda_{2g}$  to view  $\mathbb{C}^g$  as  $\mathbb{R}^{2g}$ ; then  $Z$  will become a real analytic surface in  $\mathbb{R}^{2g}$ , denoted again  $Z$ , invariant under  $\mathbb{Z}^{2g}$ . Also, as in (i), the torsion points on  $A$  will correspond to rational points in  $\mathbb{R}^{2g}$ . Then, the torsion points on  $X$  will correspond to rational points on  $Z$ . Note that, in view of the invariance of  $Z$  by integral translations, it suffices to study the rational points on  $Z$  in a bounded region of  $\mathbb{R}^{2g}$ .

(iv) Due to a method introduced by Bombieri-Pila [BP] for curves, and further developed by Pila [P] for surfaces, and by Pila-Wilkie [PW] in higher dimensions, one can get good estimates for the number of rational points with denominator  $T$  on a bounded region of a real analytic variety. As has to be expected, these estimates apply only if one confines the attention to the rational points which do not lie on any of the real semi-algebraic curves on the variety. For the number of these points, the estimates take the shape  $\ll T^\epsilon$ , for any given  $\epsilon > 0$ . This can be applied to the above defined  $Z$ ; it turns out that, since  $X$  is not a translate of an elliptic curve,  $Z$  does not contain any semi-algebraic curve.

(v) All of this still does not still yield any finiteness result, merely estimates for the number of torsion points on  $X$ , of a given order  $T$ . The crucial issue is that if  $X$  contains an algebraic point  $P$ , it automatically contains its conjugates over a field of definition. Now, Masser has proved in 1984 that the degree, over a field of definition for  $A$ , of any torsion point of exact order  $T$  is  $\gg T^\rho$  for a certain  $\rho > 0$  depending only on  $\dim A$ . Hence, if  $X$  contains a point of order  $T$ , it contains at least  $\gg T^\rho$  such points.

Then, comparing the estimates coming from (iv) and (v) we deduce that the order of the torsion points on  $X$  is bounded, concluding the argument.

In some previous proofs, one exploited not a lower bound for degrees, but rather the Galois structure of the field generated by torsion points. However this may be considered an information substantially of different nature.

To better isolate the new arguments from previous ones, we will prove the Manin-Mumford conjecture in the following weak form, which includes the case of curves.

**Theorem 1.1.** *Let  $A$  be an abelian variety and  $X$  an algebraic subvariety of  $A$ , both defined over a numberfield. Suppose that  $X$  does not contain any translate of an abelian subvariety of  $A$  of dimension  $> 0$ . Then  $X$  contains only finitely many torsion points of  $A$ .*

However, rather elementary purely geometrical considerations based on degrees allow one to get easily the following more precise version, for which we shall give a sketch at the end; in this statement by ‘torsion coset’ we mean a translate of an abelian subvariety by a torsion point:

**Theorem 1.1\*.** *Let  $A$  be an abelian variety and  $X$  an algebraic subvariety of  $A$ , both defined over a numberfield. The Zariski closure of the set of torsion points of  $A$  contained in  $X$  is a finite union of torsion cosets.*

Concerning the proof-method, a further point is that, in order for the Pila-Wilkie estimates to be applicable, we must study the real semi-algebraic curves that may lie on the inverse image  $Z$  of  $X$  under the analytic uniformization  $\mathbb{C}^g \rightarrow A$ . The set  $Z$  is analytic (defined by the vanishing of certain polynomials in the abelian functions giving the map  $\mathbb{C}^g \rightarrow A$ ) and periodic modulo the lattice  $\Lambda$ .

In §2, Theorem 2.1, we show that a connected real semi-algebraic curve contained in such a set  $Z$  must be contained in a complex linear subspace contained in  $Z$ , and moreover a subspace in which the period lattice has full rank. In other words, we prove that the “algebraic part” of  $Z$  in the sense of Pila-Wilkie (see Def. 2.1 below) corresponds precisely to the union of translates of abelian subvarieties of  $A$  of dimension  $> 0$  and contained in  $X$ . This fact seems to be not entirely free of independent interest.

With this result in hand the proof of Theorem 1.1 is concluded in §3 by combining the ingredients mentioned above.

## §2. STRUCTURE OF THE ALGEBRAIC PART OF A PERIODIC ANALYTIC SET

As announced, this section will be devoted to a complete description of algebraic subsets of an analytic subvariety of a complex torus, which corresponds to an analytic subvariety of  $\mathbb{C}^g$  periodic for a full lattice  $\Lambda$ , that is, invariant by translations in  $\Lambda$ . We express explicitly all of this in a few definitions and then state the main Theorem 2.1.

We take a (full rank) lattice  $\Lambda \subset \mathbb{C}^g$  which will be fixed throughout. We further fix a  $\mathbb{Z}$ -basis  $\lambda_1, \dots, \lambda_{2g}$  of  $\Lambda$  and use it to identify  $\mathbb{C}^g$  with  $\mathbb{R}^{2g}$ .

We shall use throughout the notation  $Z + \Lambda := \cup_{\lambda \in \Lambda} (Z + \lambda) := \{z + \lambda : z \in Z, \lambda \in \Lambda\}$  for the union of the translates of  $Z$  by the vectors in  $\Lambda$ . A set  $Z \subset \mathbb{C}^g$  will be called ( $\Lambda$ -) *periodic* if  $Z + \Lambda = Z$ .

As usual, an *analytic* set  $Z \subset \mathbb{C}^g$  will mean a set such that every point  $z \in \mathbb{C}^g$  has an open neighbourhood  $U$  in which  $Z$  is defined as the set of common zeros of a finite collection of functions (depending on  $z$ ) that are (complex) analytic (i.e. regular) in  $U$ . Such a set is readily seen to be *real analytic* as a subset of  $\mathbb{R}^{2g}$ . A *semi-algebraic* set in  $\mathbb{R}^n$  is a finite union of sets of the form  $\{x \in \mathbb{R}^n : f_1(x) = \dots = f_k(x) = 0, g_1(x) > 0, \dots, g_\ell(x) > 0\}$  where  $f_i, g_j \in \mathbb{R}[x], x = (x_1, \dots, x_n)$  (see e.g. [vdD2, p1] or [BM, Def. 1.1]).

**Definition 2.1.** Let  $Z \subset \mathbb{C}^g$ . We define the *complex algebraic part*  $Z^{\text{ca}}$  of  $Z$  to be the union of connected closed algebraic subsets of  $\mathbb{C}^g$  of positive dimension contained in  $Z$ . Viewing  $Z$  as a subset of  $\mathbb{R}^{2g}$  we define the *real algebraic part*  $Z^{\text{ra}}$  of  $Z$  to be the union of connected real algebraic sets of positive dimension contained in  $Z$ . Finally we define the *algebraic part*  $Z^{\text{alg}}$  of  $Z$  to be the union of all connected real semi-algebraic sets of positive dimension contained in  $Z$ . One readily sees that  $Z^{\text{ca}} \subset Z^{\text{ra}} \subset Z^{\text{alg}}$ .

We reserve the term *subspace* for a (respectively complex or real) vector subspace of  $\mathbb{C}^g$  or  $\mathbb{R}^{2g}$ . By a (respectively complex or real) *linear subvariety* we mean a subset of  $\mathbb{C}^g$  or  $\mathbb{R}^{2g}$  defined by the vanishing of some (complex or real) linear equations, not necessarily homogeneous. A subspace  $H$  in which  $H \cap \Lambda$  has full rank in  $H$  will be called a ( $\Lambda$ -) *full subspace*. A set of the form  $z + H$  where  $z \in \mathbb{C}^g$  and  $H$  is a complex linear subspace will be called a *coset*.

Thus a subspace  $H$  that is both complex and full corresponds precisely to a *subtorus* of  $\mathbb{C}^g/\Lambda$ , and a coset of such an  $H$  will be called a *torus coset*.

**Definition 2.2.** For a set  $Z \subset \mathbb{C}^g$  we let  $Z^{\text{torus coset}}$  be the union of torus cosets of positive dimension contained in  $Z$ .

A torus coset is evidently a complex linear subvariety, whence  $Z^{\text{torus coset}} \subset Z^{\text{ca}} \subset Z^{\text{ra}} \subset Z^{\text{alg}}$ . For a periodic analytic set  $Z$  we show that all these sets coincide.

**Theorem 2.1.** *Let  $Z \subset \mathbb{C}^g$  be a periodic analytic set. Then  $Z^{\text{alg}} = Z^{\text{torus coset}}$ .*

Our proof of this result involves several preliminaries; for the reader's convenience we explicitly subdivide the proof into three Steps.

**Step 1.** *Reduction semi-algebraic  $\rightarrow$  complex algebraic.* In this step we reduce to complex curves contained in  $Z$ , by proving that  $Z^{\text{alg}} = Z^{\text{ca}}$ . We first prove a lemma:

**Lemma 2.1.** *Let  $Z \subset \mathbb{C}^g$  be analytic. Suppose that  $x \in \mathbb{C}^g$  has a neighbourhood  $U$  such that  $x$  is a smooth point of  $Y \cap U$ , where  $Y$  is a real algebraic curve with  $Y \cap U \subset Z$  as subsets of  $\mathbb{R}^{2g}$ . Then there is a neighbourhood  $U'$  of  $x$  contained in  $U$  and a complex algebraic curve  $\Gamma$  such that  $Y \cap U' \subset \Gamma \subset Z$ .*

*Proof.* We took coordinates in  $\mathbb{R}^{2g}$  using the lattice basis, but clearly  $Y$  will remain semi-algebraic under any real linear change of coordinates. Let us here write  $(x_1, y_1, \dots, x_n, y_n)$  for the coordinates of  $\mathbb{R}^{2g}$ , where  $z_j = x_j + iy_j$  are the coordinates of  $\mathbb{C}^g$ . Also, for our purposes we can assume that  $Y$  is irreducible, by moving if necessary to a smaller neighbourhood, since the smooth point  $x$  will belong to just one component.

We may assume by translation that  $x = 0$ . If  $x_1$  (say) is a non-constant function on the real curve  $Y$  near 0 then, for  $t = x_1$  in some real neighbourhood of 0, all the functions  $x_1, y_1, \dots, x_n, y_n$  are real analytic functions of some  $m$ -th root  $u := t^{1/m}$ , and (as functions of  $u$ ), are algebraic over  $\mathbb{R}(x_1) \subset \mathbb{C}(x_1, y_1)$ . We claim that each of the functions  $x_2, y_2, \dots, x_n, y_n$  is algebraic over the field  $\mathbb{C}(z_1) = \mathbb{C}(x_1 + iy_1)$ . This is because the function  $x_1 + iy_1$  is non-constant on  $Y$  since  $x_1$  is non-constant, and  $\mathbb{C}(x_1, y_1)$  is algebraic over  $\mathbb{C}(x_1)$ , so of transcendence degree 1. Therefore, for each  $j$ , the function  $z_j = x_j + iy_j$  is also algebraic over  $\mathbb{C}(z_1)$ . The functions  $z_j(u)$  are analytic for real  $u$  in some neighbourhood of 0, hence for complex  $u$  in a complex neighbourhood of 0; the image of  $u \mapsto z(u)$  is thus a non-discrete subset of a complex algebraic curve  $\Gamma$ , which in a neighborhood of 0 contains  $Y$  and must necessarily be contained in  $Z$ . Certainly  $\Gamma$  contains a smooth point in this neighborhood, and we get  $\Gamma \subset Z$  by analytic continuation, as the smooth points of an irreducible complex curve are connected, and the remaining points belong to  $Z$  by continuity.  $\square$

We can now prove the alluded reduction:

**Proposition 2.1.** *Let  $Z \subset \mathbb{C}^g$  be a periodic analytic set. Then  $Z^{\text{alg}} = Z^{\text{ca}}$ .*

*Proof.* Suppose  $W$  is a connected real semi-algebraic set of positive dimension with  $W \subset Z$ . Then, omitting at most finitely many points,  $W$  is a union of real connected semi-algebraic curves. If  $Y$  is such a curve then,  $Y$  is real algebraic in the neighbourhood of any smooth point and we may apply Lemma 2.1 to find  $Y$  contained in a complex algebraic curve  $\Gamma \subset Z$ . This proves the statement.  $\square$

**Step 2.** *Complex curves in periodic analytic varieties and linear spaces.* In this step we prove that if a complex curve  $C$  is contained in the periodic analytic set  $Z$ , then there are 'several' complex lines  $l$  with  $C + l$  contained in  $Z$ . In turn, this will involve a few preliminaries.

Suppose  $C \subset \mathbb{C}^g$  is an irreducible complex algebraic curve. Of the coordinates  $z_1, \dots, z_g$  of  $\mathbb{C}^g$ , if  $z_i$  is not constant on  $C$  then any  $z_i, z_j$  are related by some irreducible polynomial equation  $G(z_i, z_j) = 0$ . For  $|z_i|$  sufficiently large, say  $|z_i| > R$ , the solutions  $z_j$  are given by convergent Puiseux series  $\phi_j(z_i)$ . By a *branch* of  $C$  we mean a choice of index  $i$  and a  $g$ -tuple  $\phi = (\phi_1(z_i), \dots, \phi_g(z_i))$  of algebraic Puiseux series, convergent for  $|z_i| > R$ , and such that  $\phi(z_i) = (\phi_1(z_i), \dots, \phi_n(z_i)) \in C$  for all  $|z_i| > R$ . By a suitable choice of index  $i$ , we can always obtain a branch of  $C$  such that, for all  $j$ ,

$$\phi_j(z_i) = \alpha_j z_i + \text{lower order terms}, \quad \alpha_j \in \mathbb{C}.$$

We call such a branch *linear*, and  $\alpha = (\alpha_1, \dots, \alpha_g)$  the *direction* of the branch. Observe that  $\alpha_i = 1$ , so  $\alpha \neq 0$ .

Suppose  $\phi(w) = (\phi_1(w), \dots, \phi_g(w))$  is an  $n$ -tuple of algebraic Puiseux series, convergent for  $|w| > R$ . Then, fixing  $w_0, w_1 \in \mathbb{C}$  and  $\mu = (\mu_1, \dots, \mu_g) \in \mathbb{C}^g$ , we consider (for  $\kappa \in \mathbb{C}$  such that  $|w_0 + \kappa w_1| > R$ ),

$$\kappa \mapsto \psi(\kappa) = (\psi_1(\kappa), \dots, \psi_g(\kappa)),$$

$$\psi_i(\kappa) = \phi_i(w_0 + \kappa w_1) - \kappa \mu_i.$$

From the algebraic relation  $G_i(t, \phi_i(t))$ , we find that  $G_i(w_0 + \kappa w_1, \psi_i(\kappa) + \kappa \mu_i) = 0$ . Thus the  $\psi_i(\kappa)$  are also algebraic functions of  $\kappa$ , and the locus above is Zariski dense in an irreducible algebraic curve in  $\mathbb{C}^g$ , which we denote  $\Gamma(\phi, \mu, w_0, w_1)$ . It is important to note that its degree is bounded in terms of the degree of the curve containing  $\phi$  (independently of the choice of  $w_0, w_1, \mu$ ). We now have a lemma, perhaps known but for which we have found no reference:

**Lemma 2.2.** *Let  $Z \subset \mathbb{C}^g$  be an analytic set,  $B \subset \mathbb{C}^g$  a bounded set, and  $\delta$  a positive integer. There exists a positive integer  $K = K(Z, B, \delta)$  with the following property. Suppose  $\Gamma \subset \mathbb{C}^g$  is an irreducible complex algebraic curve of degree  $\leq \delta$ , and with  $\#(Z \cap B \cap \Gamma) \geq K$ . Then  $\Gamma \subset Z$ .*

*Proof.* About each point of  $\mathbb{C}^g$  there is an open disk in which  $Z$  is defined by the vanishing of a finite number of regular functions. Taking a smaller disk, these functions may be assumed regular in a neighbourhood of the closure of the disk. Then by compactness of the closure of  $B$ , we can find a finite number of open disks  $U$  covering  $B$ , and on each disk a finite number of functions, regular in a neighbourhood of the closure of the disk, whose zero-locus defines  $Z$  in the disk. Thus, in each of the finitely many disks,  $Z$  is expressed as in intersection of finitely many hypersurfaces. As shown in a moment, this remark reduces the proof to the case of hypersurfaces, namely to the following

**Claim.** *Suppose that  $U \subset \mathbb{C}^n$  is a bounded open disk and that  $f$  is a complex-valued function that is regular in a neighbourhood of the closure of  $U$ . Let  $Y = \{z \in U : f(z) = 0\}$ , and  $\delta$  a positive integer. There is a positive integer  $K = K(U, f, \delta)$  with the following property. Suppose  $\Gamma \subset \mathbb{C}^n$  is an irreducible curve of degree  $\leq \delta$  and  $\#Y \cap \Gamma \geq K$ . Then  $f$  vanishes identically on  $\Gamma$  in  $U$ .*

Given the claim, we may establish the lemma as follows. For each disk  $U$  in the finite covering take  $K_U$  to be the maximum of the  $K(U, f, \delta)$  over all the finitely many functions  $f$  defining  $Z$  on  $U$ , and take  $K = \sum_U K_U + 1$ . For each  $U$  put  $Z_U = Z \cap U$ . Let now  $\Gamma$  be an irreducible complex algebraic curve of degree  $\leq \delta$  with  $\#Z \cap B \cap \Gamma \geq K$ . By the pigeonhole principle, one of the disks  $U$  has  $\#(Z_U \cap \Gamma) \geq K_U$ . By the Claim, each of the functions  $f$  defining  $Z$  on  $U$  vanishes identically on  $\Gamma$ , so that, restricted to  $U$ ,  $\Gamma \subset Z$ . However  $\Gamma$  is connected, so by analytic continuation we conclude that  $\Gamma \subset Z$ . It remains then to prove the Claim, as we shall now do.

Since  $\Gamma$  has degree  $\leq \delta$ , each pair of coordinates  $z_i, z_j$  satisfy on  $\Gamma$  an algebraic relation  $P_{ij}(z_i, z_j) = 0$  for some nonzero polynomial  $P_{ij} \in \mathbb{C}[x_1, x_2]$  of degree  $\leq \delta$ . These polynomials, considered up to nonzero constant factors, define an algebraic set of dimension 1, containing  $\Gamma$  as a component. This set is possibly reducible but, given the  $P_{ij}$ , by projecting to a general plane, we see that there certainly exists a hypersurface  $H$  of degree  $\leq g\delta$  that contains  $\Gamma$  but not any other component of dimension 1; this  $H$  corresponds to a further polynomial  $P_H$  of degree  $\leq g\delta$ . Now, the polynomials in two variables of degree  $\leq \delta$ , up to constants, are parametrised by a projective space  $\mathbb{P}^D$ ,  $D = (\delta + 1)(\delta + 2)/2$ , whereas  $H$  is parametrized by  $\mathbb{P}^{D'}$  for a  $D'$  depending only on  $g, \delta$ . Hence the set  $\{H, P_{ij}\}$  is parametrized by the product

$$\Delta = \mathbb{P}(\mathbb{C})^{D'} \prod_{ij} \mathbb{P}(\mathbb{C})^D,$$

a compact space, which we can assume contained in  $\mathbb{R}^m$  for some suitable  $m$ . For  $w \in \Delta$  let  $I_w$  be the ideal generated by the corresponding  $P_{ij}, P_H$ . Consider the set

$$V := \{(z, w) \in \mathbb{C}^g \times \Delta : z = (z_1, \dots, z_g) \in U, w \in \Delta, f(z) = 0, Q(z) = 0, \text{ all } Q \in I_w\}.$$

We have  $V \subset \mathbb{C}^g \times \Delta$  with projections  $\pi_1, \pi_2$  onto the factors  $\mathbb{C}^g$  and  $\Delta$  respectively.

Our very construction shows that every algebraic curve  $\Gamma \subset \mathbb{C}^g$  of degree  $\leq \delta$  is defined, up to finitely many points, by  $I_w$  for (at least one) suitable  $w = w_\Gamma \in \Delta$ ; hence, for some point  $w_\Gamma \in \Delta$ ,  $\pi_1(\pi_2^{-1}(w_\Gamma))$  equals the intersection  $Y \cap \Gamma$  plus a finite set.

Now we consider  $V$  as a subset of  $\mathbb{R}^M$  for suitable  $M$ . Since  $f$  is regular on a neighborhood of the closure of  $U$ , the set  $V$  is *subanalytic* (even *semianalytic*) in  $\mathbb{R}^M$  (for the definition see [G] or [BM, Definition 2.1]). Further,  $V$  is bounded, since  $U$  is bounded and  $\Delta$  is compact. We appeal to Gabrielov's Theorem [G, or see e.g. BM, Theorem 3.14] to conclude:

*As  $w$  varies over the bounded set  $\pi_2(V)$ , the number of connected components of  $\pi_2^{-1}(w)$  is bounded by some finite number  $N = N(U, f, D)$ ; the number of connected components of  $\pi_1(\pi_2^{-1}(w))$  is then also bounded by  $N$ .*

Put  $K = N + 1$ . If now  $\#Y \cap \Gamma \geq K$ ,  $Y \cap \Gamma$  cannot consist of isolated points and must, as a semianalytic set, have dimension  $\geq 1$ . This set must then contain some smooth real-analytic arc. If we take a point on such an arc that is a non-singular point of  $\Gamma$ , then restricting  $f$  to a neighbourhood in which we can complex analytically parameterize  $\Gamma$ , we find that  $f$  restricted to this local parameterization is an analytic function with non-isolated zeros. It therefore vanishes identically on  $\Gamma$  in this neighbourhood. But now since  $\Gamma$  is irreducible, the set of its non-singular points is connected, and we find that  $f$  vanishes identically on  $\Gamma$  in  $U$  by analytic continuation. This establishes the Claim, and concludes the proof of Lemma 2.2.  $\square$

(Readers familiar with *o-minimal structures* (see [vdD2]) will recognize that the key point underlying the Claim is the *o-minimality* of the structure  $\mathbb{R}_{\text{an}}$  generated by restricted real-analytic functions [vdD].)

The following result now follows easily:

**Corollary.** *Let  $Z \subset \mathbb{C}^g$  be a periodic analytic set and  $r > 0$ . Let  $\phi(w)$  be an  $n$ -tuple of convergent (for  $|w| > R$ ) algebraic Puiseux series. There is an integer  $K = K(Z, r, \phi)$  with the following property. Let  $B \subset \mathbb{C}^g$  be a ball of radius  $r$ , and  $w_0, w_1 \in \mathbb{C}, \tau, \mu \in \mathbb{C}^g$ . If*

$$\#(\Gamma(\phi, \mu, w_0, w_1) + \tau \cap Z \cap B) \geq K$$

*then  $\Gamma(\phi, \mu, w_0, w_1) + \tau \subset Z$ .*

*Proof.* For a fixed  $B$  the conclusion follows directly from Lemma 2.2, because  $\Gamma(\phi, \mu, w_0, w_1) + \tau$  is an algebraic curve of degree bounded only in terms of  $\phi$ . Since  $Z$  is periodic, the ball  $B$  may be assumed to be centred in a fundamental domain, so dependence on the centre of the ball may be eliminated.  $\square$

We now come to the fundamental point of this Step 2. In the following,  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbb{C}^g$ . For a point  $z = (z_1, \dots, z_g) \in \mathbb{C}^g - \{0\}$  we denote by  $\mathbb{C}z$  the complex line  $\{wz : w \in \mathbb{C}\}$ .

**Proposition 2.2.** *Let  $Z \subset \mathbb{C}^g$  be a periodic analytic set and  $C$  be an irreducible complex algebraic curve with  $C + \tau \subset Z$  for a  $\tau \in \mathbb{C}^g$ . Let  $\phi(w)$  be a linear branch of  $C$  (convergent for  $|w| > R$ ) with direction  $\mathbb{C}\alpha$  and put  $K = K(Z, 1, \phi)$ , as in the last Corollary. Finally, let  $\lambda \in \Lambda$  be such that  $\|\beta - \lambda\| < (2K)^{-1}$  for some  $\beta \in \mathbb{C}\alpha$ . Then  $C + \tau + \mathbb{C}\lambda \subset Z$ .*

*Proof.* Fix  $w_1 \in \mathbb{C}$  such that  $\|w_1\alpha - \lambda\| < (2K)^{-1}$ . Now choose  $R_1 \geq R$  such that, for any  $w \in \mathbb{C}$  with  $|w| > R_1 + K|w_1|$  and any  $w' \in \mathbb{C}$  with  $|w'| \leq K|w_1|$  we have

$$\|\phi(w + w') - \phi(w) - w'\alpha\| < 1/2.$$

This is possible by the condition on  $\phi$  that all terms apart perhaps from the leading term are sub-linear. If now  $|w_0| > R_1 + K|w_1|$  and  $k = 0, 1, \dots, K$  we have

$$\|\phi(w_0 + kw_1) - \phi(w_0) - k\lambda\| \leq \|\phi(w_0 + kw_1) - \phi(w_0) - kw_1\alpha\| + k\|w_1\alpha - \lambda\| < 1.$$

Now,  $\phi(w_0 + kw_1) + \tau - k\lambda \in Z$  because  $Z$  is periodic and  $C + \tau \subset Z$ ; also,  $\phi(w_0 + kw_1) + \tau - k\lambda \in \Gamma := \Gamma(\phi, \lambda, w_0, w_1) + \tau$  provided  $|w_0 + kw_1| > R$ , and by the above all these points lie in the ball of radius 1 about  $\phi(w_0) + \tau$  for  $k = 0, 1, \dots, K$ .

By the Corollary to Lemma 2.2, we deduce that  $\Gamma \subset Z$ .

We thus find  $\Gamma(\phi, \lambda, w_0, w_1) + \tau \subset Z$  for all  $w_0$  suitably large, but we must still show that in fact all  $C + \tau + \mathbb{C}\lambda \subset Z$ . If we set  $y = w_0 + \kappa w_1$ ,  $\kappa \in \mathbb{C}$ , we find that

$$\phi(y) + \tau - ((y - w_0)/w_1)\lambda \in Z$$

provided  $|y| > R$ . But, fixing  $y, \kappa, w_1$ , the above represents a line as  $w_0$  varies. Since a segment of this line lies in  $Z$ , the whole line does, by analytic continuation. So for  $x \in \mathbb{C}$  the points

$$\phi(y) + \tau - x\lambda \in Z$$

provided only  $|y| > R$ . Fixing now  $x$ , the curve in  $y$  is just a branch of  $C + \tau - x\lambda$ . This curve is irreducible, being a translate of  $C$ , and so  $C + \tau - x\lambda \subset Z$  for all  $x$ .  $\square$

**Step 3.** *Linear subvarieties of a periodic analytic variety.* We study general linear subvarieties of a periodic analytic variety  $Z$ , their intersections with the lattice  $\Lambda$ , and use all of this to exploit the important conclusion of the last proposition, which produces certain translates  $C + l$  of  $C$  by a line, contained in  $Z$ .

We first recall some useful simple facts from the known theory of closed subgroups of real vector spaces, and start by noting that: *if  $H$  is a real subspace of  $\mathbb{C}^g$  then the closure of  $H + \Lambda$  has the form  $K + \Lambda$  where  $K$  is a full real subspace containing  $H$ .*

This statement follows from the description of closed subgroups of real vector spaces given in [S, Lecture VI, §2], which implies that the closure in  $\mathbb{C}^g$  of  $H + \Lambda$  has the form  $K + \Lambda_0$

for  $K$  a real subspace and  $\Lambda_0$  a lattice in a space  $W$  complementary to  $K$ . Clearly  $H \subset K$ . Now, the projection of  $\Lambda$  to  $W$  (along  $K$ ) is  $\Lambda_0$ , which must then have full rank in  $W$ , because  $\mathbb{R}^{2g} = \mathbb{R}\Lambda \subset \mathbb{R}(K + \Lambda_0) = K + \mathbb{R}\Lambda_0$ . Now, lifting a basis of  $\Lambda_0$  to  $\Lambda$  we see that  $\Lambda_1 := \Lambda \cap K$  has maximal rank, i.e. equal to  $\dim K$ , as wanted.

We further note that *for any open ball  $I$  around 0 in  $K$ , the set  $H + I$  contains a set of generators for  $\Lambda_1 := \Lambda \cap K$* . To prove this, let  $\Lambda'$  denote the lattice generated by  $\Lambda \cap (H + I)$  and observe that  $H + \Lambda'$  is dense in  $K$ : in fact, by definition of  $\Lambda'$ , the closure of  $H + \Lambda'$  contains the intersection of  $I/2$  with the closure of  $H + \Lambda$ , so it contains  $I/2$ , whence it must contain the whole  $K$ . Now, let  $\lambda \in \Lambda_1$ ; it belongs to the closure  $K$  of  $H + \Lambda'$ ; hence  $\lambda - I$  intersects  $H + \Lambda'$ , so  $\lambda + \Lambda'$  intersects  $\Lambda \cap (H + I) \subset \Lambda'$ , proving that  $\lambda \in \Lambda'$  and so  $\Lambda_1 \subset \Lambda'$ , as desired.

Now we may give the following definitions:

**Definition 2.3.** Let  $H$  be a real subspace of  $\mathbb{C}^g$ .

1. We denote by  $c(H) := \mathbb{C}H$  the complex space generated by  $H$  and call it the *complex closure* of  $H$ ; we call  $H$  *complex* if  $H = c(H)$ .
2. We denote by  $f(H)$  the *full closure* of  $H$ , namely the full real subspace  $K$  such that  $H + \Lambda$  is dense in  $K + \Lambda$ , as in the above remark. So  $H$  is full just if  $f(H) = H$ .
3. We denote by  $\mathbf{fc}(H)$  the *full-complex closure* of  $Z$ , namely the union of the iterates of  $H$  under the map  $H \mapsto f(c(H))$ .

One could alternately iterate the two operations  $H \mapsto c(H)$ ,  $H \mapsto f(H)$ . In general, if  $g > 1$  we need not have  $f(c(H)) = c(f(H))$ . (In  $\mathbb{C}^2$  take for instance  $H = \mathbb{R}(1, 0)$ ,  $\Lambda = \mathbb{Z}(1, 0) + \mathbb{Z}(i, 1) + \mathbb{Z}(1, i) + \mathbb{Z}(1, \sqrt{2})$ . Then  $f(H) = H$ ,  $c(H) = \mathbb{C}(1, 0)$ . Since  $c(H)$  is not full, we have that  $f(c(H))$  contains properly  $c(f(H)) = c(H)$ .) Anyway  $\mathbf{fc}(H)$  may also be characterized as the smallest subspace containing  $H$  that is both full and complex.

**Lemma 2.3.** *Let  $Z \subset \mathbb{C}^g$  be a periodic analytic set. Suppose  $z \in \mathbb{C}^g$  and  $H$  is a real subspace of  $\mathbb{C}^g$  with  $z + H \subset Z$ . Then there is a torus coset  $z + M$  with  $z + H \subset z + M \subset Z$ , and one can take  $M = \mathbf{fc}(H)$ .*

*Proof.* Note that by Def. 2.3(2) (which amounts to the opening remarks of this Step 3), if  $H$  is a real subspace and  $z \in \mathbb{C}^g$ , then  $(z + H) + \Lambda$  is dense in some  $(z + K) + \Lambda$  where  $K$  is a full real subspace containing  $H$ . Now, if  $H$  is a real subspace with  $z + H \subset Z$  then, by periodicity, the fact just stated and continuity, we have  $z + f(H) \subset Z$ , and, by analytic continuation, we have  $z + c(H) \subset Z$ . The conclusion follows by applying these observations to  $H$  and its iterates under full and complex closure.  $\square$

We can now rapidly conclude the proof of Theorem 2.1; we prove a last lemma:

**Lemma 2.4.** *Let  $Z \subset \mathbb{C}^g$  be a periodic analytic set,  $C$  an irreducible complex algebraic curve and  $M$  a complex subspace such that  $C + M \subset Z$ . Suppose that  $C + M$  is not a coset of  $M$ . Then there is a complex subspace  $M'$  with  $M \subset M'$ ,  $\dim M < \dim M'$ , and  $C + M' \subset Z$ .*

*Proof.* Let  $N$  be a complex linear subspace supplementary to  $M$  in  $\mathbb{C}^g$ . So every translate  $z + M$ , where  $z \in \mathbb{C}^g$ , intersects  $N$  in just one point. Now,  $C + M$ , as the image of the sum-map  $C \times M \rightarrow C + M$  contains an open dense set in its Zariski closure in  $\mathbb{C}^g$ , which is an irreducible algebraic subvariety of  $\mathbb{C}^g$ ; also,  $(C + M) \cap N$  is irreducible (because it is the projection of  $C + M$  to  $N$  along  $M$ ). By hypothesis  $(C + M) \cap N$  is not equal to a point, hence it cannot be a finite set of points and therefore it contains an open dense subset of a closed irreducible algebraic curve  $C' \subset N$ ; note that  $C + M \subset C' + M \subset Z$  and  $\dim(C + M) = \dim M + 1$ .



Take now a linear branch  $\phi$  of  $C'$  with direction  $\mathbb{C}\alpha$ , so  $\mathbb{C}\alpha$  lies in  $N$ . Then there is a small open ball  $B$  around 0 in  $\mathbb{C}^g$  so that  $(\mathbb{C}\alpha + B) \cap M$  does not contain any nonzero lattice point. By a remark above,  $\Lambda \cap (\mathbb{C}\alpha + B)$  contains generators for the full closure of  $\mathbb{C}\alpha$ , so in particular it contains a lattice point  $\lambda \notin M$ .

Put  $M' = M + \mathbb{C}\lambda$ .

Then  $C' + M' \subset Z$ . For if  $\tau \in M$  we have  $C' + \tau \subset Z$ , but this curve is a translate of  $C'$ , and by Proposition 2.2 (applied to  $C'$  in place of  $C$ ) we find that for small enough  $B$ ,  $(C' + \tau) + \mathbb{C}\lambda \subset Z$ . This being true for all  $\tau \in M$ , we have  $C' + M' \subset Z$ . Now so  $C + M' \subset C' + M' \subset Z$ . This completes the proof.  $\square$

*Proof of Theorem 2.1*

Let  $C$  be an irreducible complex algebraic curve contained in  $Z$  and take a maximal complex subspace  $M$  such that  $C + M \subset Z$ . By maximality and Lemma 2.4 we deduce that  $C + M$  is a single coset of  $M$ , which must be a torus coset by maximality and Lemma 2.3.

It follows that every closed complex algebraic set of positive dimension contained in  $Z$  is contained in the union of full cosets of positive dimension contained in  $Z$ , i.e. that  $Z^{\text{ca}} \subset Z^{\text{torus coset}}$ . Proposition 2.1 now finally proves what we need.  $\square$

**Remarks.** 1. Note that the final argument easily leads to the following assertion: *If an irreducible complex algebraic set  $V$  is contained in  $Z$  then there is a torus coset  $z + M$  with  $V \subset z + M \subset Z$ .* To prove this, take a maximal torus  $M$  such that for a point  $z$  we have  $z + M \subset Z$ . Applying the last conclusion of the proof of Theorem 2.1 to all the irreducible curves  $C$  on  $V$  passing through  $z$  we see by maximality that  $z + M$  contains a neighborhood of  $z$  in  $V$ . Hence it contains  $V$ .

2. One can check that our proof of Theorem 2.1 works in fact for arbitrary complex tori  $\mathbb{C}^g/\Lambda$ , even if they are not algebraic (i.e. complex analytically isomorphic to an abelian variety). However, a simple complex torus that is not algebraic has no infinite proper analytic subsets [Ba, Py2].

### §3. MANIN-MUMFORD

Let now  $A$  be a (projective) Abelian variety, and  $X$  a subvariety of  $A$ . We have a complex analytic uniformization  $\mathbb{C}^g \rightarrow A$ , periodic with period lattice  $\Lambda$ . The preimage  $Z \subset \mathbb{C}^g$  of  $X$  is a periodic analytic set, and we have shown that all real semialgebraic subsets, connected of positive dimension, of the real reduct of  $Z$  are contained in the union of torus cosets contained in  $Z$ .

Now a subtorus of  $\mathbb{C}^g/\Lambda$  corresponds to an abelian subvariety of  $A$  (see e.g. [R, remark on page 86], or it may be argued directly using Chow's Theorem that such a subtorus is algebraic).

*Proof of Theorem 1.1.* Our interest is in the torsion points  $P$  of  $A$  that lie on  $X$ . Let  $\text{tor}(A)$  denote the torsion subgroup of  $A$ , consisting of all points of  $A$  of finite order. The *order*  $T = T(P)$  of  $P$  is the minimal positive integer with  $TP = 0$ . A torsion point  $P$  of  $A$  corresponds to a rational point  $z = z_P = (q_1, \dots, q_{2g}) \in \mathbb{Q}^{2g}$  of  $Z$  considered as a subset of  $\mathbb{R}^{2g}$ . The order of  $P$  is equal to the *denominator* of  $z$ , i.e. the minimal integer  $d > 0$  for which  $dz \in \mathbb{Z}^{2g}$ . By the present assumptions,  $X$  does not contain any translate of an abelian subvariety of dimension  $> 0$ ; hence, by Theorem 2.1 the set  $Z^{\text{alg}} = Z^{\text{torus coset}}$  is empty.

As a subset of  $\mathbb{R}^{2g}$  the set  $Z$  is  $\mathbb{Z}^{2g}$ -periodic. In considering torsion points it therefore suffices to replace  $Z$  by  $\mathcal{Z} = Z \cap [0, 1)^{2g}$ , and clearly  $\mathcal{Z}^{\text{alg}} = Z^{\text{alg}} \cap [0, 1)^{2g}$  is empty as well.

For a set  $W \subset [0, 1)^{2g}$  and a real number  $T \geq 1$  we denote by  $N(W, T)$  the number of rational points of  $W$  of denominator dividing  $T$ . By Pila-Wilkie [PW], for every  $\epsilon > 0$ ,

$$N(\mathcal{Z}, T) = N(\mathcal{Z} - \mathcal{Z}^{\text{alg}}, T) \leq c_1(\mathcal{Z}, \epsilon)T^\epsilon. \quad (3.1)$$

On the other hand, there are lower bounds for the degree of torsion points. Suppose  $A$  is defined over a number field  $K$ . For  $P \in \text{tor}(A)$  set  $d(P) = [K(P) : \mathbb{Q}]$ . Then Masser [M] proves that

$$d(P) \geq c_2(A)T^\rho$$

for some  $c_2(A) > 0$  and some  $\rho > 0$  which depends only on the dimension  $g$  of  $A$ . Note that all the conjugates of  $P$  over a number field of definition for both  $A, X$  are still torsion points on  $X$ , of the same order as  $P$ . By the lower bound of Masser just displayed, the number of such conjugates is at least  $c_3(A)T^\rho$ . Hence we get at least

$$c_3(A)T^\rho$$

distinct points  $z \in \mathcal{Z}$  of height  $\leq T$ , corresponding to  $P$  and its conjugates. Choosing in (3.1)  $\epsilon = \rho/2$  and comparing the estimates so obtained we conclude that  $T$  is bounded.  $\square$

As anticipated in the Introduction, we note that the same arguments may be used to prove the stronger version given by Theorem 1.1\*, at the cost of adding some simple geometrical facts on subvarieties of abelian varieties and cosets contained in them. Let us briefly recall this here. The point is to describe the cosets contained in  $X$ . Call such a coset  $b + B$  ( $b$  a point,  $B$  an abelian subvariety of  $A$ ) *maximal* if it is not included in any other larger torus coset still contained in  $X$ . Then one may prove that the set of abelian subvarieties  $B$  is finite, for  $b + B$  running through maximal cosets. A simple proof of this is in [BZ], Lemma 2. (It uses only rather standard considerations involving degrees.) With this in mind, we proceed to illustrate the proof of Theorem 1.1\* by induction on  $\dim A$ , the case of dimension 0 (or 1) being indeed trivial. Using exactly the same method of proof of the weaker version, it suffices, through Theorem 2.1, to deal with the torsion points which lie in some coset  $b + B$  contained in  $X$  and having dimension  $> 0$ . We may then assume that this coset is maximal, and thus that  $B$  lies in a certain finite set. Thus for our purposes we may assume that  $B$  is fixed. The quotient  $A/B$  has the structure of an abelian variety  $A'$  and we have  $\dim A' < \dim A$ . Let  $\pi : A \rightarrow A'$  be the natural projection. The set of  $b \in A$  such that  $b + B$  is contained in  $X$  is easily seen to be an algebraic variety, which projects to a variety  $X'$  under  $\pi$ . The coset  $b + B$  contains a torsion point if and only if  $\pi(b)$  is torsion on  $A'$ . Now, the induction assumption applied to  $A', X'$  easily concludes the proof.

**Final remarks.**

1. The Manin-Mumford conjecture for  $A$  and  $X$  defined over  $\mathbb{C}$  follows from the above by specialization arguments, as in the original papers of Raynaud. Moreover, it is known that a version that is uniform (regarding the number of cosets required to contain all the torsion points) as  $X$  varies over a family of subvarieties of fixed dimension and degree of a given abelian variety follows from the above version, as described in [BZ] or see Hrushovski [H] or Scanlon [Sc] (so-called “automatic uniformity”).
2. It seems that an argument along the present lines can also be given for the easier multiplicative version of Manin-Mumford mentioned in §1. Here the role of our Theorem 2.1 would be played by the theorem of Ax [Ax] establishing Schanuel’s conjecture for power series.
3. Quantitative versions of the Manin-Mumford conjecture (and indeed of the Mordell-Lang conjecture) giving upper estimates for the number of torus cosets are given by Rémond [Re]. Good dependencies e.g. on the degree of  $X$  when  $A$  is fixed also follow from Hrushovski’s proof [H]. Our method does not for the present yield any new quantitative information. The result of Pila-Wilkie holds in an arbitrary o-minimal structure. In this generality it seems one could not hope for good dependence e.g. on the degree of  $X$  in the Pila-Wilkie result. For the specific case of sets defined

by algebraic relations among theta-functions one might hope for good bounds, but we have not pursued this. For the multiplicative version, bounds with good dependencies on the degrees of the varieties were obtained by Rémond [Re2]; see also Beukers-Smyth (for curves) and Aliev-Smyth (in general) [BS, AS].

4. It seems clear that the constant  $c_1$  can be taken uniformly as  $X$  varies over all subvarieties of  $A$  of fixed dimension and degree, by standard uniformity properties in o-minimal structures.

5. In a paper in progress by Masser and Zannier, the present method has been applied to prove the following: *For  $l \neq 0, 1$ , let  $E_l$  be the elliptic curve  $y^2 = x(x-1)(x-l)$  and let  $P_l, Q_l$  be two points on  $E_l$  with  $x$ -coordinate resp.  $2, 3$ . Then there are only finitely many complex values of  $l$  such that both  $P_l, Q_l$  are torsion on  $E_l$ .* This kind of result is related to Silverman's specialization theorem, a special case of which implies that the  $l \in \mathbb{C}$  such that  $P_l$  or  $Q_l$  is torsion form a set of algebraic numbers of bounded height. The finiteness statement however seems not to follow directly from any known result. (The Manin-Mumford statement, applied to a suitable subvariety of  $E \times E$ , would work if  $P_l, Q_l$  would be taken as variable  $\mathbb{Z}$ -independent points on a fixed elliptic curve  $E$ .)

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