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Note on Carlson's theorem

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Abstract. Carlson's well known theorem gives conditions under which a function, holomorphic in the right half plane and of exponential type, is uniquely determined by its value sequence on $\mathbb{N}$. This note gives a variant in which the function is permitted (slightly) faster than exponential growth on the positive real axis.

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Carlson's well known theorem [2] may be stated as follows. *Suppose that a function $f(z)$, holomorphic in $\mathrm{Re}(z) \geq 0$, is of exponential type and has type $< \pi$ on the imaginary axis. If $f(n) = 0$ for $n \in \mathbb{N} = \{0, 1, 2 \ldots\}$ then $f(z)$ vanishes identically* (see e.g. [1, 9.2.1], or a slightly weakened version in [6, 5.81]). Thus a function satsifying the stated growth conditions is uniquely determined by its value sequence on $\mathbb{N}$.

This note is concerned with the following question: suppose a function $f(z)$, holomorphic in $\mathrm{Re}(z) \geq 0$, has exponential growth of type $< \pi$ on the imaginary axis. Can the growth of $f(z)$ on the real axis be permitted to be somewhat faster than exponential type and still preserve the property of unique determination by the value sequence on $\mathbb{N}$?

It turns out there is a simple observation that can be made. Namely, by means of the Gamma function $\Gamma(z)$, growth of type $\exp(\alpha|z|\pi/2)$ on the imaginary axis, where $\alpha \in \mathbb{R}$, may be exchanged for growth of type $\exp(\alpha|z|\log|z|)$ on the real axis. After stating and proving the result I say a little more about the context in which this question arose. The case $c = 0$ is a slight strengthening of Carlson's theorem, while the case $c = 1/2$ has been given by Yoshino [8] under somewhat more stringent hypotheses.

Theorem. *Let $c, \gamma, \delta \in \mathbb{R}$ with $c + \gamma < 1$ and $\delta > 0$. Write $z = x + iy$. Suppose $f(z)$ is holomorphic in the region $x \geq 0$ and satisfies*

$$\limsup_{|y|\to\infty} \frac{\log|f(iy)|}{\pi|y|} \leq \gamma, \qquad \limsup_{x\to\infty} \frac{\log|f(x)|}{2x\log x} \leq c$$

and that, throughout $x \geq 0$, as $|z| \to \infty$ (i.e. uniformly in the argument of z),

$$\log|f(z)| = O(|z|^{2-\delta}).$$

Suppose $f(n) = 0$ for all $n \in \mathbb{N}$. Then $f(z)$ vanishes identically.

Remarks.

1. The conclusion may fail if $c+\gamma = 1$, or if $\delta = 0$. As witnesses for the former failure, the functions $\sin \pi z$, for which $\gamma = 1, c = 0, \delta = 1$ (the usual example showing Carlson's theorem is sharp), $\Gamma(z+1)/\Gamma(-z)$, for which $\gamma = 0, c = 1, 0 < \delta < 1$; indeed, taking any $t \in \mathbb{R}$,

$$\frac{1}{\Gamma(-z)\Gamma(z+1)^{2t-1}}$$

(see the start of the proof for the definition of powers of $\Gamma(z+1)$) gives an example for which $\gamma = 1-t, c = t, 0 < \delta < 1$. As witness for the latter failure: the function

$$\exp(2\pi(ze^{-i\pi/4})^2) - 1$$

for which $c = \gamma = \delta = 0$.

2. Note that γ, c are permitted to be negative. The motivating problem discussed below involves a function T that is bounded on vertical lines (so that $\gamma = 0$), and $c = 1/2$. The class of functions regular in $x \geq 0$ with $\gamma = 0, c < 1$ seems to be an interesting class of functions that enjoy unique determination by their values on $\mathbb{N}$. Classes with $\gamma < 0$ might also be of interest in certain problems related to entire arithmetic functions (since e.g. $\Gamma(z+1)$ is in the class with $\gamma = -1/2, c = 1/2$) but I do not know of specific applications. Classes with $c < 0$ would seem to be less interesting in this regard since, if such functions are integer valued on $\mathbb{N}$, they are forced to vanish on all but finitely many points of $\mathbb{N}$.

Proof. To prove the theorem, one could proceed using Phragmén-Lindelöf directly, following the lines of Carlson's original demonstration. However, it is easier simply to transport the problem into Carlson's setup, by an initial application of Phragmén-Lindelöf.

Let us first remark that the function $\Gamma(z+1)$ is regular and never zero in $x \geq 0$. So $\log \Gamma(z+1)$ can be defined in a regular manner there, and thence the function $\Gamma(z+1)^\alpha$ for any $\alpha \in \mathbb{C}$. The principal value of the logarithm will be taken. By Stirling's formula (see [7, 13.6]),

$$\log \Gamma(z+1) = \big(z + \frac{1}{2}\big) \log z - z + \frac{1}{2} \log(2\pi) + O(z^{-\frac{1}{2}})$$

as $z \to \infty$, uniformly in $x \geq 0$.

Let $c' > c$ with $\gamma + c' < 1$. Then the function

$$g(z) = f(z)/\Gamma(z+1)^{2c'}$$

is regular in $x \geq 0$, bounded on the real axis and satisfies

$$\limsup_{|y|\to\infty} \frac{\log |g(iy)|}{|y|} \ \leq \ \gamma\pi + \frac{2c'\pi}{2} \ < \ \pi.$$

Further, for any $\delta' < \delta$,

$$|g(z)| = O(\exp(|z|^{2-\delta'}))$$

as $|z| \to \infty$ for $x \geq 0$, uniformly in the argument of z.

Now consider the function

$$h(z) = g(z)\exp(-\sqrt{2}\pi z e^{-\pi i/4})$$

in the first quadrant. This function is bounded on the lines $x = 0, y = 0$, and satisfies, for any $\delta'' < \delta'$,

$$|g(z)| = O(\exp(|z|^{2-\delta''}))$$

as $|z| \to \infty$ in the quandrant, uniformly in the argument of z.

By Phragmén-Lindelöf (see e.g. [6, 5.61]), $h(z)$ is bounded in the first quadrant, and so $g(z)$ is of exponential type there. By a similar argument $g(z)$ is of exponential type in the fourth quadrant, and thus $g(z)$ is of exponential type throughout $x \geq 0$.

So $g(z)$ satisfies the hypotheses of Carlson's theorem, and since g vanishes on $\mathbb{N}$ it must vanish identically. Hence $f(z)$ vanishes identically, proving the result. □

Let me say a little more about the context in which this question arose. Let $k \in \mathbb{N}$. In [4] I consider entire functions $f(z)$ whose value sequence on $\mathbb{N}$ has the following property: On every subset of $\mathbb{N}$ consisting of $\leq k+1$ points, the values of $f(z)$ may be interpolated by an element of $\mathbb{Z}[x]$. This generalizes the "integer valued entire function" ($k = 0$) case that is the subject of Pólya's paper [5]. I prove that if the exponential type of such a function is below a certain type (depending on k) on the real axis, and $< \pi$ on the imaginary axis, then the function reduces to a polynomial. However, I am not able to exhibit a "smallest" entire function with the requisite property, as does Pólya, for $k = 0$, with the function 2^z.

The "$k = \infty$" property can also be considered: namely, consider entire functions that admit interpolation by an element of $\mathbb{Z}[x]$ on *every* finite subset of $\mathbb{N}$. By [4, Theorem 1.4], a (near) "smallest" transcendental entire function having this property but not able to be interpolated by a polynomial on *all* $\mathbb{N}$ is provided by

$$T(z) = \int_0^\infty (1+t)^z e^{-t} dt.$$

Note that $T(z) = e\Gamma(z+1, 1)$, where $\Gamma(z, w)$ is the so-called *complementary incomplete Gamma function* (see e.g. [3, Ch.2 §5]).

Repeated integration by parts verifies that $T(z)$ interpolates the following series, which is convergent only for $z \in \mathbb{N}$ (when it reduces to a finite sum):

$$T(z) = 1 + z + z(z-1) + z(z-1)(z-2) + \dots.$$

The value sequence on $\mathbb{N}$ of this series, and hence of $T(z)$, is in some sense (see [4]) canonical for the "$k = \infty$" property. While $T(z)$ is not of exponential type on the real axis (it grows like $e\Gamma(x+1)$), it is bounded (even slowly decaying) on vertical lines. The question arose: Within which class(es) of functions does $T(z)$ provide the unique interpolation of its values on $\mathbb{N}$? The Theorem and the above observations show that the class of functions satisfying the growth hypotheses of the Theorem with (c, γ, δ), where $\gamma + c < 1, \delta > 0$, is such a class if $1/2 \leq c < 1$.

The question considered here can also be considered for functions holomorphic in the whole plane, i.e. entire functions. Thus an entire function of exponential type that is of type $< \pi$ on the imaginary axis is determined by its value sequence on $\mathbb{Z}$, being already determined by its value sequence on $\mathbb{N}$. Suppose an entire function has exponential growth of type $< \pi$ on the imaginary axis. How fast can it grow on the real axis while preserving its unique determination by its values on $\mathbb{Z}$?

Since the Gamma function is not entire, the observation made in the present note does not extend to the entire function situation. This question seems to be interesting in itself, but it is also relevant to the corresponding problem of entire functions with the abovedescribed arithmetic conditions holding on all $\mathbb{Z}$.

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