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Integer points on the dilation of a subanalytic surface

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Abstract

Let $\Omega \subset \mathbb{R}^n$ be a compact subanalytic set of dimension 2 and $t \geq 1$. This paper gives an upper bound as $t \to \infty$ for the number of integer points on the homothetic dilation $t\Omega$ of Ω that do not reside on any connected semialgebraic subset of $t\Omega$ of positive dimension. Implications for the density of rational points on Ω are also elaborated.

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1. Introduction

For a set of points $\Omega \subset \mathbb{R}^n$ let $\Omega(\mathbb{Z})$ denote the subset consisting of points with integer coordinates and $\Omega(\mathbb{Q})$ the subset of points with rational coordinates. For $H \geq 1$ let $\Omega(\mathbb{Q}, H)$ denote the subset of $\Omega(\mathbb{Q})$ of rational points P of *height* $H(P) \leq H$, where, if $P = (a_1/b, \ldots, a_n/b)$ with $a_j, b \in \mathbb{Z}, b > 0$ and $\gcd(a_1, \ldots, a_n, b) = 1, H(P) = \max\{|a_j|, b\}$. The *homothetic dilation* $t\Omega$ of Ω by t is defined by

$$t\Omega = \{(tx_1, \ldots, tx_n) : (x_1, \ldots, x_n) \in \Omega\}.$$

It will always be assumed that $t \geq 1$. The cardinality of a set S will be denoted $\#S$.

When Ω is an algebraic variety, an elaborate complex of results and conjectures asserts that the geometry of Ω exerts significant control on the structure of $\Omega(\mathbb{Z})$ and $\Omega(\mathbb{Q})$, and on the behaviour of $\#\Omega(\mathbb{Q}, H)$ as $H \to \infty$. See for example [9] or [6, §F.5] ("Geometry Governs Arithmetic") and the references therein. In particular, the conjectures of Lang [9, I §3] assert that a variety has only finitely many rational points outside its *special set*.

Suppose that Ω is a *subanalytic* set (a definition of subanalytic sets, and statements of the key properties to be used, are set out in §2, following [2]). If the *dimension* of Ω (see §2) is ≥ 2 then Ω may contain subsets of positive dimension that are semialgebraic even if Ω itself is not semialgebraic. Let Ω^{alg} denote the union of all connected subanalytic subsets of Ω of positive dimension that are semialgebraic (defined over $\mathbb{R}$). The distribution of integer and rational points on Ω^{alg} will be governed by the geometry of its semialgebraic constituents (in general Ω^{alg} will not itself be semialgebraic, or even subanalytic (see below)).

This paper is concerned with the complementary subset $\Omega^{\mathrm{trans}} = \Omega - \Omega^{\mathrm{alg}}$. Note that $(t\Omega)^{\mathrm{alg}} = t(\Omega^{\mathrm{alg}})$, and so likewise $(t\Omega)^{\mathrm{trans}} = t(\Omega^{\mathrm{trans}})$. By analogy with the philosophy of the special set, it might be expected that strong paucity properties should hold for the rational and integral points of Ω^{trans}. It seems natural to make the following conjectures. Note that these conjectures are trivial for semialgebraic Ω.

Conjecture 1.1. *For compact subanalytic $\Omega \subset \mathbb{R}^n$ and $\epsilon > 0$ there is a constant $c_1(\Omega, \epsilon)$ such that*

$$\# t\Omega^{\mathrm{trans}}(\mathbb{Z}) \le c_1(\Omega, \epsilon)\, t^\epsilon.$$

Conjecture 1.2. *For compact subanalytic $\Omega \subset \mathbb{R}^n$ and $\epsilon > 0$ there is a constant $c_2(\Omega, \epsilon)$ such that*

$$\#\Omega^{\mathrm{trans}}(\mathbb{Q}, H) \le c_2(\Omega, \epsilon)\, H^\epsilon.$$

A subanalytic set $\Omega \subset \mathbb{R}^n$ will be called a *subanalytic curve* if it has dimension 1, and a *subanalytic surface* if it has dimension 2. For compact subanalytic curves that are graphs of functions in $\mathbb{R}^2$, Conjecture 1.1 is proved in [3] and Conjecture 1.2 is proved in [10] (see also [4]). It is not difficult to extend these results to general compact subanalytic curves in $\mathbb{R}^n$; proofs are included here (in §7) as the more general formulation of 1.1 is needed for the present purposes. The main result of this paper is to establish Conjecture 1.1 for compact subanalytic surfaces.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a compact subanalytic surface and $\epsilon > 0$. There is a constant $c_3(\Omega, \epsilon)$ such that*

$$\#\, t\,\Omega^{\mathrm{trans}}(\mathbb{Z}) \le c_3(\Omega, \epsilon) t^\epsilon.$$

While the primary focus in this paper will be the number of integer points on homothetic dilations, results about $t\Omega(\mathbb{Z})$ give information about $\Omega(\mathbb{Q}, H)$: For *any* set $\Omega \subset \mathbb{R}^n$, if $P \in \Omega(\mathbb{Q}, H)$ then $bP \in b\Omega(\mathbb{Z})$, where $P = (a_1/b, \ldots, a_n/b)$ as above. Thus

$$\#\Omega(\mathbb{Q}, H) \;\le\; \sum_{h=1}^{H} \# h\,\Omega(\mathbb{Z}).$$

Corollary 1.4. *For Ω as in the Theorem and $\epsilon > 0$ there is a constant $c_4(\Omega, \epsilon)$ such that*

$$\#\Omega^{\mathrm{trans}}(\mathbb{Q}, H) \le c_4(\Omega, \epsilon) H^{1+\epsilon}. \square$$

Let $\Omega \subset \mathbb{R}^n$ be subanalytic. The example $\Omega = \{(x, y, z) \in \mathbb{R}^3, z = \exp(y \log(x + 1)), (x, y) \in [0, 1]^2\}$ shows that Ω^{alg} is not in general subanalytic: in this case Ω^{alg} is the subset with y rational, and consists of a dense set of curves of unbounded degree (see §2: a relatively compact subanalytic set has only finitely many connected components). It is easy to construct examples where Ω^{alg} consists of all but finitely many points of Ω (e.g. the surface of revolution of a transcendental curve); at the other extreme, Ω may have the property that its intersection with every algebraic space curve consists of a finite number of points, so that Ω^{alg} is empty. An intermediate possibility, for subanalytic surfaces, is that Ω^{alg} is semianalytic of dimension 1 (and hence semialgebraic). The following result can be deduced by appealing to results of [3] to control the points on Ω^{alg}.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a compact analytic submanifold of $\mathbb{R}^n$ of dimension 2. Suppose that Ω^{alg} is semialgebraic of dimension 1. Let $\epsilon > 0$. There is a constant $c_5(\Omega, \epsilon)$ such that*

$$\#\, t\,\Omega(\mathbb{Z}) \le c_5(\Omega, \epsilon)\, t^\epsilon.$$

Corollary 1.6. *For Ω as in the Theorem and $\epsilon > 0$ there is a constant $c_6(\Omega, \epsilon)$ such that*

$$\#\Omega(\mathbb{Q}, H) \le c_6(\Omega, \epsilon) H^{1+\epsilon}. \square$$

To contextualize these results consider first the trivial bounds available for these quantities. If $\Omega \subset \mathbb{R}^n$ is a compact subanalytic set of dimension k it is straightforward to show that

$$\#\ t\,\Omega(\mathbb{Z}) \le c_7(\Omega) t^k, \qquad \#\Omega(\mathbb{Q}, H) \le c_8(\Omega) H^{k+1}.$$

The results of [3, 10] for planar Ω of dimension one can be used to obtain results in higher dimensions by slicing. In this case the presence of embedded semialgebraic sets is immaterial, however it is necessary to assume, in addition to the hypotheses of 1.3, that the selected family of slices are almost all transcendental: *Let K be a compact convex subanalytic subset of $\mathbb{R}^2$ and $\Omega = \{(x, y, z) : (x, y) \in K, z = f(x, y)\}$ where f is real analytic on K. Suppose that, for all but finitely many y such that the set $\{x : (x, y) \in K\}$ is an interval of positive length, the function $f(x, y)$ is a transcendental function of x. Let $\epsilon > 0$. There are constants $c_9(\Omega, \epsilon), c_{10}(\Omega, \epsilon)$ such that*

$$\#t\Omega(\mathbb{Z}) \le c_9(\Omega, \epsilon)\, t^{1+\epsilon}, \qquad \#\Omega(\mathbb{Q}, H) \le c_{10}(\Omega, \epsilon)\, H^{2+\epsilon}.$$

The proof of these statements requires Gabrielov's Theorem (2.4) to accommodate the finitely many exceptional slices. The exponents in both estimates are optimal since the hypotheses do not preclude Ω containing a line segment.

Various upper estimates for the integer points on (hyper)surfaces have been given under hypotheses of a differential-geometric nature (that are thus more accessible than hypotheses controlling Ω^{alg}, or hypotheses that slices are transcendental). Andrews [1] considers the integer points on the surface Ω of a strictly convex closed body in $\mathbb{R}^n, n > 1$. If $S(\Omega)$ denotes the surface content of Ω, Andrews shows that

$$\#\Omega(\mathbb{Z}) \le c_{11}(n) S(\Omega)^{n/(n+1)}.$$

For surfaces in $\mathbb{R}^3$, $S(t\Omega) = t^2 S(\Omega)$, leading to an estimate with exponent $3/2$; note that the constant $c_{11}(n)$ is independent of Ω. For a strictly convex arc $\Omega = \{(x, y), y = f(x)\} \subset \mathbb{R}^2$ such an estimate

$$\#\Omega(\mathbb{Z}) \le 3(4\pi)^{-1/3} S(\Omega)^{2/3} + O(S(\Omega)^{1/3}),$$

in which the constant is best possible, is due to Jarnik [7]. Results for surfaces (and hypersurfaces of higher dimension) are obtained by Schmidt [11, 12]. For example it is shown that, for a surface $\Omega \subset \mathbb{R}^3$ that is sufficiently smooth and not a cylinder (see [11, Theorem 2] for the precise formulation) in a box of side $t \ge 1, \#\Omega(\mathbb{Z}) \ \le c_{12}\ t^{3/2}$ where c_{12} is absolute.

The present paper adapts the methods of [3], showing in the first instance, in §3 and §4, that the points of $t\Omega(\mathbb{Z})$ lie on very few algebraic hypersurfaces. Motivation for this adaptation was provided by the recent work of Heath-Brown [5], wherein a generalization of this same main tool of [3] to higher dimensions is effected by rather different means in the algebraic setting. This step can be carried out for any $\Omega \subset \mathbb{R}^n$ that is compact subanalytic of dimension $< n$, and for rational points as well as dilation-integer points (see 4.3, 4.4 and 4.5).

In §5 and §6, Gabrielov's Theorem (2.4) on subanalytic sets is used to gain uniform control over certain numerical quanta of such hypersurface intersections with $t\Omega$. In §7, Conjectures 1.1 and 1.2 are established for compact subanalytic curves. The proofs of the main theorems are then given in §8. Essential use is made there of a result implicit in [3] that gives a uniform upper estimate for the number of integer points on a sufficiently smooth plane curve for certain classes of curves. This result is recalled in §8. The absence of an analogous result for rational points of bounded height is the obstruction to establishing Conjecture 1.2 for compact subanalytic surfaces. Some further remarks are made at the end of §8, including a discussion of the effectivity of the constants in the main results.

The problems considered here arose from a conversation in which it was asked whether better bounds would be expected for $\#t\Omega(\mathbb{Z})$, for compact Ω of dimension > 1, when Ω was not semialgebraic. (For Ω of dimension 1 this question is answered affirmatively by the results of [3].) This prompted the conjectures formulated here without any particular applications in view. The results obtained here show similar behaviour in dimension 2 to dimension 1, namely that $\#t\Omega(\mathbb{Z})$ can only have growth that is a positive power of t as $t \to \infty$ if Ω contains semialgebraic subsets of positive dimension.

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2. Subanalytic sets

The following characterization of subanalytic sets, and statements of the key properties of subanalytic sets to be used in the sequel, are taken from [2]. Let M be a real analytic manifold.

Definition 2.1. ([2, §3.13 (3)]) A subset $X \subset M$ is *subanalytic* if each point P of M has a neighbourhood U such that $X \cap U$ belongs to the class of subsets of U obtained using finite intersection, finite union and complement, from the family of closed subsets of U of the form $f(A)$, where A is a closed analytic subset of a real analytic manifold N, $f : N \to U$ is real analytic, and $f|A$ is proper.

Definition 2.2. ([2, §3.3, §3.5, §7.1]) Let X be a subanalytic subset of M and $P \in X$. Then P is a *smooth point* of X (of *dimension* k) if, in some neighbourhood of P in M, X is an analytic submanifold (of dimension k). The *dimension* of X is the highest dimension of its smooth points. The *singular set* of X, denoted $\mathrm{sing}(X)$, is the complement in X of the smooth points of highest dimension. A subanalytic set X is *smooth* if every point of X is smooth, i.e., if X is an analytic submanifold of M.

Uniformization Theorem 2.3. ([2, §0.1]) *Let X be a closed subanalytic subset of M. Then there is real analytic manifold N of the same dimension as X and a proper real analytic mapping $\psi : N \to M$ with $\psi(N) = X$.* □

If X is a subanalytic set then the number of connected components of X is locally finite ([2, §3]), and hence is finite if X is relatively compact. The number of connected components of a (relatively compact) subanalytic set X will be denoted $\mathrm{cc}(X)$.

Gabrielov's Theorem 2.4. ([2, §3.14]) *Let N, Y be real analytic manifolds and $p : N \times Y \to Y$ the projection on the second factor. Let $X \subset N \times Y$ be relatively compact and subanalytic. There exists a constant $c_{13}(N, Y, X)$ such that $\mathrm{cc}\big(X \cap p^{-1}(y)\big) \leq c_{13}$ for every $y \in Y$.* □

Tamm's Theorem 2.5. ([2, §7.2]) *Let $X \subset M$ be subanalytic. For each $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the set of smooth points of X of dimension k is subanalytic. In particular, $\mathrm{sing}(X)$ is subanalytic.* □

3. A consequence of Taylor's Formula

The setup below follows that given in [8, Chapter 2]. Let $k \in \mathbb{N}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{N}^k$. Set $|\mu| = \mu_1 + \mu_2 + \ldots + \mu_k$, $\mu! = \mu_1!\mu_2! \ldots \mu_k!$ and, when $x = (x_1, x_2, \ldots, x_k)$, write x^μ for $x_1^{\mu_1} x_2^{\mu_2} \ldots x_k^{\mu_k}$. Thus, for $d \in \mathbb{N}$, monomials of exact degree d in k variables are indexed by elements of

$$\Lambda_k(d) = \{\mu \in \mathbb{N}^k, |\mu| = d\}.$$

Let $L_k(d) = \#\Lambda_k(d)$. Monomials of degree $\leq d$ in k variables are indexed by elements of

$$\Delta_k(d) = \{\mu \in \mathbb{N}^k, |\mu| \leq d\} = \bigcup_{\delta=0}^{d} \Lambda_k(\delta).$$

Set $D_k(d) = \#\Delta_k(d)$. For variables $x = (x_1, x_2 \ldots, x_k)$ put

$$\frac{\partial^\mu}{\partial x^\mu} = \frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \frac{\partial^{\mu_2}}{\partial x_2^{\mu_2}} \cdots \frac{\partial^{\mu_k}}{\partial x_k^{\mu_k}}.$$

For $z = (z_1, z_2, \ldots, z_k), y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$ write $z - y$ for $(z_1 - y_1, z_2 - y_2, \ldots, z_k - y_k)$.

With this notation, suppose $\phi : \mathbb{R}^k \to \mathbb{R}$ is defined and has $b + 1$ continuous derivatives at each point of the line segment joining $y, z \in \mathbb{R}^k$. According to Taylor's Formula ([8, Theorem 2.2.5]) there is a point ξ on this line segment such that

$$\phi(z) = \sum_{\mu \in \Delta_k(b)} \frac{1}{\mu!} \frac{\partial^\mu}{\partial x^\mu} \phi(y)(z - y)^\mu + \sum_{\mu \in \Lambda_k(b+1)} \frac{1}{\mu!} \frac{\partial^\mu}{\partial x^\mu} \phi(\xi)(z - y)^\mu.$$

The quantities $L_k(d), D_k(d)$ satisfy the relations

$$D_k(d) = \sum_{\delta=0}^{d} L_k(\delta), \qquad L_{k+1}(d) = D_k(d).$$

Note that $L_k(d) = \binom{k-1+d}{k-1}$, from which it follows that, for fixed k, $L_k(d) \in \mathbb{Q}[d]$ with degree $k - 1$ and leading coefficient $1/(k - 1)!$, while $D_k(d) \in \mathbb{Q}[d]$ with degree k and leading coefficient $1/k!$.

Let $k, n, d \in \mathbb{N}$. There is a unique $b = b(k, n, d)$ such that $D_k(b) \leq D_n(d) < D_k(b+1)$. Set

$$B(k, n, d) = \sum_{\beta=0}^{b} L_k(\beta)\beta + \left(D_n(d) - \sum_{\beta=0}^{b} L_k(\beta)\right)(b+1).$$

Lemma 3.1. *Let $k, n, d \in \mathbb{N}$ and put $D = D_n(d)$. Suppose $\phi_1, \ldots, \phi_D : \mathbb{R}^k \to \mathbb{R}$ are functions possessing continuous derivatives up to order $b(k, n, d) + 1$ on a compact convex set J. There is a constant $c_{14}(J, \phi_1, \ldots, \phi_D)$ with the following property. Let $U \subset \mathbb{R}^n$ be a disk of radius $r \leq 1$ and $z^{(1)}, \ldots, z^{(D)} \in J \cap U$. Then*

$$|\det(\phi_i(z^{(j)}))| \leq c_{14}(J, \phi_1, \ldots, \phi_D)\, r^{B(k,n,d)}.$$

Proof. The intersection $J \cap U$ is a convex set, and there is a point $z^{(0)}$ of $J \cap U$ such that every other point of $X \cap U$ is at a distance $\leq r$ (take a point of $J \cap U$ nearest to the center of U). Write each entry of $\det(\phi_i(z^{(j)}))$ using Taylor's Formula with remainder term of order $b + 1$ about $z^{(0)}$. In expanding the determinant, consider the terms corresponding to a particular specification of the number of terms of each order of derivative.

Consider a minor of size $h \times h$ of $\det(\,\phi_i(z^{(j)})\,)$ comprising the expansion terms of degree $\beta \leq b$ only. That is, select h of the points $\zeta^{(j)}$ from among the $z^{(j)}$, and h functions ψ_i from among the ϕ_i and consider

$$\det\left(\sum_{\mu \in \Lambda_k(\beta)} \frac{1}{\mu!}\frac{\partial^\mu}{\partial x^\mu}\psi_i(\zeta^{(0)})(\zeta^{(j)} - z^{(0)})^\mu\right).$$

If $h > L_k(\beta)$ then the columns are dependent, and the minor vanishes.

Thus if, for a particular such specification of orders, there are more than $L_k(\beta)$ terms of degree β to be taken for some β, then the totality of terms corresponding to this choice vanishes. Therefore the order of the lowest order nonvanishing term is $B(k, n, d)$, and so

$$|\det(\phi_i(z^{(j)}))| \leq c_{14}\, r^{B(k,n,d)}$$

where c_{14} is a certain function of the maximum sizes of the derivatives of the ϕ_i (up to order $b(k, n, d) + 1$) on $J \cap U$, and powers of r. Since $r \leq 1$, c_{14} can be taken to depend only on the other quantities. □

4. Exploring with algebraic hypersurfaces

The first step in the proof of Theorem 1.3 is to show that the integer points of $t\Omega$ lie on very few algebraic hypersurfaces in $\mathbb{R}^n$. This is essentially a higher dimensional version of the "Main Lemma" of [3]. A result of this nature (but proved rather differently) has been given by Heath-Brown [5, Theorem 14] in the algebraic case. This step does not require analyticity of Ω, only that it is the union of finitely many images in $\mathbb{R}^n$ of C^∞ maps of convex compact subsets of $\mathbb{R}^k$.

It is convenient to work within Ω rather than its dilations $t\Omega$, so set

$$\Omega(\mathbb{Z},t)=\{Q\in\Omega : tQ\in t\Omega(\mathbb{Z})\}.$$

Let $k,n,d\in\mathbb{N}$. Set $B=B(k,n,d)$ as previously and put

$$V=V(k,n,d)=\sum_{\beta=0}^{d}L_n(\beta)\beta, \qquad\qquad \epsilon(k,n,d)=\frac{kV}{B}.$$

It follows from the observations in §3 that, k,n being fixed and $d\to\infty$,

$$b(k,n,d)=\left(\frac{k!d^n}{n!}\right)^{1/k}(1+o(1)),\qquad B(k,n,d)=\frac{1}{(k+1)(k-1)!}\left(\frac{k!}{n!}\right)^{(k+1)/k}d^{n(k+1)/k}(1+o(1))$$

while

$$V(k,n,d)=\frac{1}{(n+1)(n-1)!}d^{n+1}(1+o(1)),$$

whence, if $k<n$, $\epsilon(k,n,d)\to 0$.

Proposition 4.1. *Let $J\subset\mathbb{R}^k$ be a convex compact set, $d\in\mathbb{N}$, and $\phi=(\phi_1,\dots,\phi_n):J\to\mathbb{R}^n$ where the functions $\phi_j:J\to\mathbb{R}$ have $b(k,n,d)+1$ continuous derivatives. Let $\Omega\subset\mathbb{R}^n$ be the image of J under ϕ and $t\geq 1$. There is a constant $c_{15}(J,\phi,d)$ such that $\Omega(\mathbb{Z},t)$ is contained in the union of at most*

$$c_{15}(J,\phi,d)\,t^{\epsilon(k,n,d)}$$

algebraic hypersurfaces (possibly reducible) of degree $\leq d$.

Proof. Let U be a disk in $\mathbb{R}^k$ of radius $r\leq 1$ and consider points $Q_j\in J\cap U\cap\Omega(\mathbb{Z},t)$, $j=1,\dots,D_n(d)$. Applying Lemma 3.1 with the functions

$$\phi^\mu=(\phi_1(x_1,\dots,x_k),\dots,\phi_n(x_1,\dots,x_k))^\mu,$$

$\mu\in\Delta_n(d)$ there is a constant $c_{14}(J,\phi,d)$ such that

$$|\det(\phi_\mu(Q_j)|\leq c_{14}(J,\phi,d)r^{B(k,n,d)}.$$

However, the assumption that the points $Q_i\in\Omega(\mathbb{Z},t)$ implies that

$$t^{V(k,n,d)}\det\big(\phi_\mu(Q_j)\big)\in\mathbb{Z}.$$

Thus if $r<(c_{14}t^V)^{-1/B}$ then $\det\big(\phi_\mu(Q_j)\big)=0$ for any selection of $Q_j\in J\cap U$ for which $Q_j\in\Omega(\mathbb{Z},t)$. Then by the argument of [3, Lemma 1], all these points lie on a single hypersurface of degree $\leq d$. Now J may be covered by at most $c_{15}(J,\phi,d,c_{14})\,t^{kV/B}$ closed disks of radius $\leq(1/2)(c_{14}t^V)^{-1/B}$. □

For $k, n, d \in \mathbb{N}$ set $\epsilon'(k,n,d) = kdD_n(d)/B(k,n,d)$. Observe that, k, n being fixed with $k < n$, $\epsilon'(k,n,d) \to 0$ as $d \to \infty$.

Proposition 4.2. *Let $J \subset \mathbb{R}^k$ be a convex compact set, $d \in \mathbb{N}$, and $\phi = (\phi_1, \ldots, \phi_n) : J \to \mathbb{R}^n$ where the functions $\phi_j : J \to \mathbb{R}$ have $b(k,n,d) + 1$ continuous derivatives. Let $\Omega \subset \mathbb{R}^n$ be the image of J under ϕ and $t \geq 1$. There is a constant $c_{16}(J, \phi, d)$ such that $\Omega(\mathbb{Q}, H)$ is contained in union of at most*

$$c_{16}(J, \phi, d)\, H^{\epsilon'(k,n,d)}$$

algebraic hypersurfaces (possibly reducible) of degree $\leq d$.

Proof. The proof proceeds exactly as that for 4.1. Let U be a disk in $\mathbb{R}^k$ of radius $r \leq 1$ and consider points $Q_j \in J \cap U \cap \Omega(\mathbb{Q}, H), j = 1, \ldots, D_n(d)$. Suppose $b_j\phi(Q_j) \in b_j\Omega(\mathbb{Z})$ where $|b_j| \leq H$. Then

$$\prod_{j=1}^{D_n(d)} b_j^d \det(\phi_\mu(Q_j) \in \mathbb{Z}.$$

Now $\prod_j b_j^d \leq H^{dD_n(d)}$ and so all such points Q_j in a disk of radius $r < \left(c_{14}H^{-dD_n(d)}\right)^{-1/B}$ in $\mathbb{R}^k$ lie on a single hypersurface of degree $\leq d$. Now J may be covered by at most $c_{16}(J, \phi, d, c_{14})\, H^{kdD_n(d)/B}$ closed disks of radius $\leq (1/2)(c_{14}t^{D_n(d)})^{-1/B}$. □

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a compact subanalytic set of dimension $k < n$, and $\epsilon > 0$. There are constants $c_{17}(\Omega, \epsilon)$ and $d_1(k, n, \epsilon)$ such that $t\Omega(\mathbb{Z})$ is contained in the intersection of $t\Omega$ with at most*

$$c_{17}(\Omega, \epsilon)t^\epsilon$$

algebraic hypersurfaces in $\mathbb{R}^n$ of degree $\leq d_1(k, n, \epsilon)$.

Proof. Let $d \in \mathbb{N}$. By the Uniformization Theorem there is a real analytic manifold N of dimension k, and a proper analytic mapping $\psi = (\psi_1, \ldots, \psi_n) : N \to \mathbb{R}^n$ with $\psi(N) = \Omega$. Since ψ is proper, N is compact. Now N is contained in the union of finitely many subsets homeomorphic to the closed unit ball in $\mathbb{R}^k$, the homeomorphism analytic in an open neighbourhood of the ball. Applying 4.1. to the map from each such ball into $\mathbb{R}^n$ finds the points of $\Omega(\mathbb{Z}, t)$, and hence those of $t\Omega(\mathbb{Z})$, contained in the union of at most

$$c_{15}(\Omega, d)t^{\epsilon(k,n,d)}$$

hypersurfaces of degree $\leq d$. The observation that $\epsilon(k, n, d) \to 0$ as $d \to \infty$ completes the proof. □

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a compact subanalytic set of dimension $k < n$, and $\epsilon > 0$. There are constants $c_{18}(\Omega, \epsilon)$ and $d_2(k, n, \epsilon)$ such that $\Omega(\mathbb{Q}, H)$ is contained in the intersection of Ω with at most*

$$c_{18}(\Omega, \epsilon)H^\epsilon$$

algebraic hypersurfaces in $\mathbb{R}^n$ of degree $\leq d_2(k, n, \epsilon)$.

Proof. Let $d \in \mathbb{N}$. Apply the Uniformization Theorem as in the proof of 4.3, and apply 4.2 to find the points of $\Omega(\mathbb{Q}, H)$ contained in the union of at most

$$c_{16}(\Omega, d)H^{\epsilon'(k,n,d)}$$

hypersurfaces of degree $\leq d$. But $\epsilon'(k, n, d) \to 0$ as $d \to \infty$. □

Remark 4.5. In fact the same conclusion obtains (with different constants $c_{19}(\Omega, \epsilon), d_3(k, n, \epsilon)$) if, instead of the height H induced from the embedding of $\mathbb{R}^n$ in $\mathbb{P}^n$, the height H_* corresponding to its embedding in $(\mathbb{P}^1)^n$ is used. Namely, if $P = (a_1/b_1, \ldots, a_n/b_n) \in \mathbb{Q}^n$ with $a_j, b_j \in \mathbb{Z}$, $b_j > 0$ and $\gcd(a_j, b_j) = 1$ for each j, then $H_*(P) = \max\{|a_j|, b_j\}$. To prove this, proceed as in the proofs of 4.3 and 4.4 but with $\epsilon''(k, n, d) = nkdD_n(d)/B$.

5. Algebraic hypersurface intersections

Let $\Omega \subset \mathbb{R}^n$ be a closed subanalytic surface. By the Uniformization Theorem (2.3), Ω admits a uniformization $\psi = (\psi_1, \psi_2, \ldots, \psi_n) : N \to \mathbb{R}^n$, $\psi(N) = \Omega$ by a proper analytic map ψ, where N is a real analytic manifold of dimension 2. In this section assume additionally that N is connected.

Let Υ be an algebraic hypersurface in $\mathbb{R}^n$ and put $\Psi = \Omega \cap \Upsilon$. Then Ψ is a subanalytic set. Let $V = \{x \in N : \psi(x) \in \Upsilon\}$. If Υ is the zeroset of $H \in \mathbb{R}[x_1, x_2, \ldots, x_n]$, $H \neq 0$, then V is the zeroset in N of $H(\psi_1, \ldots, \psi_n)$, and is thus an analytic subset of N. The set V admits a decomposition into subanalytic subsets (in fact into semianalytic subsets: see [2]) as described below. This decomposition will be essential in the proof of the main results.

Let $P \in V$. It may be assumed that $P = (0, 0)$ in some local coordinate system (ξ, η) on N. The defining equation H of Υ gives an equation

$$K(\xi, \eta) = H(\psi_1(\xi, \eta), \ldots, \psi_n(\xi, \eta)) = 0$$

defining V in the local coordinates. Define V_s locally as the set of points (ξ, η) with

$$K_\xi(\xi, \eta) = K_\eta(\xi, \eta) = 0.$$

Then V_s is a subanalytic set (in fact analytic).

If V has dimension 2 then this means $K(\xi, \eta)$ vanishes on some open set in N and so, as N is connected, K vanishes identically on N. So $V_s = V$ completes the decomposition in this case.

Otherwise, V has dimension ≤ 1 and it may be assumed (after a possible rotation of coordinates) that $K(\xi, 0)$ is not identically zero. Then by the Weierstrass Preparation Theorem (see [8, Theorem 6.3.1]) it is possible to write

$$K = QU$$

where U does not vanish in a neighbourhood of P and Q is a *distinguished polynomial*, that is, of the form

$$Q(\xi, \eta) = \xi^m + A_1(\eta)\xi^{m-1} + \ldots + A_m(\eta)$$

where $A_i(\eta)$ are analytic in a neighbourhood of P and vanish at P. The set V_s of singular points of V, i.e. the points where $m \geq 2$.

The set $V_{ns} = V - V_s$ of nonsingular points of V is subanalytic. It admits the following further decomposition into subanalytic subsets depending on a finite set $\mathbf{\Pi}$ of coordinatized planes in $\mathbb{R}^n$ and a positive integer M.

Let $\Pi \subset \mathbb{R}^n$ be a plane with coordinates (u, v) and let $\pi : \mathbb{R}^n \to \Pi$ be the orthogonal projection of $\mathbb{R}^n$ onto Π. Composing ψ with π gives a map $(u(\xi, \eta), v(\xi, \eta))$ from N to Π locally at P.

Suppose $P \in V_{ns}$. Then, writing as above $K = QU$ with U nonvanishing in a neighbourhood of P and ($m = 1$ and so) $Q(\xi, \eta) = \xi + A(\eta)$, V is locally parametrized by $\xi = -A(\sigma), \eta = \sigma$. Let V_u denote the subset of V_{ns} of points P at which $du/d\sigma = 0$ and $dv/d\sigma = 0$. Note that

$$du/d\sigma = 0 \iff u_\xi A_\sigma - u_\eta = 0 \iff u_\xi K_\eta - u_\eta K_\xi = 0,$$

$$dv/d\sigma = 0 \iff v_\xi A_\sigma - v_\eta = 0 \iff v_\xi K_\eta - v_\eta K_\xi = 0$$

so that V_u is subanalytic.

At points of $V_{ns} - V_u$, the slope du/dv is well defined (possibly infinite). Let V_a be the set of $P \in V_{ns} - V_u$ at which the slope du/dv belongs to $\{0, \pm 1, \infty\}$. In the local coordinates, these points correspond to the vanishing of

$$\det\begin{pmatrix} u_\xi & u_\eta \\ K_\xi & K_\eta \end{pmatrix}, \ \det\begin{pmatrix} v_\xi & v_\eta \\ K_\xi & K_\eta \end{pmatrix}, \ \det\begin{pmatrix} u_\xi - v_\xi & u_\eta - v_\eta \\ K_\xi & K_\eta \end{pmatrix}, \ \det\begin{pmatrix} u_\xi + v_\xi & u_\eta + v_\eta \\ K_\xi & K_\eta \end{pmatrix}$$

for slope $0, \infty, 1, -1$, respectively. The set V_a is subanalytic.

At points of $V_{ns} - V_u - V_a$, the image of V in Π is locally a graph with respect to both u and v axes. Write $u = f(v), v = g(u)$, where f, g are real analytic functions. Repeated implicit differentiation yields expressions for the derivatives of f, g to any order in the local coordinates ξ, η at P. Explicit expressions may be obtained essentially following [3, Proof of Lemma 5]. Specifically, the dependence $v = v(\xi(\sigma), \eta(\sigma))$ may be inverted locally to obtain, for suitable F, G,

$$u = u(F(v), G(v)), \quad v = v(F(v), G(v)), \quad K(F(v), G(v)) = 0.$$

The successive derivatives of the second and third relations may be used to successively solve for and eliminate $F^{(n)}, G^{(n)}$. Write these equations, for $n \geq 1$, in the form

$$K_\xi F^{(n)} + K_\eta G^{(n)} = P_n(K, v), \quad v_\xi F^{(n)} + v_\eta G^{(n)} = Q_n(K, v),$$

where P_n, Q_n are the appropriate differential rational functions. The the determinant of the system is $K_\xi v_\eta - K_\eta v_\xi$ at every stage. Differentiating the relation for u and substituting finds

$$(K_\xi v_\eta - K_\eta v_\xi)^{2n-1} \frac{d^n u}{dv^n} = R_n(u, v, K), \qquad (K_\xi v_\eta - K_\eta v_\xi)^{2n-1} \frac{d^n v}{du^n} = S_n(u, v, K)$$

for suitable differential polynomials R_n, S_n.

Thus (and here enters the parameter M) the set of points $P \in V_{ns} - V_u - V_a$ at which

$$\frac{d^M u}{dv^M}(\pi(\psi(P))) \cdot \frac{d^M v}{dv^M}(\pi(\psi(P))) = 0$$

is a subanalytic set V_b. Finally the residual set $V_c = V_{ns} - V_u - V_a - V_b$ is subanalytic.

Now suppose $\mathbf{\Pi}$ is a finite collection of coordinatized planes in $\mathbb{R}^n$ and $M \in \mathbb{N}$. For each $\Pi \in \mathbf{\Pi}$ the set V_{ns} admits a decomposition into subsets $V_u^\Pi, V_a^\Pi, V_b^\Pi, V_c^\Pi$. For $\theta : \mathbf{\Pi} \to \{u, a, b, c\}$ let

$$V_\theta = \bigcap_{\Pi \in \mathbf{\Pi}} V_{\theta(\Pi)}^\Pi.$$

Then each V_θ is subanalytic, and the collection of sets V_θ over all θ provides a decomposition of V_{ns} with respect to $\mathbf{\Pi}$ and M. This is the required decomposition.

The discussion above may be summarized as follows, together with one further observation.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a closed subanalytic surface. Let N be a real analytic manifold of dimension 2, with $\psi : N \to \mathbb{R}^n$ a proper real analytic map such that $\psi(N) = \Omega$. Suppose that N is connected. Let $\mathbf{\Pi}$ be a finite collection of coordinatized planes in $\mathbb{R}^n$, and $M \in \mathbb{N}$. Let $\Upsilon \subset \mathbb{R}^n$ be an algebraic hypersurface and $V = \{P \in N, \psi(P) \in \Upsilon\}$.*

Suppose V is of dimension 2. Then $V_s = V$, so that $V_{ns} = V - V_s$ is empty.

Suppose V is of dimension ≤ 1. Then V_s is the set of singular points of V. The complementary set V_{ns} may be decomposed into subanalytic subsets V_θ with respect to $\mathbf{\Pi}$ and M as described above.

Suppose $\Pi \in \mathbf{\Pi}$, with $\pi : \mathbb{R}^n \to \Pi$ the orthogonal projection, and that $\gamma \subset \Pi$ is an algebraic curve. Let $\Gamma = \pi^{-1}(\gamma)$ be the right cylinder in $\mathbb{R}^n$ over $\gamma \subset \Pi$. Put $W = \psi^{-1}(\Gamma)$. Then the intersections $V_\theta \cap W$ are subanalytic. □

6. Application of Gabrielov's Theorem

If Ω is compact and uniformized by N then the sets V, V_s, V_{ns} and the various subsets V_θ under consideration are all relatively compact subanalytic sets. So they have finitely many connected components (see §2). However it will be necessary to have bounds for the number of connected components that depend only on Ω and the *degrees* of the algebraic surfaces Υ, Γ involved (and the parameter M where relevant), but that are otherwise independent of the particular algebraic surfaces. Since the relevant spaces of algebraic surfaces are compact, such bounds may be obtained by appealing to Gabrielov's Theorem (2.4).

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a compact subanalytic surface. Let N be a connected real analytic manifold of dimension 2, and $\psi : N \to \mathbb{R}^n$ a proper real analytic map with $\psi(N) = \Omega$. Let $\mathbf{\Pi}$ be a finite collection of coordinatized planes in $\mathbb{R}^n$. Let $d, M, \delta \in \mathbb{N}$. There exist constants*

$$c_{20}(\Omega, d), \quad c_{21}(\Omega, d, \mathbf{\Pi}, M), \quad c_{22}(\Omega, d, \mathbf{\Pi}, M, \delta)$$

with the following property.

Let $\Upsilon \subset \mathbb{R}^n$ be an algebraic hypersurface of degree $\leq d$. Let $V = \psi^{-1}(\Upsilon)$. Let V_s be as described in §5 and let $\{V_\theta, \theta : \mathbf{\Pi} \to \{u, a, b, c\}\}$ be the decomposition of $V_{ns} = V - V_s$ corresponding to $\mathbf{\Pi}, M$ as described §5. Let further $\Pi \in \mathbf{\Pi}$, Γ the right cylinder over an algebraic curve $\gamma \subset \Pi$ of degree $\leq \delta$, $W = \psi^{-1}(\Gamma)$ and $\theta : \mathbf{\Pi} \to \{u, a, b, c\}$. Then

$$cc(V_s) \leq c_{20}(\Omega, d), \quad cc(V_\theta) \leq c_{21}(\Omega, d, \mathbf{\Pi}, M), \quad cc(V_\theta \cap W) \leq c_{22}(\Omega, d, \mathbf{\Pi}, M, \delta).$$

Proof. The hypersurfaces Υ correspond to elements $H \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ of degree $\leq d$ modulo multiplication by elements of $\mathbb{R} - \{0\}$, and are thus the points of a projective space $Y = \mathbb{P}^{D_n(d)}(\mathbb{R})$. Observe that Y is a compact real analytic manifold. Let $p : N \times Y \to Y$ be projection. Let $X = \{(P, H) \in N \times Y, H(\psi(P)) = 0\}$. If P has local coordinates (ξ, η) then X is described locally in N by

$$K(\xi, \eta) = H(\psi_1(\xi, \eta), \psi_2(\xi, \eta), \ldots, \psi_n(\xi, \eta)) = 0.$$

Thus X is compact and analytic, and the fibre $X_H = p^{-1}(H)$ is the set $V = \psi^{-1}(\Upsilon)$ associated to the hypersurface Υ defined by $H = 0$. Let $X_s \subset X$ be the set of points P at which, in the local coordinates,

$$K_\xi(\xi, \eta) = 0, \quad K_\eta(\xi, \eta) = 0.$$

Then X_s is compact analytic and the fibre $(X_s)_H = V_s$. The application of Gabrielov's Theorem (2.4) to $X_s \subset N \times Y$ yields the constant $c_{20}(\Omega, d) = c_{13}(N, Y, X_s)$.

Let $X_{ns} = X - X_s$. Thus X_{ns} is subanalytic in fact semianalytic). For $\Pi \in \mathbf{\Pi}$ with coordinates (u, v) let X_u^Π be the subset of X_{ns} at which

$$u_\xi K_\eta - u_\eta K_\xi = 0, \quad v_\xi K_\eta - v_\eta K_\xi = 0.$$

Then X_u is relatively compact subanalytic and the fibre $(X_u^\Pi)_H = V_u^\Pi$. Next set, for each $\Pi \in \mathbf{\Pi}$, $X_a^\Pi \subset X_{ns} - X_u^\Pi$ the points at which

$$\det\begin{pmatrix} v_\xi & v_\eta \\ K_\xi & K_\eta \end{pmatrix} . \det\begin{pmatrix} u_\xi & u_\eta \\ K_\xi & K_\eta \end{pmatrix} . \det\begin{pmatrix} u_\xi - v_\xi & u_\eta - v_\eta \\ K_\xi & K_\eta \end{pmatrix} . \det\begin{pmatrix} u_\xi + v_\xi & u_\eta + v_\eta \\ K_\xi & K_\eta \end{pmatrix} = 0$$

Then X_a^Π is a relatively compact subanalytic subset of $N \times Y$ and the fiber $(X_a^\Pi)_H = V_a^\Pi$. Set now $X_b^\Pi \subset X_{ns} - X_u^\Pi - X_a^\Pi$ to be the points at which

$$R_M(u, v, K).R_M(v, u, K) = 0.$$

Then X_b^Π is a relatively compact subanalytic subset of $N \times Y$ and the fiber $(X_b^\Pi)_H = V_b^\Pi$. Finally set $X_c^\Pi = X_{ns} - X_u^\Pi - X_a^\Pi - X_b^\Pi$, relatively compact subanalytic with fiber $(X_c^\Pi)_H = V_c^\Pi$.

Now for $\theta : \mathbf{\Pi} \to \{u, a, b, c\}$ put

$$X_\theta = \bigcap_{\Pi \in \mathbf{\Pi}} X_{\theta(\Pi)}^\Pi.$$

Then $X_\theta \subset N \times Y$ is relatively compact subanalytic and $(X_\theta)_H = V_\theta$. The application of Gabrielov's Theorem to the sets X_θ for all θ yields the constant $c_{21}(\Omega, d, \mathbf{\Pi}, M)$.

To deal with the cylinder Γ it is necessary to bring the space $T = \mathbb{P}^{D_2(\delta)}(\mathbb{R})$ of plane curves of degree δ in $\Pi \in \mathbf{\Pi}$ into the picture. So let $Z \subset N \times Y \times T$ be the set of (P, H, G) with $(P, H) \in X$ as above and having no dependence on the defining equation G of $\gamma \subset \Pi$. Define Z_θ, for $\theta : \mathbf{\Pi} \to \{u, a, b, c\}$ as above with no dependence on G. Now define Z_q to be the subset of Z_c for which (P, H, G) lies in the cylinder Γ corresponding to G. The fibre over (H, G) is $V_\theta \cap W$. Applying Gabrielov's Theorem to these fibres (over all $\Pi \in \mathbf{\Pi}$) yields the constant $c_{22}(\Omega, d, \mathbf{\Pi}, M, \delta)$. □

7. Subanalytic curves

The following simple observation will be used at several junctures in this and the subsequent section to control the relations between a subanalytic set Ω and semialgebraic sets; both in the case that Ω *contains* semialgebraic sets, and the case that Ω *is contained in* semialgebraic sets.

Let $\Omega \subset \mathbb{R}^n$ be a subanalytic set. Suppose $\{x_1, x_2, \ldots, x_n\}$ is a coordinate system on $\mathbb{R}^n$. For a subset $\sigma \subset \{1, 2, \ldots, n\}$ let Π_σ denote the linear coordinate subspace of $\mathbb{R}^n$ whose coordinates are $\{x_i, i \in \sigma\}$, and let π_σ be the orthogonal projection of $\mathbb{R}^n$ onto Π_σ.

Definition 7.1. Let $\Omega \subset \mathbb{R}^n$ be a subanalytic set. Define $\kappa(\Omega)$ to be the largest $h \in \mathbb{N}$ such that there exists $\sigma \subset \{1, 2, \ldots, n\}$ with $\#\sigma = h$ and such that the projection $\pi_\sigma(\Omega)$ has the property that it is not contained in any algebraic hypersurface (defined over $\mathbb{R}$) in Π_σ.

Observe that $\kappa(\Omega)$ is the maximal size of a set $\{x_j, j \in \sigma\}$ that is "algebraically independent" on Ω; Thus Ω is contained in a closed algebraic subset of $\mathbb{R}^n$ of dimension $\kappa(\Omega)$. So if Ω has dimension k then $\kappa(\Omega) \geq k$.

Proposition 7.2. *Let $\Omega \subset \mathbb{R}^n$ be a subanalytic set of dimension k. If $\kappa(\Omega) = k$ then there is a subanalytic $B \subset \Omega$ of dimension $\leq k - 1$ such that if P is a smooth point of Ω of dimension k with $P \notin B$ then $P \in \Omega^{\text{alg}}$.*

Proof. Let A be a closed algebraic subset of $\mathbb{R}^n$ of dimension $\kappa(\Omega) = k$ with $\Omega \subset A$. The set $\text{sing}(A)$ of singular points of A has dimension $\leq k - 1$. Put $B = \Omega - \text{sing}(A)$, so that B has dimension $\leq k - 1$. Now suppose $P \in \Omega - B$ is a smooth point of Ω of dimension k. Then P is also a smooth point of dimension k of A, and in a neighbourhood of P the sets Ω and A are real analytic manifolds of the same dimension with $\Omega \subset A$. So locally they coincide, whence $P \in \Omega^{\text{alg}}$. □

Remark 7.3. The quantity $\kappa(\Omega)$ suffices for the purposes of this paper. However, a more refined quantity is $\kappa'(\Omega)$ defined as follows. If Ω is connected and smooth, set $\kappa'(\Omega) = \kappa(\Omega)$. Otherwise set $\kappa' = \max_Z \kappa(Z)$ where Z is a connected component of the smooth points of Ω of highest dimension. Proposition 7.2 holds with $\kappa'(\Omega)$ in place of $\kappa(\Omega)$.

Proof of Conjecture 1.1 and 1.2 for subanalytic curves. The connected components of $\Omega - \text{sing}(\Omega)$ are subanalytic of dimension 1, while $\text{sing}(\Omega)$ has dimension ≤ 0. It thus suffices to prove the conclusion for connected nonsemialgebraic Ω of dimension 1 that are the closure of their smooth points of dimension 1. If $n = 1$ then Ω is an interval, whence semialgebraic (in any case in this situation Ω^{trans} is empty).

Now suppose $n > 2$. Consider the images of Ω under the projections π_σ for $\sigma \subset \{1, 2, \ldots, n\}, \#\sigma = 2$. If these images are all semialgebraic then Ω is semialgebraic by 7.1, so it may be assumed that at least one such projection has a transcendental image. Then the restriction $\pi_\sigma : \Omega \to \pi_\sigma(\Omega)$ has finite fibres, uniformly bounded by compactness (or Gabrielov's Theorem).

So it suffices to prove the result for $n = 2$. Let $\epsilon > 0$ be given. According to Lemma 4.3, the points in question in Conjecture 1.1 (respectively Lemma 4.4 for Conjecture 1.2) lie on $\le c_{17}(\Omega, \epsilon)t^\epsilon$ algebraic hypersurfaces of degree $\le d_1(1, n, \epsilon)$ (respectively $\le c_{18}(\Omega, \epsilon)H^\epsilon$ algebraic hypersurfaces of degree $\le d_2(1, n, \epsilon)$). The intersection of Ω with a hypersurface consists of finitely many points, and by compactness (or Gabrielov's Theorem) there is a uniform bound on the number of intersection points over all hypersurfaces of degree $d_1(1, n, \epsilon)$ (respectively $d_2(1, n, \epsilon)$). □

Remark 7.4. The following strengthening of 1.2 can be proved pursuing Remark 4.5: *Let $\Omega \subset \mathbb{R}^n$ be a compact subanalytic curve and $\epsilon > 0$. Then*

$$\#\{P \in \Omega^{\text{trans}}(\mathbb{Q}), H_*(P) \le H\} \le c_{23}(\Omega, \epsilon)H^\epsilon.$$

In fact this is proved in [10] for graphs $\Omega \subset \mathbb{R}^2$. It seems natural to conjecture a similar strengthening for any compact subanalytic $\Omega \subset \mathbb{R}^n$.

The following examples, worked out in discussion with E. Bombieri, elaborate a remark made in [3]. The first shows that the assertion of 1.1 cannot be improved in general for curves, and thus also in higher dimensions. The second example shows that the assertion of 1.1 can fail for the graph of a function that is analytic on an interval that is bounded but open at one end (i.e., for a set that fails to be subanalytic "at just one point").

Example 7.5. Let $\epsilon(t) : [1, \infty) \to \mathbb{R}$ be a strictly decreasing function with $\epsilon(t) \to 0$ as $t \to \infty$. Define a sequence $\{N_j, j \in \mathbb{N}\}$ of positive integers inductively as follows. Set $N_0 = 1$. Supposing $N_0, N_1, \ldots, N_{k-1}$ defined, let N_k be defined so that $N_k \ge k, N_{k-1}|N_k$ and

$$\epsilon\big(N_k^{N_{k-1}}2^{k-1}\big) \le \frac{1}{2N_{k-1}}.$$

Set now $t_k = N_k^{N_{k-1}}2^{k-1}$, $X_k = \{i/N_k : i \in \mathbb{Z}, 0 \le i \le N_k\}$ for $k \in \mathbb{N}$ and define, for $x \in [0, 1]$,

$$f(x) = \sum_{k=0}^{\infty} 2^{-k} \prod_{z \in X_k} (x - z).$$

Then f is analytic on $[0, 1]$. Let $\Omega = \{(x, f(x), x \in [0, 1]\}$. If $x \in X_k$ then $N_k x \in \mathbb{Z}$ and $t_k f(x) \in \mathbb{Z}$ so that

$$\#t_k\Omega(\mathbb{Z}) \ge N_k \ge \exp\big(\frac{\log t_k}{2N_{k-1}}\big) = t_k^{1/(2N_{k-1})} \ge t_k^{\epsilon(t_k)}.$$

Example 7.6. Let again $\epsilon(t) : [1, \infty) \to \mathbb{R}$ be a strictly decreasing function with $\epsilon(t) \to 0$ as $t \to \infty$. Define a sequence $\{t_j, j \in \mathbb{N}\}$ of positive integers inductively as follows. Set $t_0 - 1$. Supposing

$t_0, t_1, \ldots, t_{k-1}$ defined, let t_k be defined so that $t_{k-1}|t_k$ and $\epsilon(t_k) \leq 2^{-k-1}$. Construct a function $f(x)$, transcendental analytic on $(0,1]$, such that, in each interval $(2^{-k-1}, 2^{-k}]$, if $t_k x \in \mathbb{Z}$ then $t_k f(x) \in \mathbb{Z}$. Let $\Omega = \{(x, f(x)), x \in (0,1]\}$. Then

$$\#t_k\Omega(\mathbb{Z}) \geq \sum_{j=0}^{k} \frac{t_j}{2^{j+1}} \geq \frac{t_k}{2^{k+1}} \leq \epsilon(t_k)t_k.$$

Note that this example is essentially optimal for functions analytic on a bounded semiopen interval: Consider a function g analytic on $(0,1]$. For any $\epsilon > 0$ the interval $[\epsilon, 1]$ is compact, so that if Ω_ϵ is the graph of g on $[\epsilon, 1]$ then $\#t\Omega_\epsilon(\mathbb{Z}) \leq c_1(\Omega_\epsilon, \epsilon)t^\epsilon$, while over the dilation of $(0,1)$ there can be at most ϵt integer points.

8. Subanalytic surfaces

The following result, implicit in the proof of [3, Theorem 8], is the final ingredient required in the proof of Theorem 1.3.

Proposition 8.1. *Let $\delta \in \mathbb{N}, \delta \geq 4$ and set $M = D_2(\delta)$. There is a constant $c_{24}(\delta)$ with the following property. Let $N \geq 1$ and $I \subset \mathbb{R}$ a closed interval of length $\leq N$. Let ϕ be a function possessing M continuous derivatives on I, with $|\phi'| \leq 1$ and $\phi^{(M)} \neq 0$ on I. Let α be the graph of ϕ. Then the points of $\alpha(\mathbb{Z})$ lie on the union of at most $c_{24}(\delta)N^{8/(3\delta+9)}$ algebraic curves of degree at most δ.* □

Proof of Theorem 1.3. Suppose $\Omega \subset \mathbb{R}^n$. The Uniformization Theorem (2.3) provides a proper real analytic map $\psi : N \to \mathbb{R}^n$, where N is a real analytic manifold of dimension 2 and $\psi(N) = \Omega)$. Since Ω is compact and ψ is proper, N is compact.

If $\Omega = A \cup B$ then $A^{\text{alg}} \cup B^{\text{alg}} \subset \Omega^{\text{alg}}$. Thus if the assertion of the Theorem holds for A and B it also holds for Ω (this remark applies in any dimension). Now N consists of finitely many connected components, so it suffices to consider the case in which N is connected.

Since Ω has dimension 2, clearly $n \geq 2$. If $n = 2$ then, at its smooth points, Ω is locally a subset of $\mathbb{R}^2$. So the smooth points of Ω are contained in Ω^{alg}. Thus Ω^{trans} is not only contained in the singular set $\text{sing}(\Omega)$, but in $(\text{sing}(\Omega))^{\text{trans}}$. However, $\text{sing}(\Omega)$ has dimension ≤ 1 and is subanalytic by Tamm's Theorem (2.5). The conclusion holds since Conjecture 1.1 holds for compact subanalytic curves. So it may be assumed that $n \geq 3$.

Suppose $\kappa(\Omega) \leq 2$. Then by 7.1, Ω^{trans} is contained in a subanalytic set B of dimension ≤ 1, and hence indeed in B^{trans}. The conclusion of the theorem again follows since Conjecture 1.1 holds for compact subanalytic curves. So it may be assumed that $\kappa(\Omega) \geq 3$.

Choose $\sigma \subset \{1, 2, \ldots, n\}$ with $\#\sigma \geq 3$ such that the projection $\pi_\sigma(\Omega)$ is not contained in any algebraic hypersurface in Π_σ. Let ϵ be given and choose d, δ such that

$$\epsilon(2, n, d) + 8/(3\delta + 9) \leq \epsilon.$$

Here d will be the degree of hypersurfaces used to apply 4.3, while δ will be the degree of plane algebraic curves used in an application of 8.1.

By the proof of Lemma 4.3, $(\pi_\sigma(\Omega))(\mathbb{Z}, t)$ is contained in the union of at most $c_{17}(\pi_\sigma(\Omega), d)t^{\epsilon(2,n,d)}$ sets of the form $\pi_\sigma(\Omega) \cap \Upsilon_\sigma$ where Υ_σ is an algebraic hypersurface of degree $\le d$ in the subspace Π_σ. Suppose Υ_σ is the zeroset of a polynomial H in the variables corresponding to Π_σ. Then, in $\mathbb{R}^n$, the equation $H = 0$ determines a hypersurface $\Upsilon = \pi_\sigma^{-1}(\Upsilon_\sigma)$ of degree d, and $\Omega(\mathbb{Z}, t)$ is contained in the union of the sets $\Omega \cap \Upsilon$. Moreover, the definition of $\kappa(\Omega)$ ensures that none of these intersections is all of Ω, and thus the corresponding subset $V = \psi^{-1}(\Upsilon) \subset N$ is not all of N. Since N is a connected real analytic manifold, such V may not contain a neighbourhood of any point of N, and thus has dimension ≤ 1.

Let $\Psi = \Omega \cap \Upsilon$ be one of these sets and V be the corresponding subset of N. To prove the Theorem it suffices to show that, for a suitable constant $c_{25}(\Omega, d, \delta)$,

$$\#\Psi^{\text{trans}}(\mathbb{Z}, t) \le c_{25}(\Omega, d, \delta)t^{8/(3\delta+9)}.$$

Let now $S = \{\tau \subset \{1, 2, \ldots, n\}, \#\tau = 2\}$. Let $\mathbf{\Pi} = \{\Pi_\tau, \tau \in S\}$. Put $M = D_2(\delta)$. With respect to $\mathbf{\Pi}$ and M, V_{ns} admits a decomposition into subanalytic sets V_θ as described in §5. The number of subsets V_θ is $\#\{\theta\} = 4^{\#S} = 4^{\binom{n}{2}}$. By 6.1, V has at most $c_{20}(\Omega, d)$ singular points (i.e. the number of connected components of V_s) while each set V_θ has at most $c_{21}(\Omega, d, \mathbf{\Pi}, M)$ connected components. At most c_{21} such components reduce to a single point in Ω, as this is only possible if $\theta(\Pi_\tau) = u$ for every $\tau \in S$. Therefore, all but at most $c_{20} + c_{21}$ points of Ψ lie on a connected subanalytic curve β that is the image under ψ of a connected component of one of the sets V_θ.

Consider such a set $\beta \subset V_\theta$. Let $\tau \in S$. If $\theta(\Pi_\tau) = u$ then β lies in the inverse image under π_τ of a point in Π_τ. If $\theta(\Pi_\tau) = a$ then β lies in the inverse image under π_τ of lines in Π_τ, and if $\theta(\Pi_\tau) = b$ then β lies in the inverse image under π_τ of a polynomial of degree $\le M$ with respect to one of the coordinate axes of Π_τ. So in all these cases, $\Pi_\tau(\beta)$ is contained in an algebraic hypersurface in Π_τ. If this is the case for every $\tau \in S$ then β must itself be algebraic by 7.1. Since β is connected of dimension 1, the points of β would then belong to $\Psi^{\text{alg}} \subset \Omega^{\text{alg}}$.

Therefore, if $P \in \Psi$ does not belong to Ω^{alg} then it lies in a set β as above with the property that, for some $\tau \in S$, $\theta(\Pi_\tau) = c$ and the image of β in Π_τ is not semialgebraic. Fix such β, τ. The image α of β under π_τ is a graph with respect to both coordinate axes of x_i, x_j of Π_τ. Thus

$$\alpha = \{(x_i, x_j) : x_i = f(x_j)\} = \{(x_i, x_j) : x_j = g(x_i)\}$$

for appropriate functions f, g on appropriate domains. Since $\theta(\Pi_\tau) = c$, $f^{(M)}, g^{(M)}$ are nonvanishing. Further, one of these functions, say $h \in \{f, g\}$, has the property that $|h'| < 1$ on its domain. The domain of h is an interval of length at most $c_{26}(\Omega)$, where $c_{26}(\Omega)$ is the maximum diameter of $\pi_\tau(\Omega)$ over $\tau \in S$. The domain of h may not be closed, but for each t there is a closed subinterval on which the graph includes all the points of $\alpha(\mathbb{Z}, t)$. Now by 8.1, the integer points of $t\alpha$ lie on the union of at most

$$c_{24}(\delta)\Big(c_{26}(\Omega)t\Big)^{8/(3\delta+9)}$$

algebraic curves of degree $\le \delta$.

Let $\gamma \subset \Pi_\tau$ be an algebraic curve of degree $\leq \delta$, and W the preimage of γ in N. The number of connected components of $V_\theta \cap W$ is bounded by $c_{22}(\Omega, d, \mathbf{\Pi}, M, \delta)$. Now α is the graph of a real analytic function and is nonsemialgebraic, and so $\alpha \cap \gamma$ consists of point components only. Thus $\#\alpha \cap \gamma \leq c_{22}$. Therefore

$$\#\Psi^{\mathrm{trans}}(\mathbb{Z}, t) \leq c_{20} + c_{21} + 4^{n-1} c_{21} c_{22} c_{24} \Big(c_{26} t\Big)^{8/(3\delta+9)}$$

and putting $c_{25} = c_{20} + c_{21} + 4^{n-1} c_{21} c_{22} c_{24} c_{26}^{8/(3\delta+9)}$ (recall $t \geq 1$ always) completes the proof. □

Proof of Theorem 1.5. Under the assumptions Ω can be covered by a finite number of subsets that are graphs over convex compact semianalytic plane domains ([8, Theorem 2.7.3]). So an estimate of the desired form holds for Ω^{trans}. Now Ω^{alg} is contained in a finite union of algebraic space curves. Let Γ be one such curve. Then $\Omega \cap \Gamma$ comprises a finite number of compact connected components of Γ, whose images γ under projection onto a plane Π are compact. Such γ cannot admit a rational parametrization by polynomials, and for such curves the bound $\#t\gamma(\mathbb{Z}) \leq c_{27}(\gamma, \epsilon) t^{\epsilon}$ is established in [3, Theorem 2]. □

Final Remarks 8.2.

1. Some comments on the effectivity of the constants in the main results. The initial argument, showing that the points in question reside on rather few hypersurface intersection, is completely effective. The constants in 4.3 and 4.4 depend explicitly on suitable norms of the functions uniformizing the surface. Likewise the last part of the argument, 8.1, is completely explicit. The application of Gabrielov's Theorem, however, introduces constants of a more subtle nature. In general not much can be said about them. Even in the one-dimensional situation, where Gabrielov's Theorem may be replaced by simple compactness, the examples given in §7 show that there is little control over the resulting constants. However, additional information about the surface (e.g. that the uniformizing functions solve algebraic differential equations, or that certain Wronskian determinants do not vanish) can be used to gain effective control over the requisite constants. See similar considerations in [10] and the related discussion in [13].

2. Consider subanalytic $\Omega \subset \mathbb{R}^n$ of dimension n. Define, for each $d \in \mathbb{N}$, the subset $\Omega_d^{\mathrm{alg}} \subset \Omega$ consisting of all semialgebraic subsets of pure positive dimension that are contained in some algebraic hypersurface of degree $\leq d$. The example of §1 shows that Ω_d^{alg} is not in general semianalytic. However the assumption that Ω_d^{alg} is semialgebraic for all d suffices in Theorem 1.5.

3. Theorem 1.3 may also be used to give results in higher dimensions by slicing. Suppose $\Omega \subset \mathbb{R}^n$ is compact subanalytic of dimension k. If $\Gamma \subset \Omega$ then $\Gamma^{\mathrm{alg}} \subset \Omega^{\mathrm{alg}}$. Slicing Ω into subanalytic surfaces using coordinate linear subspaces of dimension $n - k + 2$ from a compact set K, and applying Gabrielov's Theorem over the K fibres shows that, for $\epsilon > 0$ and suitable $c_{28}(\Omega, \epsilon)$,

$$\#t\Omega^{\mathrm{trans}}(\mathbb{Z}) \leq c_{28}(\Omega, \epsilon)\, t^{k-2+\epsilon}.$$

The constant $c_{28}(\Omega, \epsilon)$ evidently depends on the choice of slices.

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