1a)

Supplementary lecture in PDEs

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Topics: • Distributions — rigorous version of delta-functions, etc.

(will include some cross-talk between topics) • ~~Transform methods~~ ~~Fourier Transforms, etc.~~ Fourier Theory

Distributions (some resources: books by Stakgold, Butkov)

- consider $f: R_n \xrightarrow{\subset \mathbb{R}^n} \mathbb{R}$; $f(x) = y$ is a rule that assigns $y \in \mathbb{R}$ to $x \in R_n$

- an indirect description of f can be very useful

  - instead of giving f at every point, we give $\int_{R_n} f(x)\phi(x)dx$ $\forall$ functions ϕ of some type (to be specified) ($\phi \in K$ = space of approximating functions)

  - we thereby view f as a functional on K (i.e., f associates w/ each $\phi \in K$ the real number $\int_{R_n} f(x)\phi(x)dx$]

- this idea will allow us to expand the types of functions we can consider; in particular, with a good choice of K, will be able to deal with "singular" functions like δ-functions

- the K will use is the set of "test functions" which are extremely smooth

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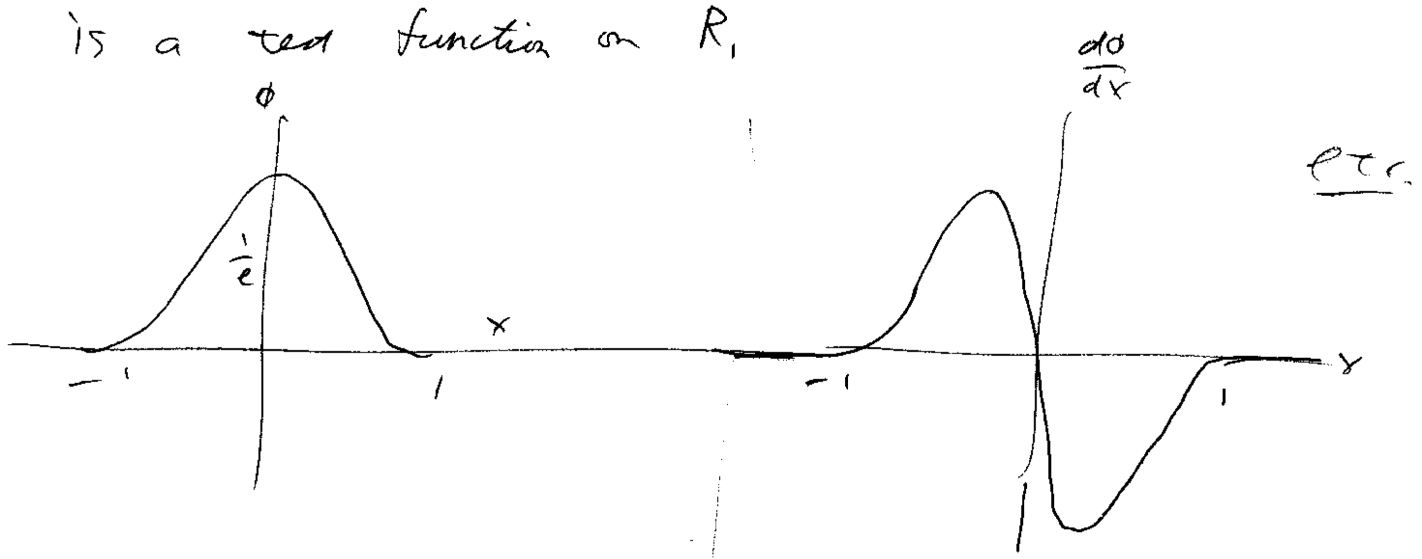
- def: a test function $\phi(x) = \phi(x_1, \dots, x_n)$ on R_n is ~~which~~ is a function which is infinitely differentiable on R_n (i.e., all partial derivatives of all orders exist) and vanishes outside some bounded region (

1b) • the space of all test functions on R_n is denoted as $K = C_0^\infty(R_n)$

e.g.1) $$\phi(x) = \begin{cases} \exp\left(\dfrac{1}{x^2-1}\right), & |x|<1 \\ 0, & |x|\ge 1 \end{cases}$$

is a test function on R_1



• ϕ is certainly C^∞ (denotes infinitely differentiable) except possibly at ± 1; ϕ is even, so we can just look at $x=1$ & will also see what happens at $x=-1$

• $\lim\limits_{x\to 1^-} \exp\left(\dfrac{1}{x^2-1}\right) = 0 \Rightarrow \Phi$ is continuous at $x=1$

• to show that all derivatives of ϕ are 0 at $x=1$, simply note that $\forall m$, $\lim\limits_{x\to 1^-} \dfrac{1}{(x^2-1)^m} \exp\left(\dfrac{1}{x^2-1}\right) = 0$

$\Longrightarrow \Phi \in C_0^\infty(R_1)$ ✓
(i.e., it's a test function)

e.g.2) $$\phi(x) = \begin{cases} \exp\left(\dfrac{1}{|x|^2-1}\right), & |x|<1 \\ 0, & |x|\ge 1, \end{cases}$$ $x \in R_n$, $|x| = r$ = distance from origin

A. 2a) Some properties

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(1)
- $\Phi_1(x), \Phi_2(x) \in C_0^\infty(R_n) \Rightarrow c_1\Phi_1 + c_2\Phi_2 \in C_0^\infty(R_n)$

(i.e., linearity holds)

(2)
- if $\phi(x) \in C_0^\infty(R_n)$, then $\phi\left(\frac{x-x_0}{\epsilon}\right)$ is also ~~a function~~ in $C_0^\infty(R_n)$; it vanishes outside the ball of radius ϵ & center x_0

(3)
- if $\phi \in C_0^\infty(R_n)$, then so is every partial derivative of ϕ

(4)
- if $\phi \in C_0^\infty(R_n)$ & $a(x)$ is infinitely differentiable, then $a(x)\phi(x) \in C_0^\infty(R_n)$

(5)
- if $\phi(x_1, \cdots, x_m) \in C_0^\infty(R_m)$ & $\psi(x_{m+1}, \cdots, x_n) \in C_0^\infty(R_{n-m})$, then $\phi(x_1, \cdots, x_m)\psi(x_{m+1}, \cdots, x_n) \in C_0^\infty(R_n)$

- Convergence in the space of test functions

- def: $k = (k_1, \cdots, k_n)$ is a multi-index of dim. n;

$|k| = k_1 + \cdots + k_n$,

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = \frac{\partial^{k_1 + \cdots + k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

- e.g., $n = 3$, $k = (2, 0, 5) \Rightarrow D^k = \frac{\partial^7}{\partial x_1^2 \partial x_3^5}$

- def: the support of fn $f(x)$ is the closure of the set of points in R_n ~~on~~ which $f(x) \neq 0$.

(i.e., $C_0^\infty(R_n)$ = space of infinitely differentiable functions of compact support) *def*. a sequence of test functions $\{\phi_1(x), \dots, \phi_m(x), \dots\}$ is a <u>null sequence</u> in $C_0^\infty(R_n)$ iff:

(1) $\exists$ a common bounded region outside of which all $\phi_m(x)$ vanish (i.e., the support of all ϕ_m is contained in a sufficiently large ball)

(2) $\forall$ multi-index k of dim n, $\lim_{m\to\infty} \max_{x\in R_n} |D^k \phi_m(x)| = 0$

- Thus, $\{\phi_m(x)\} \to 0$ <u>uniformly</u> in R_n & so does $\{D^k \phi_m(x)\}$

  ↳ hence, the approach to 0 is rather strong

Distributions

- *def*: f is a <u>linear functional</u> on $C_0^\infty(R_n)$ if $\exists$ a rule that assigns to each $\phi \in C_0^\infty(R_n)$ a real number (denoted $\langle f, \phi\rangle$) s.t. $\langle f, \alpha_1\phi_1 + \alpha_2\phi_2\rangle = \alpha_1\langle f,\phi_1\rangle + \alpha_2\langle f,\phi_2\rangle$ $\forall \alpha_1, \alpha_2 \in \mathbb{R}$ & $\forall \phi_1, \phi_2 \in C_0^\infty(R_n)$.

note: $\langle f, 0\rangle = 0$, &, $\langle f, \sum_{k=1}^m \alpha_k \phi_k\rangle = \sum_{k=1}^m \alpha_k \langle f, \phi_k\rangle$

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def: a linear functional on $C_0^\infty(R_n)$ is <u>continuous</u> if whenever $\{\phi_m(x)\}$ is a null sequence in $C_0^\infty(R_n)$, ~~the sequence~~ $\langle f, \phi_m\rangle \to 0$ as $m\to\infty$

~~Such a continuous lin f is a distribution~~ such a linear functional is called a <u>distribution</u>

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def: a function $f(x)$ on R_n is locally integrable if $\int_\Omega |f| dx$ exists on every bounded domain $\Omega \subset R_n$

A locally integrable function $f(x)$ on R_n defines an n-dim distribution f through the rule

$$\langle f, \phi \rangle = \int_{R_n} f(x)\phi(x)\,dx = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n)\,\phi(x_1, \cdots, x_n)\,dx_1 \cdots dx_n$$

(i.e., you can think of any locally integrable function as a distribution) (this gives a regular dist.)

- if f_1 & f_2 are different, they generate different distributions

- if f_1 & f_2 coincide except at a finite number of points, they generate the same distribution

  ↳ more generally, 2 functions are equal "almost everywhere" if $\int_\Omega |f_1 - f_2| dx = 0 \ \forall$ bounded domain Ω

- we can also consider singular distributions for "generalized functions" (which are not locally integrable)

- examples:

(1) let $\Omega \subset R_n$ & consider $\langle I_\Omega, \phi \rangle = \int_\Omega \phi(x)\,dx$, ~~$= \int_{R_n} I_\Omega \phi(x) dx$~~

where $I_\Omega(x) = \begin{cases} 1, & x \in \Omega \\ 0, & x \notin \Omega \end{cases}$

(over)

(e.g. 1 continued)

- $\langle I_\Omega, \phi \rangle = \int_\Omega \phi(x)\,dx = \int_{R_n} I_\Omega(x)\,\phi(x)\,dx \Rightarrow$ I_Ω is a distribution; it is "regular" (i.e., not singular) because it is piecewise continuous & hence locally integrable

- special case of particular interest: $n = 1$ & $\Omega = (0, \infty)$

$$\Rightarrow I_\Omega(x) = H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}$$

"Heaviside function"

(e.g. 2) let $\xi \in R_n$ & consider the linear functional δ_ξ defined from $\langle \delta_\xi, \phi \rangle = \phi(\xi)$, which assigns to each test function its value at ξ

- if $\{\phi_m(x)\}$ is a null sequence in $C_0^\infty(R_n)$, then $\{\phi_m(\xi)\} \to 0$, so δ_ξ is a distribution

(this is the "Dirac distribution" with "pole" at ξ)

- let's show that δ_0 is a singular distribution; if it were regular, there would exist a locally integrable function $f(x)$ s.t. $\int_{R_n} f(x)\phi(x)\,dx = \phi(0)$ $\quad \forall \phi \in C_0^\infty(R_n)$

- consider the test functions $\cancel{\phi_a(x) = \phi\left(\tfrac{x}{a}\right)}$,

~~where $\phi(x) = \begin{cases} \exp(\ldots), & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$~~

7a

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$$\Rightarrow \Psi_a(x) = \begin{cases} \exp\left(\dfrac{a^2}{|x|^2 - a^2}\right), & |x| < a \\ 0, & |x| \geq a \end{cases}$$

- note that $\Psi_a(0) = \frac{1}{e}$, $|\Psi_a(x)| \leq \frac{1}{e}$

$$\Rightarrow \left|\int_{R_n} f(x)\Psi_a(x)\,dx\right| = \left|\int_{|x|<a} f(x)\exp\left(\frac{a^2}{|x^2| - a^2}\right)dx\right| \leq \frac{1}{e}\int_{|x|<a}|f(x)|\,dx$$

- if $f(x)$ is locally integrable, we must have $\lim_{a\to 0}\int_{|x|<a}|f(x)|\,dx = 0$ & hence $\lim_{a\to 0}\int_{R_n} f(x)\Psi_a(x)\,dx = 0$

however, by def of δ_0, we actually have
$\int_{R_n} f(x)\Psi_a(x)\,dx = \frac{1}{e} \quad \forall a$

- this gives a contradiction, so δ_0 must be singular

- note: by ~~translation~~, distributions can be translated:

$$\langle f(x-a), \phi(x)\rangle = \langle f(x), \phi(x+a)\rangle$$

$$\left(\xrightarrow{e.g.} \delta_\xi(x) = \delta_0(x - \xi)\right)$$

(e.g. 3) Scale expansion & contraction: (similarity transformation)

- if $f(x)$ is locally integrable, so is $f(\alpha x)$ $\forall$ real $\alpha \neq 0$

$\downarrow$

the corresponding distribution is $\int_{R_n} f(\alpha x)\phi(x)\,dx$;

we can do this for singular distr. [letting $y = \alpha x$ & noting that the integration limits get reversed if $\alpha < 0$, we get $\langle f(\alpha x)\, \phi(x)\rangle = \frac{1}{|\alpha|^n}\langle f(x), \phi(\frac{x}{\alpha})\rangle$

7b)

differentiation of distributions

if f is differentiable on R, w/ locally integrable first derivative f', then we get the following distribution for f':

$$\langle f', \phi \rangle = \int_{-\infty}^{\infty} f'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx = \langle f, -\phi' \rangle$$

use integration by parts & the fact that $\phi \equiv 0$ outside a bounded interval

- this suggests defining the derivative f' of *any* distribution f through $\langle f', \phi \rangle = \langle f, -\phi' \rangle$ (*)

- we need to confirm that this gives us a distribution:

  ~~ϕ is a test function~~

  $\phi \in C_0^\infty \Rightarrow \phi' \in C_0^\infty$ & the RHS is well-defined because f is a distribution

  - linearity is clear, so we just need to check continuity: if $\{\phi_m\}$ is a null sequence in $C_0^\infty(R,)$, so is $\{-\phi_m'\} \Rightarrow \langle f, -\phi_m' \rangle \to 0$ as $m \to \infty$ ✓

- we can similarly get other derivatives:

$$\langle D^k f, \phi \rangle = (-1)^{|k|} \langle f, D^k \phi \rangle$$

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e.g.) consider $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

- if we try to differentiate it as a regular function, we run into problems

⟨over⟩

5a) <cont.>

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- however, we can do this by defining it as a distribution: $\langle H, \phi \rangle = \int_0^\infty \phi(x)\,dx$

$\Rightarrow H'$ is defined as a distribution by

$$\langle H', \phi \rangle = \langle H, -\phi' \rangle = -\int_0^\infty \phi'(x)\,dx = \phi(0)$$

$$\Rightarrow H'(x) = \delta_0(x) \underbrace{\equiv \delta(x)}_{\text{use this notation}}$$

Convergence of Sequences and Series of Distributions

def: let $\{f_\alpha\}$ be a family of distributions; we say that $\{f_\alpha\}$ converges distributionally to f as $\alpha \to \alpha_0$ (& write $f_\alpha \to f$) if $\lim_{\alpha \to \alpha_0} \langle f_\alpha, \phi \rangle = \langle f, \phi \rangle$ $\forall\, \phi \in C_0^\infty(R_n)$

~~Thm~~: If $\lim_{\alpha \to \alpha_0} \langle f_\alpha, \phi \rangle$ exists $\forall\, \phi \in C_0^\infty(R_n)$, $\exists!$ distribution f s.t. $f_\alpha \to f$ as $\alpha \to \alpha_0$; i.e., $\lim_{\alpha \to \alpha_0} \langle f_\alpha, \phi \rangle = \langle f, \phi \rangle$ $\forall \phi \in C_0^\infty(R_n)$

unique limit

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note: every convergent sequence or series of distributions can be differentiated term by term

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e.g.) in R_1, consider the sequence $\{S_k(x)\}$ given by

$$S_k(x) = \begin{cases} 0, & |x| > \frac{1}{2k} \\ k, & |x| < \frac{1}{2k} \end{cases}$$

over

5b | <cont.>

then, denoting $I_k := (-\frac{1}{2k}, \frac{1}{2k})$, we have

$$\langle S_k, \phi \rangle = \int_{I_k} k\phi(x)dx = \phi(0) + \int_{I_k} k[\phi(x) - \phi(0)]\,dx$$

and $$\left| k\int_{I_k} [\phi(x) - \phi(0)]dx \right| \le k\int_{I_k} |\phi(x) - \phi(0)|dx \le \max_{x\in I_k} |\phi(x) - \phi(0)|$$

- if ϕ is continuous at $x=0$, the maximum $\to 0$ as $k\to\infty$ $\therefore \lim_{k\to\infty} \langle S_k, \phi\rangle = \phi(0)$ $\forall$ $\phi \in C_0^\infty$ (in fact, $\forall$ continuous ϕ)

$\Rightarrow \lim_{k\to\infty} S_k(x) = \delta(x)$ [in distributional sense]

- more generally, let $\{f_\alpha(x)\}$ be a family of locally integrable functions in R_n with the property

$$\lim_{\alpha\to\alpha_0} \int_{R_n} f_\alpha(x)\phi(x)dx = \phi(0) \quad \forall \phi \in C_0^\infty(R_n)$$

- $\{f_\alpha\}$ is called a delta sequence (construct simultaneously squeezing & increasing peak)

e.g. (1) the example above

~~(2) $f(x) = \frac{1}{\pi(1+x^2)}$~~

occurs in potential theory (2) $f_y(x) = \dfrac{y}{\pi(y^2+x^2)}$ $(y\to 0^+)$

Féjer kernel from theory of Fourier integrals (3) $f_y(x) = \dfrac{\sin^2 yx}{\pi y x^2}$ $(y\to\infty)$

6a (more ~~of~~ delta sequences)

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(4) Dirichlet kernel for Fourier integrals,

$$\frac{1}{2\pi}\int_{-R}^{R} e^{i\omega x}\,d\omega = \frac{\sin Rx}{\pi x}$$

(δ-sequence as $R\to\infty$)

• note that we can write ~~$\sin \pi x$~~

$$\frac{\sin Rx}{\pi x} = R f(Rx), \text{ where } f(x) = \frac{\sin x}{\pi x}, \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$(5)\quad f_k(x) = \begin{cases} \displaystyle\sum_{n=-k}^{k} \frac{1}{2\pi} e^{inx} = \frac{\sin(k+\frac{1}{2})x}{2\pi \sin\frac{1}{2}x}, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$$

(δ-sequence as $k\to\infty$)

Dirichlet kernel for Fourier series

$\boxed{\text{Fourier Series}}$

(we'll start by reviewing classical results, but they're done in a way that's probably more rigorous than what you've seen before)

• ~~brief~~ review of ~~'classical' results~~
• recall:

~~f is a ... valued~~ fun

• $f: \mathbb{R} \longrightarrow \mathbb{C}$ with $f(x+2\pi) = f(x)\ \forall x$

• $f \in L_1(-\pi,\pi)$, so $\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$

~~we~~ one often wants $f \in L_2(-\pi,\pi)$, so $\int_{-\pi}^{\pi} |f(x)|^2\,dx < \infty$ $\quad \subset L_1(-\pi,\pi)$

6b)

a ~~trigonometric~~ Fourier series is of the type $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ (*), $c_k \in \mathbb{C}$

- The sequence ~~of partial sums~~ $\{S_n(x)\}$ of partial sums is defined symmetrically: $S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$, $n = 0, 1, 2, \ldots$

- ~~we want to show~~ Suppose that $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges uniformly for $x \in [-\pi, \pi]$

- look at $\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx$

$\rightarrow$ can integrate term by term to get $c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx$ $(m \in \mathbb{Z})$

~~then are the coefficients if the series converges~~

Let $f \in L_2(-\pi,\pi)$ & $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx$; then

(1) $\lim_{k\to\infty} c_k = 0$

(2) $\sum_{k=-\infty}^{\infty} |c_k|^2$ converges

(3) $\sum_{k=-\infty}^{\infty} |c_k|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$

Proof of (1) – (3):

$$0 \le \int_{-\pi}^{\pi} \Big| f - \sum_{k=-n}^{n} c_k e^{ikx}\Big|^2 dx = \int_{-\pi}^{\pi} \Big(f - \sum_{k=-n}^{n} c_k e^{ikx}\Big)\overline{\Big(f - \sum_{k=-n}^{n} c_k e^{ikx}\Big)}\,dx$$

$$= \int_{-\pi}^{\pi} |f|^2 dx - 2\pi \sum_{k=-n}^{n} |c_k|^2$$

(over)

7a (proof of (1) – (3) cont.)

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$\Rightarrow \forall n, \displaystyle\sum_{k=-n}^{n} |c_k|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 dx$

- let $\{t_n\}$ be the sequence of partial sums

$t_n = \displaystyle\sum_{k=-n}^{n} |c_k|^2$; t_n increases monotonically w/ n & is bounded above by the const. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2 dx$

$\Rightarrow t_n$ converges, so (2) & (3) hold

$\Rightarrow |c_k|^2 \to 0$ as $|k| \to \infty$, so (1) holds

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- Remarks: (1) holds even for $f \in L_1(-\pi,\pi)$ & in that case this result is known as the Riemann–Lebesgue lemma

- $f \in L_1 \Rightarrow \forall \epsilon > 0$, we can write $f = g + h$, where g is bounded and $\int_{-\pi}^{\pi} |h(x)| dx < \epsilon$

$\Rightarrow \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x) e^{-ikx} dx \right| < \frac{\epsilon}{2\pi} \quad \forall k$

$\Rightarrow$ by (1) applied to g, we get $\frac{1}{2\pi}\int_{-\pi}^{\pi} g e^{-ikx} dx \to 0$ as $|k| \to \infty$

$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f e^{-ikx} dx \to 0$ as $|k| \to \infty$ (as required) ✓

$\Rightarrow$ for $f \in L_1(a,b)$, $\int_a^b f(x) e^{-ikx} dx$, $\int_a^b f(x)\cos kx\, dx$, $\int_a^b f(x) \sin kx\, dx$

all $\to 0$ as $|k| \to \infty$ $(k \in \mathbb{R})$

- (2): the smoother f is, the faster the Fourier coefficients vanish

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- $f \in C^k(S)$ if it is continuous w/ continuous derivatives up to order k on the unit circle S (all the functions are periodic, so they're defined on the circle)

- $f \in D(S)$ ("obeys Dirichlét conditions") if it is piecewise continuous & if $\exists$ a finite subdivision of S s.t. f has a continuous derivative w/in each subinterval & s.t. the left- & right-hand derivatives at each end of the subintervals

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Thm) $f \in C^p(S) \Rightarrow$ the Fourier coeffs $\{c_m\}$ satisfy $\lim_{m\to\infty} c_m m^p = 0$

(try to prove this at home)

(maybe assign as hw?) ← [hint: do integration by parts many times]

Thm) $f \in C^{p-1}$, $f^{(p)} \in D \Rightarrow$ the Fourier coeffs $\{c_m\}$ of f satisfy $|c_m m^{p+1}| \leq M \;\forall m$

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e.g.) $f(x) = \cos \alpha x$ $(\alpha \in \mathbb{R})$, $-\pi \leq x < \pi$, $f(x+2\pi) = f(x)$

(note: unless $\alpha \in \mathbb{Z}$, this is not the same as putting this on $x \in \mathbb{R}$)

- $\alpha \notin \mathbb{Z} \Rightarrow a_m = (-1)^{m+1} \dfrac{2\alpha}{\pi} \dfrac{\sin \alpha\pi}{m^2 - \alpha^2}$, $m = 0, 1, 2, \ldots$ (calculation)

$\Longrightarrow m^2 a_m$ is bounded as $m \to \infty$

- f is continuous but has discontinuous derivative at odd multiples at π

- $f \in \mathbb{Z} \Rightarrow a_m = 0$ except for $m = \alpha$ $(a_\alpha = 1)$

↳ i.e., the Fourier series is just the single cosine term

↳ in this case, f is infinitely differentiable, so $m^p a_m \to 0$ as $|m| \to \infty$ $\forall p$

8a

$\boxed{\text{Thm.}}$ $f \in L_1(S)$ & $x_0 \in S$; $f(x_0^+)$, $f(x_0^-)$ (and also the left- & right-hand derivatives of f) exist. Then the Fourier series of f converges to the value $\frac{1}{2}[f(x_0^+) + f(x_0^-)]$ at x_0

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Proof: Let $S_n(x_0) = \sum_{k=-n}^{n} c_k e^{ikx_0}$, where $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx$

$\hookrightarrow$ we need to show: $\lim_{n\to\infty} S_n(x_0) = \frac{1}{2}[f(x_0^+) + f(x_0^-)]$

- rotation leaves S invariant, so we can take $x_0 = 0$ wlog

- note that $\sum_{k=-n}^{n} e^{ikx} = \sum_{k=-n}^{n} e^{-ikx} = -1 + \sum_{k=0}^{n} e^{ikx} + \sum_{k=0}^{n} e^{-ikx}$
  - each sum on the RHS is a geometric series, so we compute:

$$(*)\quad \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{-ikx} = \frac{\sin[(n+\frac{1}{2})x]}{2\pi \sin(\frac{x}{2})} := D_n(x)$$

$$\Longrightarrow \int_{-\pi}^{\pi} D_n(x)\,dx = 1, \quad \int_{0}^{\pi} D_n(x)\,dx = \frac{1}{2}$$

- $D_n(x)$ is called the <u>Dirichlet kernel</u>; it has period 2π; $D_n(0)$ is understood to be the limit of $(*)$ as $x \to 0$ (i.e., $\frac{2n+1}{2\pi}$)
- $S_n(0) = \sum_{k=-n}^{n} c_k = \sum_{k=-n}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = \int_{-\pi}^{\pi} D_n(x) f(x)\,dx$
- to prove the desired result, we basically need to show that $D_n(x)$ is a delta sequence on $-\pi < x < \pi$
  - it will be convenient to split the interval into $(-\pi, 0)$ & $(0, \pi)$

(over)

(8b) <cont.>

- on $(0,\pi)$, we get $\int_0^\pi D_n(x) f(x)\,dx = \frac{f(0^+)}{2} + \int_0^\pi D_n(x)[f(x)-f(0^+)]\,dx$ (**)

↳ writing $\frac{f(x)-f(0^+)}{\sin(\frac{x}{2})} = \left[\frac{f(x)-f(0^+)}{x}\right]\left[\frac{x}{\sin(\frac{x}{2})}\right]$, (***)

we observe that the first factor has a limit as $x \to 0^+$ because the right-hand derivative exists at $x=0$

$$\Longrightarrow \frac{[f(x)-f(0^+)]}{x} \in L_1(0,\pi)$$

- $\frac{x}{\sin(\frac{x}{2})}$ is bounded on $(0,\pi)$, so the product on the RHS of (***) is in $L_1(0,\pi)$

- from the Riemann-Lebesgue lemma, we get that the RHS of (**) $\to \frac{1}{2} f(0^+)$ as $n \to \infty$

- doing the same argument on $(-\pi,0)$ completes the proof ✓

Thm.) $f \in C'(S) \Rightarrow$ its Fourier series converges uniformly to f

Proof: do this at home if you're interested; start w/ expression for Fourier coeffs, integrate by parts, use Schwarz inequality, & verify convergence of relevant quantities

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Corollary: $f(x) \in C'(S) \Longrightarrow$ its Fourier series converges in the L_2 sense to f

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Proof: we have $\int_{-\pi}^{\pi} |f|^2 dx - 2\pi \sum_{k=-m}^{m} |C_k|^2 = \int_{-\pi}^{\pi} \left| f(x) - \sum_{k=-m}^{m} C_k e^{ikx} \right|^2 dx$ (****)

& because of the uniform convergence of the Fourier series to f, we can pick M large enough s.t. $\left| f - \sum_{k=-m}^{m} C_k e^{ikx} \right| < \epsilon$ for $m > M$

(over)

9a) ⟨cont⟩ ✓ MAP 1/10/09 B56 - sum

$\Rightarrow$ the RHS of (****) $\to 0$ as $m \to \infty$

- this implies immediatly that

$$\sum_{m=-\infty}^{\infty} |C_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2\,dx \quad (\text{"Parseval's identity"})$$

for $f \in C'(S)$

- one can show that Parseval's identity holds for $f \in L_2(-\pi,\pi)$ by approximating it in the L_2 sense by a function $g \in C'(S)$

- Gibbs phenomenon: The Fourier series of a discontinuous function cannot converge uniformly on an interval enclosing the discontinuity
  - this will give an overshoot phenomenon near jump
  - you might see this again later (e.g., on hw)... (for now, we're moving on)

Fourier Series as Distributions

- first, we extend our previous notions to hold for cplx-valued functions; all the earlier results survive

---

- a Fourier series $\sum_{n=-\infty}^{\infty} C_n e^{inx}$ converges uniformly for $|C_n| \le \frac{M}{n^2}$ for large $|n| \Rightarrow$ it also converges in the sense of distributions

  how should we interpret $\sum_{n=-\infty}^{\infty} C_n e^{inx}$ if all we know is that for large $|n|$, $|C_n| \le M n^\alpha$ for some $\alpha \in \mathbb{Z}$?

9b]

- this class includes series that diverge badly in the pointwise sense

  - consider $\sum_{n\neq 0}(in)^{-\alpha-2} c_n e^{inx}$, which converges uniformly (& hence distributionally) to some function (call it $g(x)$)

  ($n \in \mathbb{Z}$)

  - the $(\alpha+2)$nd distributional derivative of

$$g(x) = \sum_{n\neq 0}(in)^{-\alpha-2} c_n e^{inx} \text{ gives}$$

$$g^{(\alpha+2)}(x) = \sum_{n\neq 0} c_n e^{inx} \Rightarrow \sum_{n=-\infty}^{\infty} c_n e^{inx} = c_0 + g^{(\alpha+2)}(x)$$

---

e.g.) $f(x) = x - \pi$, $0 \le x < 2\pi$, $f(x) = f(x+2\pi)$; $f \in D$ & has Fourier series

$$f(x) = -2\sum_{n=1}^{\infty} \frac{\sin nx}{n} = i\frac{\sum_{n\neq 0} e^{inx}}{|n|} \quad (**)$$

- $(**)$ does not converge uniformly to f but does converge distributionally to f

  ↳ we can see this by looking at the integrated series (term by term) $2\sum_{n=1}^{\infty}\frac{(\cos nx)}{n^2}$,

  which converges uniformly (and hence distributionally) to a function $F(x)$ whose derivative is $f(x)$; we can then differentiate this series termwise to recover $(**)$ in the distributional sense

→ differentiating again gives $f'(x) = -2\sum_{n=1}^{\infty} \cos(nx)$

↳ one can show that $f' = 1 - \sum_{k=-\infty}^{\infty} 2\pi\delta(x - 2k\pi)$

[using one of the HW problems]

(over)

p.10a (1.9) (cont.)

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BSB – supp

$$\Longrightarrow \sum_{k=-\infty}^{\infty} \delta(x-2k\pi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos nx \qquad (***)$$

- in a classical sense, the series on the RHS doesn't converge

↳ w/ distributions, (***) means that $\forall$ test function ϕ,

$$\sum_{k=-\infty}^{\infty} \phi(2k\pi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}(\cos nx)\,\phi(x)\,dx$$

- We can rewrite (***) as

$$\sum_{k=-\infty}^{\infty} \delta(x-2k\pi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-inx}$$

$$\overset{x \to x-\xi}{\Longrightarrow} \sum_{k=-\infty}^{\infty} \delta(x-\xi-2k\pi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\xi-x)} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\begin{bmatrix}\cos n\xi \cos nx \\ + \\ \sin n\xi \sin nx\end{bmatrix} \qquad (****)$$

- thinking of $\xi \in (-\pi,\pi)$ as fixed, let's apply (****) to a test function w/ support in $(-\pi,\pi)$:

$$\Longrightarrow \phi(\xi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in\xi}\int_{-\pi}^{\pi} e^{-inx}\phi(x)\,dx, \quad -\pi<\xi<\pi,$$

which is just the Fourier series expansion formula for ϕ

e.g.) Poisson summation formula

- let $\phi(x)$ be a (cplx-valued) test function on R_1; then $\phi\left(\frac{\lambda x}{2\pi}\right)e^{ixt/2\pi}$ $(\lambda, t \in \mathbb{R})$ is also a test function

- w/ the above stuff, we can write
$$\sum_{k=-\infty}^{\infty}\phi(\lambda k)e^{ikt} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx}\phi\left(\frac{\lambda x}{2\pi}\right)e^{ixt/(2\pi)}dx$$

- letting $y = \frac{\lambda x}{2\pi}$, we get
$$\sum_{k=-\infty}^{\infty}\phi(\lambda k)e^{ikt} = \frac{1}{|\lambda|}\sum_{n=-\infty}^{\infty}\hat{\Phi}\left(\frac{t+2\pi n}{\lambda}\right), \quad (i)$$

where $\hat{\Phi}(\omega) = \int_{-\infty}^{\infty} e^{i\omega x}\phi(x)dx$ is the Fourier transform of $\phi(x)$ (we'll discuss this more later)

- putting $t=0$ in (i) gives $\sum_{k=-\infty}^{\infty}\phi(\lambda k) = \frac{1}{|\lambda|}\sum_{n=-\infty}^{\infty}\hat{\Phi}\left(\frac{2\pi n}{\lambda}\right)$ (ii)

- putting $\lambda = 1$ in (ii) gives $\sum_{k=-\infty}^{\infty}\phi(k) = \sum_{n=-\infty}^{\infty}\hat{\Phi}(2\pi n)$

- (i), (ii), (iii) all go by the name Poisson's summation formula

- remark: we derived (i)–(iii) for test functions, but they in fact hold for any piecewise smooth ϕ s.t. $\phi(\infty) = \phi(-\infty) = 0$ & $\int_{-\infty}^{\infty}|\phi|dx$ is finite

(over)

11a] (cont)

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- if we put (as an e.g.) $\phi(x) = e^{-x^2}$ in (ii), we get

$$\underbrace{\sum_{k=-\infty}^{\infty} e^{-\lambda^2 k^2}}_{\substack{\text{converges rapidly} \\ \text{for large } \lambda}} = \underbrace{\frac{\sqrt{\pi}}{\lambda} \sum_{n=-\infty}^{\infty} e^{-n^2\pi^2/\lambda^2}}_{\substack{\text{converges rapidly} \\ \text{for small } \lambda}}$$

(so the relation is useful because numerically use the LHS for large λ but the RHS for small λ

$\boxed{\text{Fourier Transforms and Integrals}}$

- let $f(x)$ vanish 'sufficiently fast' as $|x| \to \infty$ & let $f_T(x)$ be a T-periodic function that coincides w/ f on $-\frac{T}{2} \le x < \frac{T}{2}$, then if f_T obeys Dirichlet conditions, we can write $(\forall x)$,

$$f_T(x) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k x/T}, \quad C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(\xi) e^{-2k i \xi/T} d\xi$$

$$\Longrightarrow \text{ for } |x| < \frac{T}{2}, \quad f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left[\frac{2\pi}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(\xi) e^{i2\pi k(x-\xi)/T} d\xi\right]$$

- letting $\omega_k = \frac{2\pi k}{T}$, $\Delta\omega = \frac{2\pi}{T}$, & $g(x,\omega,T) = \int_{-\frac{T}{2}}^{T/2} f(\xi) e^{i\omega(x-\xi)} d\xi$,

we get $f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} (\Delta\omega) g(\omega_k, x, T), \quad |x| < \frac{T}{2}$,

which for large T can be regarded as a Riemann sum over a partition (of spacing $\frac{2\pi}{T}$) of the ω-axis (so we basically have an integral here)

- taking limits gives a <u>Fourier integral expansion</u> for $f(x)$ in $L_1(-\infty,\infty)$:

(over)

11b <cont>

- $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x}\, dx,$ (*)

where $\hat{f}(\omega) := \int_{-\infty}^{\infty} f(\xi) e^{i\omega\xi}\, d\xi$ is the Fourier transform of f

- (*) shows how to reconstruct f from its continuous spectrum (you can imagine important numerical uses of time series...)

(there are other types of transforms as well: wavelets, etc.)

- The integral in (*) needs to be intepreted as

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} \hat{f}(\omega) e^{-i\omega x}\, d\omega \qquad (**)$$

- To verify (**), first note that $\hat{f}$ exists (because f is L^1); substituting the def. of $\hat{f}$ gives

$$\frac{1}{2\pi}\int_{-R}^{R}\hat{f}(\omega)e^{-i\omega x}\,d\omega = \frac{1}{2\pi}\int_{-R}^{R} d\omega\, e^{-i\omega x}\int_{-\infty}^{\infty} f(\xi)e^{i\omega\xi}\,d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi)\frac{\sin R(\xi - x)}{\pi(\xi - x)}\,d\xi = \int_{-\infty}^{\infty} f(x+y)\frac{\sin Ry}{\pi y}\,dy$$

- To calculate the limit as $R\to\infty$, we use the fact that $\frac{\sin Ry}{\pi y}$ is a delta-sequence, so that the limit is $f(x)$ for a large class of functions

2a

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Thm

- Fourier Integral Thm.: Let $f \in L_1(-\infty,\infty)$ and let f satisfy Dirichlet conditions; then $\hat{f}(\omega)$ exists and the inversion formula (*) holds in the sense

$$\frac{f(x^+)+f(x^-)}{2} = \lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} \hat{f}(\omega)e^{-i\omega x}\,d\omega$$

- Using the def. of Fourier transforms, you can show (using integration by parts) Parseval's formulas:

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega \\ \displaystyle\int_{-\infty}^{\infty} \hat{f}(x)g(x)dx = \int_{-\infty}^{\infty} f(x)\hat{g}(x)dx \end{cases}$$

e.g.) Band-limited functions

- these provide a good demo. of the interrelation between Fourier series & integrals

- a function $\phi(x)$ $(x\in\mathbb{R})$ is band limited if $\hat{\phi}(\omega)$ vanishes for $|\omega|$ larger than some critical frequency (i.e., $\hat{\phi}$ has bounded support)

- e.g., suppose $\text{supp}(\hat{\phi}) \subset (-2\pi, 2\pi)$

$\Longrightarrow$ only the $n=0$ term occurs in the Poisson summation formula (iii)

$\Downarrow$

$$\underbrace{\sum_{k=-\infty}^{\infty} \phi(k)}_{\text{trapezoid approx}} = \hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(x)dx$$

(which is thus exact in this case!)

(over)

126 (cont).

an important property of band-limited functions is described by the <u>sampling formula</u>:

↳ let $\lambda = 1$ in Poisson summation formula (i), multiply both sides by e^{-ixt}, & integrate over $t \in [-\pi, \pi]$ to get

$$\sum_{k=-\infty}^{\infty} \phi(k) \frac{2\sin((x-k)\pi)}{x-k} = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-ixt} \hat{\phi}(t + 2\pi n)\,dt$$

- if $\hat{\phi}$ vanishes for $|\omega| \geq \pi$, only one term remains on the RHS: $\int_{-\pi}^{\pi} e^{-ixt}\hat{\phi}(t)\,dt$, which by the Fourier inversion formula is just $2\pi\phi(x)$

$$\therefore \phi(x) = \sum_{k=-\infty}^{\infty} \phi(k) \frac{\sin \pi(x-k)}{\pi(x-k)}, \quad -\infty < x < \infty$$

↳ i.e., a band-limited function is completely determined by "sampling" at discrete values

---

e.g.) <u>Heisenberg Uncertainty Principle</u>

- in QM, this gives a quantitative bound on the possible precision of simultaneous measurements of the position & momentum of a particle

- mathematically, it is the relationship between f & $\hat{f}$

  - consider $f(t)$ s.t. $\int_{-\infty}^{\infty} |f(t)|^2 dt = 1$; by Parseval's formula, we have $\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = 1$

    - both $|f(t)|^2$ & $\frac{1}{2\pi}|\hat{f}(\omega)|^2$ can thus be regarded as probability densities on their respective axes

Korevs

13a) <cont.>

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- we may assume wlog that the mean of each distribution is zero

$\Longrightarrow$ the variances are $\begin{cases} \sigma_t^2 := \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \\ \sigma_\omega^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega, \end{cases}$

where σ_t can be interpreted as a measure of the duration of a time signal & σ_ω as a measure of the frequency spread

Thm. If $|t|^{1/2} f$, $|t^{1/2}| f' \to 0$ as $t \to \infty$, then $\sigma_t \sigma_\omega \geq \frac{1}{2}$, and equality holds iff f is Gaussian (i.e., $f = Ce^{-\alpha t^2}$ $\forall \alpha$ & for C that preserves the normalization)

Proof (we'll do this for <u>real</u> $f(t)$; I'll leave it to you to generalize to make the necessary adjustments for cplx f)

- the Fourier transform of $f'(t)$ is $-i\omega \hat{f}(\omega)$, so Parseval's formula gives $\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} (f'(t))^2 dt$

- from the Schwarz inequality that

$$\left(\int_{-\infty}^{\infty} \frac{1}{2} t \frac{d}{dt}(f^2) dt \right)^2 = \left(\int_{-\infty}^{\infty} t f f' dt \right)^2 \leq \int_{-\infty}^{\infty} t^2 f^2 dt \int_{-\infty}^{\infty} (f')^2 dt = \sigma_t^2 \sigma_\omega^2$$

w/ equality iff $f' = ctf$ (c = const.)

(over)

13b (cont.)

- integrating LHS by parts & using the condition on f at ∞, we get

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}\left(\int_{-\infty}^{\infty} f^2\, dt\right)^2 = \frac{1}{4},$$ w/ equality iff $f' = cft$ (i.e., iff f is Gaussian)

$\boxed{\text{QED}}$

- Functions of Slow Growth

  - def: a function $f(x)$ on $\mathbb{R}$ is of slow growth if:
    (1) f is locally integrable — i.e., $\int_I |f(x)|\,dx < \infty$ $\forall$ bounded intervals I

    (2) $\exists$ const. C, n, R s.t. $|f(x)| < C|x|^n$ for $|x| > R$

(i.e., at ∞, it grows slower than some poly)

↳ to deal w/ their Fourier transforms, will need to consider one-sided functions

- $f(x)$ s.t. $f(x<0)=0$ is right-sided (or causal); this will be denoted $f_+(x)$

- similarly, $f(x)$ s.t. $f(x>0)=0$ is left-sided & denoted $f_-(x)$

- consider f_+ which is $O(e^{\alpha x})$ at $x=\infty$ (i.e., $\exists$ const C s.t. $|f_+(x)| < Ce^{\alpha x}$ for sufficiently large x)

$\Longrightarrow$ $f_+ e^{-vx} \in L_1(-\infty,\infty)$ for $v > \alpha$ & the (cplx) Fourier transform $\hat{f}_+(\omega)$ is analytic in the upper half-plane ($v > \alpha$)

note: vanishes identically for $x<0$

$\Longrightarrow$ $\hat{f}_+(\omega) = \int_0^\infty f_+(x) e^{i\omega x}\,dx \quad (v > \alpha)$, $\quad f_+(x) = \frac{1}{2\pi}\int_{iv-\infty}^{iv+\infty} \hat{f}_+(\omega) e^{-i\omega x}\,d\omega \quad (v>\alpha)$

(*) (you'll look at cplx FTs the HW)

14a

- for $f_-(x)$ which is $O(e^{\beta x})$ at $x = -\infty$, $f_-(x)e^{-vx} \in L_1(-\infty,\infty)$ for $v < \beta$ & $\hat{f}_-(\omega)$ is analytic in lower half-plane ($v < \beta$)

$$\Longrightarrow \begin{cases} \hat{f}_-(\omega) = \int_{-\infty}^{0} f_-(x) e^{i\omega x}\, dx \quad (v < \beta) \\ f_-(x) = \frac{1}{2\pi}\int_{iv-\infty}^{iv+\infty} \hat{f}_-(\omega) e^{-i\omega x}\, d\omega \quad (v < \beta) \end{cases}$$

vanishes identically for $x > 0$

- an arbitrary function $f(x) = f_+(x) + f_-(x)$,

where $f_+ = \begin{cases} f, & x > 0 \\ 0, & x < 0 \end{cases}$, $f_- = \begin{cases} 0, & x > 0 \\ f, & x < 0 \end{cases}$

$\therefore$ if f is $O(e^{\alpha x})$ at $x = +\infty$ & $O(e^{\beta x})$ at $x = -\infty$, then

$$\begin{cases} \hat{f}_+(\omega) = \int_0^{\infty} f(x) e^{i\omega x}\, dx \quad (v > \alpha) \\ \hat{f}_-(\omega) = \int_{-\infty}^{0} f(x) e^{i\omega x}\, dx \quad (v < \beta) \\ f(x) = \frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty} \hat{f}_+(\omega) e^{-i\omega x}\, d\omega + \frac{1}{2\pi}\int_{ib-\infty}^{ib+\infty} \hat{f}_-(\omega) e^{-i\omega x}\, d\omega, \quad (*) \end{cases}$$

where $a > \alpha$ & $b < \beta$

- functions of slow growth satisfy the needed properties at $|x| = \infty$: $f_+ e^{-vx} \in L_1(-\infty,\infty)\ \forall\ v > 0$ and $f_- e^{-vx} \in L_1(-\infty,\infty)\ \forall\ v < 0$

$\Longrightarrow$ we can use any $a > 0$ & $b < 0$ in (*)

(over)

(cont)

$$\Rightarrow f(x) = \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \left[\int_{i\epsilon - \infty}^{i\epsilon + \infty} \hat{f}_+(\omega) e^{-i\omega x} d\omega + \int_{-i\epsilon - \infty}^{-i\epsilon + \infty} \hat{f}_-(\omega) e^{-i\omega x} d\omega \right]$$

$$= \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} e^{-iux} \left[e^{\epsilon x} \hat{f}_+(u + i\epsilon) + e^{-\epsilon x} \hat{f}_-(u - i\epsilon) \right] du$$

- one can use the theory of distributions to show that $\hat{f}(u)$ exists and

$$\hat{f}(u) = \lim_{\epsilon \to 0^+} \left[\hat{f}_+(u+i\epsilon) + \hat{f}_-(u - i\epsilon) \right] = \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(x) e^{iux} e^{-\epsilon |x|} \, dx$$

e.g.) $f(x) = 1$, $x \in \mathbb{R}$; $f(x)$ is clearly of slow growth and

$$\begin{cases} \hat{f}_+(u+i\epsilon) = \int_0^\infty e^{i(u+i\epsilon)x} dx = \dfrac{i}{u+i\epsilon} \\ \hat{f}_-(u-i\epsilon) = \int_{-\infty}^0 e^{i(u-i\epsilon)x} dx = -\dfrac{i}{u-i\epsilon} \end{cases}$$

$$\therefore \hat{f}_+(u+i\epsilon) + \hat{f}_-(u - i\epsilon) = \frac{2\epsilon}{u^2+\epsilon^2}$$

- $\displaystyle \lim_{\epsilon \to 0} \frac{2\epsilon}{u^2+\epsilon^2} = 2\pi\delta(u)$

$$\therefore \hat{1}(u) = 2\pi \delta(u)$$

Transforms of Distributions on the Line

- we would like to do this w/ the eqn

$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$, but e^{iux} is not a test function in $C_0^\infty(R_1)$, so the action of f on e^{iux} isn't defined (over)

15a (cont)

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BSb-Supp

- we could also try to use Parseval's formula to define $\hat{f}$ from $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$, but this doesn't work either because $\hat{\phi}$ is not a test function

↳ we'll fix this problem by introducing a new class of test functions (& hence a new class of distributions) [we'll start w/ the 1D case]

- def: a cplx-valued function $\phi(x)$ $(x \in \mathbb{R})$ belongs to $C_\downarrow^\infty(R_1)$, the space of test functions of rapid decay, if:

(1) $\phi(x)$ is C^∞
(2) $\phi(x)$ & all its derivatives vanish at $|x| = \infty$ faster than any negative power of x; thus, $\forall$ non-negative integers k, l, we have $\lim_{|x| \to \infty} \left| x^k \frac{d^l \phi}{dx^l} \right| = 0$

- this is a larger class of functions than $C_0^\infty(R_1)$, as they decay really fast at ∞ rather than being constrained to vanish outside a finite interval

- a sequence $\{\phi_m(x)\}$ of functions in $C_\downarrow^\infty(R_1)$ is a null sequence in $C_\downarrow^\infty(R_1)$ if $\forall$ non-neg integers k, l,

$$\lim_{m \to \infty} \max_{x \in \mathbb{R}} \left| x^k \frac{d^l \phi_m}{dx^l} \right| = 0$$

def: a distribution of slow growth is a continuous linear functional on $C_\downarrow^\infty(R_1)$; hence, $\forall \phi \in C_\downarrow^\infty(R_1)$ $\exists$ a cplx number $\langle f, \phi \rangle$ s.t.:

$$\begin{cases} \langle f, \alpha_1 \phi_1 + \alpha_2 \phi_2 \rangle = \alpha_1 \langle f, \phi_1 \rangle + \alpha_2 \langle f, \phi_2 \rangle, \\ \lim_{m \to \infty} \langle f, \phi_m \rangle = 0 \ \forall \text{ null sequence in } C_\downarrow^\infty(R_1) \end{cases}$$

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Thm: Every function $f(x)$ of slow growth generates a distribution of slow growth via the formula

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

remarks:

- most distributions on $C_0^\infty(R_1)$ are also distributions on $C_\downarrow^\infty(R_1)$ [only the ones that grow rapidly at ∞ can't be extended]

- the previous theory can mostly be extended to this new class of distributions

Thm: If $\phi(x) \in C_\downarrow^\infty(R_1)$, then $\hat{\Phi}(u)$ exists & is also in $C_\downarrow^\infty(R_1)$

↳ **Proof**: the rapid decay of $\phi(x)$ at $|x| = \infty \Rightarrow$ absolute convergence of:

$$\frac{d^k\hat{\phi}}{du^k}(u) = \int_{-\infty}^{\infty} (ix)^k e^{iux} \phi(x)dx, \quad k = 0, 1, 2, \dots$$

$$\Rightarrow \left| \frac{d^k\hat{\phi}}{du^k}(u) \right| \le \int_{-\infty}^{\infty} |x^k \phi| dx \Rightarrow \text{LHS is bounded } \forall u$$

- also, $(iu)^p \dfrac{d^k\hat{\phi}}{du^k} = \displaystyle\int_{-\infty}^{\infty} (ix)^k \phi(x) \frac{d^p}{dx^p}(e^{iux})dx$ &

integration by parts on the RHS gives

$$(-1)^p \int_{-\infty}^{\infty} \underbrace{\left[\frac{d^p}{dx^p}(ix)^k \phi(x)\right]}_{\text{this is in } C_\downarrow^\infty(R_1)\text{, so the integrand is absolutely integrable}} e^{iux} dx$$

$$\Rightarrow \left| u^p \frac{d^k\hat{\phi}}{du^k} \right| \text{ is bounded } \forall u$$

- because p, k are arbitrary, we get $\hat{\Phi}(u) \in C_\downarrow^\infty(R_1)$ ✓

$\boxed{\text{QED}}$

16a

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- if f is a distribution of slow growth, then $\hat{f}$ is the distribution of slow growth defined from $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$

- you can check that the necessary properties of Fourier transform hold (maybe on the hw?)

---

e.g.) define the pseudo-function of $\frac{1}{x}$, $pf(\frac{1}{x})$ as the distribution $\left\langle pf\,\frac{1}{x}, \phi \right\rangle = \lim_{\epsilon \to 0} \left[\int_{\epsilon}^{\infty} \frac{\phi(x)}{x} dx + \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} dx \right]$

(recall this from HW#1)

- we're going to calculate the transform of $H(x)$ in 3 different ways:

$$(a) \langle \hat{H}, \phi \rangle = \langle H, \hat{\phi} \rangle = \int_0^\infty \hat{\phi}(x) dx = \int_0^\infty dx \int_{-\infty}^{\infty} \phi(y) e^{ixy} dy$$

$$= \lim_{R \to \infty} \int_{-\infty}^{\infty} \phi(y) dy \int_0^R e^{ixy} dx$$

$$= \lim_{R \to \infty} \int_{-\infty}^{\infty} \frac{e^{iRy} - 1}{iy} \phi(y) dy$$

- one can show that $\lim_{R \to \infty} \left[\frac{(1 - \cos Ry)}{y} \right] = pf\left(\frac{1}{y}\right)$; also, $\lim_{R \to \infty} \left(\frac{\sin Ry}{y} \right) = \pi \delta(y)$

$$\Longrightarrow \langle \hat{H}, \phi \rangle = \langle i\, pf\left(\tfrac{1}{y}\right) + \pi \delta(y), \phi(y) \rangle$$

$$\therefore \hat{H}(y) = i\, pf\left(\tfrac{1}{y}\right) + \pi \delta(y)$$

(b) $H'(x) = \delta(x)$; one can thus calculate $\widehat{(H')}(x) = -ix \hat{H}(x)$, so $\hat{H}$ satisfies the distributional eqn. $1 = -ixf(x)$ (*)

(over)

(16b) (cont)

- a particular solution of (*) is $f(x) = i\,pf\left(\frac{1}{x}\right)$

- plugging into (*), we find using results from hw #1 that

$$\langle 1, \phi\rangle = \langle x\,pf\tfrac{1}{x}, \phi\rangle = \langle pf\tfrac{1}{x}, x\phi\rangle$$
$$= \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}\frac{1}{x}\,x\,\phi(x)\,dx + \int_{\epsilon}^{\infty}\frac{1}{x}\,x\,\phi(x)\,dx\right] = \int_{-\infty}^{\infty}\phi(x)\,dx$$

(which is an identity)

- To find the general sol. of (*), we add to the particular sol. above the general sol. to the homogeneous equ. $ixf = 0$, which is $C\delta(x)$ (where C = const.)
(we'll discuss differential equ. in distributions shortly)

$$\Longrightarrow \hat{H}(x) = i\,pf\left(\tfrac{1}{x}\right) + C\delta(x) \quad (**)$$

- To find C, consider the equ. $H(x) + H(-x) = 1$;
- Fourier transforming this gives

$$\hat{H}(u) + \hat{H}(-u) = 2\pi\delta(u)$$; using $(**) \Rightarrow C = \pi$

$$\boxed{\therefore\ \hat{H}(x) = i\,pf\left(\tfrac{1}{x}\right) + \pi\delta(x)}$$

(c) we can think of $H(x)$ as the limit as $\epsilon\to 0^+$ of $H(x)e^{-\epsilon x}$ $\Longrightarrow \hat{H} = \lim_{\epsilon\to 0^+}\widehat{\left[He^{-\epsilon x}\right]}$

- $He^{-\epsilon x}$ has a Fourier transform in the ordinary sense:

$$\widehat{\left(He^{-\epsilon x}\right)} = \int_0^\infty e^{-\epsilon x}e^{iux}\,dx = \frac{1}{\epsilon - iu} = \frac{\epsilon + iu}{\epsilon^2 + u^2}$$

$$\Longrightarrow \begin{cases} \lim_{\epsilon\to 0^+}\dfrac{\epsilon}{\epsilon^2+u^2} = \pi\delta(u) \\ \lim_{\epsilon\to 0^+}\dfrac{iu}{\epsilon^2+u^2} = i\,pf\left(\dfrac{1}{u}\right) \end{cases} \quad \boxed{\therefore\ \hat{H}(x) = i\,pf\left(\tfrac{1}{x}\right) + \pi\delta(x)}$$

Differential Equations in Distributions

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- *local properties of distributions*

  - when a distribution f is generated by a continuous function, one can recover the pt. values of f from a knowledge of its actions on test functions

  - if we know that a distribution is generated by a locally integrable function, we can determine the function up to equality almost everywhere

---

def: the distribution f is said to vanish on an open set Ω if $\langle f, \phi \rangle = 0$ $\forall$ test function ϕ with support in Ω; two distributions f_1 & f_2 are equal in Ω if $f_1 - f_2$ vanishes on Ω

e.g.) $f(x) = \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$ has action on $\phi \in C_0^\infty(R_1)$ given by $\sum_{k=-\infty}^{\infty} \phi(2k\pi)$. let $\Omega = (-\pi, \pi)$. then if $\text{supp}(\phi) \subset \Omega$, $\langle f, \phi \rangle = \phi(0)$, so f coincides w/ $\delta(x)$ in Ω

e.g.) let $a > 0$; the distributions $f(x) = pf\left(H(x)\frac{1}{x}\right)$ & $f(x) = \dfrac{H(x-a)}{x}$ coincide for $x > a$

---

the DE $u' = f$ in R_1

- consider $u' = 0$ regarded as an eqn for distributions on $\mathbb{R}$

  ↳ by def, we are looking for distributions s.t.
  $\langle u, \phi' \rangle = 0 \quad \forall \phi \in C_0^\infty(R_1) \quad (*)$

$(*)$ means the action of u is 0 $\forall$ test function which is the derivative of another test function

17b) let $M \subset C_0^\infty(R_1)$ consist of elements which are first derivatives of elements of $C_0^\infty(R_1)$

$\boxed{\text{Lemma 1}}$ let $\phi \in C_0^\infty(R_1)$. Then $\phi \in M$ iff $\int_{-\infty}^{\infty} \phi\, dx = 0$.

$\boxed{\text{Lemma 2}}$ let $\phi_0(x)$ be a fixed (but arbitrary) test function s.t. $\int_{-\infty}^{\infty} \phi_0(x)\,dx = 1$; then $\forall \phi(x) \in C_0^\infty(R_1)$ $\exists!$ const. a &

$\exists! \psi \in M$ s.t. $\phi(x) = a\phi_0(x) + \psi(x)$ (**)

- now let's look at $u' = 0$; (**) $\Rightarrow \langle u, \phi\rangle = a\langle u, \phi_0\rangle + \langle u, \psi\rangle$
  - if u satisfies $u' = 0$, then $\langle u, \psi\rangle = 0$ for $\psi \in M$,

    so $\forall \phi \in C_0^\infty(R_1)$, $\langle u, \phi\rangle = a\langle u, \phi_0\rangle = \langle u, \phi_0\rangle \int_{-\infty}^{\infty} \phi\, dx = \langle c, \phi\rangle$,

    where $c = \langle u, \phi_0\rangle = $ const.

    ↳ i.e., only constant distributions can be sol. to $u' = 0$ (as expected)
    (& you can plug in & check that they actually are solutions)

- now we look at $u' = f$ (***)
  (f is an arbitrary distribution)

- to find the general sol. of (***), we use the decomposition $\langle u, \phi\rangle = a\langle u, \phi_0\rangle + \langle u, \psi\rangle$, where $\psi \in M$
  (say, $\psi = \chi'$, where $\chi \in C_0^\infty(R_1)$)

- the explicit expression for χ in terms of ϕ is

$$\chi = \int_{-\infty}^{x} \psi(s)\,ds = \int_{-\infty}^{x} \phi(s)\,ds - \langle 1, \phi\rangle \int_{-\infty}^{x} \phi_0(s)\,ds$$

- because u satisfies (***), we have $\langle u, \psi\rangle = \langle u, \chi'\rangle = -\langle f, \chi\rangle$

$$\Longrightarrow \langle u, \phi\rangle = \langle u, \phi_0\rangle\langle 1, \phi\rangle - \langle f, \chi\rangle$$ (over)

18a (cont)

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- claim: it is legitimate to define a distribution u_p from $\langle u_p, \phi \rangle = -\langle f, \chi \rangle$ (****)

- χ is a test function depending linearly on ϕ, so this gives a linear functional on the space of test functions $\phi(x)$; if $\{\phi_m\}$ is a null sequence in $C_0^\infty(R_1)$, then so is $\{\psi_m\}$ & hence $\{\chi_m\}$ $\Rightarrow$ the functional defined by (****) is continuous — ie, u_p is a distribution

  $\therefore$ every sol. of (***) is of the form $\langle u, \phi \rangle = c\langle 1, \phi \rangle + \langle u_p, \phi \rangle$ (& you can verify that everything of this form is indeed a sol.)

Green's Formula and Lagrange's identity

- consider the linear ordinary differential operator of order 2 given by $L = a_2(x)D^2 + a_1(x)D + a_0(x)$, where $D = \frac{d}{dx}$ & $a_k(x) \in C^2(R_1)$

- look at $\int_a^b vLu\,dx = \int_a^b (va_2u'' + va_1u' + va_0u)dx$, where $u, v \in C^2(R_1)$

- integrate by parts until all the differentiations are on v

$$\Rightarrow \underbrace{\int_a^b vLu\,dx - \int_a^b uL^*v\,dx = J(u,v)\Big|_a^b}_{\text{"Green's formula"}} \quad (*)$$, where

the formal adjoint L^* of L is given by

$$L^* = a_2 D^2 + (2a_2' - a_1)D + (a_2'' - a_1' + a_0)$$

(over)

(18b) <cont.>

- the bilinear form J (the conjunct of u & v) is

$$J(u,v) = a_2(vu' - uv') + (a_1 - a_2')uv$$

- (*) is valid $\forall\, b$, so we can differentiate w.r.t. b & set $b = x$ to get $\boxed{vLu - uL^*v = \frac{d}{dx} J(u,v)}$

"Lagrange's identity"

- if $L = L^*$, we say that L is <u>formally self-adjoint</u>
  - for a linear 2nd order op, L is formally self-adjoint iff $a_2' = a_1$, so that $Lu = D(a_2 Du) + a_0 u$
  - in this case, we get:
    - $J(u,v) = a_2(vu' - uv')$
    - $vLu - uLv = \frac{d}{dx} J(u,v)$
    - $\int_a^b (vLu - uLv)\,dx = J(u,v)\Big|_a^b$

- let's go up to order p:

$$L = a_p(x)D^p + \cdots + a_1(x)D + a_0(x), \text{ where } a_i(x) \in C^p(R_1)$$

- $u, v \in C^p(R_1) \Longrightarrow$

$$D\left[vD^{m-1}u - v'D^{m-2}u + \cdots + (-1)^{m-1}(D^{m-1}v)u\right] =$$

$$= vD^m u + (-1)^{m-1} u D^m v$$

$$\Longrightarrow vD^m u = (-1)^m u D^m v + D \sum_{\substack{j+k=m-1 \\ j,k\geq 0}} (-1)^k (D^k v)(D^j u)$$

(the sum disappears for $m=0$)

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- substituting $a_m v$ for v & summing from $m=0$ to p, we get

(Lagrange's identity)

$$vLu - uL^*v = \frac{d}{dx} J(u,v), \text{ where}$$

$$\begin{cases} L^* v = \sum_{m=0}^{p} (-1)^m D^m (a_m v), \\ J(u,v) = \sum_{m=1}^{p} \sum_{j+k=m-1} (-1)^k D^k (a_m v) D^j u \end{cases}$$

- we also get Green's formula: $\int_a^b (vLu - uL^*v)\,dx = J(u,v)\Big|_a^b$

- J has derivatives up to order $p-1$ (these are the boundary terms from I by P)

- for L to be formally self-adjoint ($L = L^*$), it needs to be of even order & writable in the form

$$L = D^r(b_r D^r) + D^{r-1}(b_{r-1} D^{r-1}) + \cdots + D(b_1 D) + b_0,$$

where $p = 2r$ & $b_0, \ldots, b_r$ are arbitrary functions

→ now let's think about pde's; let's start w/ the Laplacian Δ on R_n, which satisfies

$$v\Delta u - u\Delta v = \operatorname{div}(v\vec{\nabla u} - u\vec{\nabla v}), \text{ or in integrated}$$

form, $\int_\Omega (v\Delta u - u\Delta v)\,dx = \int_\Gamma \hat{n}\cdot(v\vec{\nabla u} - u\vec{\nabla v})\,dS$, $\Gamma = \partial\Omega$

unit normal vector

(This is the classical Green's formula, which you've probably seen before)

- for an arbitrary linear operator of order p, we get:

(over)

19b (cont)

$$\begin{cases} vLu - uL^*v = \operatorname{div} \vec{J}(u,v) \\ \int_\Omega (vLu - uL^*v)\,dx = \int_\Gamma \hat{n}\cdot \vec{J}(u,v)\,dS \end{cases}$$

- $Lu = \sum_{|k|\le p} a_k(x) D^k u$, $\quad L^*v = \sum_{|k|\le p} (-1)^{|k|} D^k (a_k v)$

(we get L^* by integration by parts)

- ($L = L^*$ is again formal self-adjointness)

- the expression for $\vec{J}$ is messy, so I'll just show it in specific examples

---

e.g.) $x = (x_1, \dots, x_n)$, $p = 4$, $L = \Delta\Delta$; $L = L^*$, & from direct calculation we get

$$v\,\Delta\Delta u - u\,\Delta\Delta v = \operatorname{div}\left[v\vec{\nabla}(\Delta u) - u\vec{\nabla}(\Delta v) + (\Delta v)\vec{\nabla} u - (\Delta u)\vec{\nabla} v \right]$$

---

e.g. diffusion operator $L = \frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$

- $L^* = -\frac{\partial}{\partial t} - \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$ & $vLu - uL^*v = \operatorname{div} J$,

  where $J = \hat{e}_t uv - (v\vec{\nabla}_x u - u\vec{\nabla}_x v)$, where

  $\hat{e}_t$ is unit vector in t-direction & $\vec{\nabla}_x$ means the gradient w.r.t. x directions

- Green's formula becomes

$$\int_\Omega (vLu - uL^*v)\,dx\,dt = \int_\Gamma \hat{n}\cdot(\hat{e}_t uv + u\vec{\nabla}_x v - v\vec{\nabla}_x u)\,dS,$$

(over)

 (cont)

where Ω is a domain in space-time,
$\Gamma = \partial\Omega$

Classical, weak, & distributional solutions

- consider $\frac{du}{dx} = f(x)$ on $\Omega = (a,b)$ (**)

- if f is a continuous function, then $u(x)$ is a classical (or strict) solution if it has continuous derivative that satisfies (**) pointwise

- for any such classical solution,

$$\int_\Omega f\phi\, dx = -\int_\Omega u \frac{d\phi}{dx}\, dx \quad \forall \phi \in C_0^\infty(\Omega) \qquad (***)$$

(this comes from I by P & $\phi \equiv 0$ in a nbhd of the boundary)

- The 2 sides of (***) make sense even for locally integrable f & u

(def) ↳ this then defines a weak solution: if f is locally integrable, then a locally integrable function u is a weak sol. of (**) iff it satisfies (***) $\forall \phi \in C_0^\infty(\Omega)$ (we also say $\frac{du}{dx} = f$ in the weak sense)

- (**) can be interpreted distributionally:

  - if f is a distribution, then a distribution u is a solution of (**) iff

$$\underbrace{-\left\langle u, \frac{d\phi}{dx}\right\rangle}_{\text{this is the def of the distribution } u'} = \langle f, \phi\rangle \quad \forall \phi \in C_0^\infty(\Omega)$$

- now let L be any linear diff. op. of order p in variables $x_1, \dots, x_n$:

$$L = \sum_{|k| \le p} a_k(x) D^k, \quad \begin{cases} k = (k_1, \dots, k_n) \\ |k| = k_1 + \dots + k_n \end{cases}$$

- assuming $a_k(x) \in C^\infty$, then the distribution Lu exists for any distribution u; it is defined by $\langle Lu, \phi \rangle = \langle u, L^* \phi \rangle$,

$$\text{where } L^* \phi = \sum_{|k| \le p} (-1)^{|k|} D^k (a_k \phi)$$

- we can thus give a distributional meaning to

$$(***) \quad Lu = f, \; x \in \Omega \quad \text{for a given distribution } f$$

def: a distribution u is a solution of $(***)$ on Ω if $\underset{(****)}{\,} \langle u, L^* \phi \rangle = \langle f, \phi \rangle \; \forall \phi \in C_0^\infty(\Omega)$

def: let f be locally integrable; a locally integrable u that satisfies $(****)$ is a <u>weak solution</u> to $(***)$ on Ω

Thm. let $f(x)$ be continuous on Ω. Then (a) a classical sol of $(***)$ is also a weak sol. & (b) a weak sol on Ω that has p continuous derivatives is also a classical sol.

- naturally, one can have weak sol. that aren't classical solutions

e.g. $\Omega = R_1$, $x \dfrac{du}{dx} = 0$ has weak sol $u(x) = H(x)$;

$$\langle x H', \phi \rangle = \langle H', x\phi \rangle = \langle \delta, x\phi \rangle = 0$$

! the weak sol. is associated w/ the fact that the DE has a singular point e.g.) in R_2 $(x=(x_1,x_2))$, consider $\frac{\partial u}{\partial x_1}=0$ (*)

- the classical solutions are $u=f(x_2)$ w/ f differentiable

- $u=H(x_2)$ is a weak sol of (*) because:

$$\left\langle \frac{\partial H(x_2)}{\partial x_1}, \phi(x_1,x_2)\right\rangle = \left\langle H(x_2), -\frac{\partial\phi}{\partial x_1}\right\rangle = -\int_0^\infty dx_2 \int_{-\infty}^{\infty} dx_1 \frac{\partial\phi}{\partial x_1}$$

$$= -\int_0^\infty dx_2 \left[\phi(+\infty,x_2)-\phi(-\infty,x_2)\right]=0$$

---

e.g.) Homogeneous wave eqn in one space dim x:

$$\Box^2 u := \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (**)$$

- if $f\in C^2$, then $u=f(x-t)$ is a sol of (**) (wave travelling to right w/ speed 1)

- $u=H(x-t)$ is a weak sol of (**); $\Box^2$ is formally self-adjoint, so we need to show that

$$\langle H(x-t), \Box^2\phi(x,t)\rangle = 0 \quad \forall\, \phi\in C_0^\infty(R_2)$$

or, equivalently that $\displaystyle\iint_{x>t}\left(\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}\right)dx\,dt=0 \quad \forall\,\phi\in C_0^\infty(R_2)$

- change vars: $x_1=x-t$, $x_2=x+t$

$\Rightarrow$ the last integral becomes integration of $\frac{\partial^2\phi}{\partial x_1\partial x_2}$ over half plane $\{x_1>0\}$

- then, because $\displaystyle\int_{-\infty}^{\infty}\frac{\partial^2\phi}{\partial x_1\partial x_2}dx_2=0$, we're done

  - as this e.g. shows, weak sol. can have jump discontinuities on the <u>characteristics</u> $x=\pm t$

21b) Fundamental Solutions

- $(\Omega = R_n)$

- def: a fundamental solution for L w/ pole at ξ is a solution of the equ

$$Lu = \delta_\xi(x) = \delta(x - \xi), \quad (***)$$

(where ξ is regarded as a parameter)

- Remarks:

(1) (***) can be interpreted in the sense of distributions
(the sol. $E(x,\xi)$ satisfies $\langle E, L^*\phi\rangle = \phi(\xi) \ \forall \phi \in C_0^\infty(R_n)$)
(****)

(2) (***) typically has many solutions that differ from each other by a sol. of $Lu=0$

(3) if L has const. coeffs, it is sufficient to find a fundamental sol. w/ pole at 0 & then to translate:
$E(x,\xi) = E(x-\xi, 0)$

- One determines a fundamental sol. in 2 steps:

(1) construct (often by intuitive means — symmetry, etc.) a likely candidate

(2) check that (****) is satisfied

- part (1) itself has 2 steps:
(a) solve $Lu=0$ for $x \neq \xi$
(b) build in the singularity at $x=\xi$ by using (****) near $x=\xi$ (in 1D, this means matching on both sides of ξ)

22a)

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e.g.) find a fundamental sol. E for $\frac{-d^2}{dx^2} + \varepsilon^2$

- coeffs are const., so we can look at

$$-\frac{d^2E}{dx^2} + \varepsilon^2 E = \delta(x), \qquad x \in \mathbb{R}$$

- we require continuity of E at $x=0$ & a jump of the derivative: $E'(0+,0) - E'(0-,0) = -1$

- we also demand $E(|x|=\infty)$ [this would be put in for physical considerations in this case, but should be somehow given in a well defined problem]

$$\Longrightarrow E(x) = \frac{e^{-\varepsilon|x|}}{2\varepsilon}, \quad E(x,\xi) = \frac{e^{-\varepsilon|x-\xi|}}{2\varepsilon}$$

- To check that $E(x)$ is a fundamental sol. w/ pole at 0, we must verify (***):

$$\langle E, L^*\phi\rangle = \int_{-\infty}^{0} \frac{e^{\varepsilon x}}{2\varepsilon} L^*\phi\,dx + \int_{0}^{\infty} \frac{e^{-\varepsilon x}}{2\varepsilon} L^*\phi\,dx$$

- using Green's Thm. at each interval, we find that

$$\langle E, L^*\phi\rangle = \phi(0) \quad \forall \phi \in C_0^\infty(\mathbb{R}_1)$$

(you can alternatively start w/ $E(x)$ & differentiate it as a distribution) e.g.) Consider the linear, ordinary diff. op. L of order p: we'll find the <u>causal</u> fundamental sol. $E(t, \tau)$ which vanishes for $t < \tau$ & satisfies

$$(*) \quad LE = a_p \frac{d^p E}{dt^p} + \cdots + a_1 \frac{dE}{dt} + a_0 E = \delta(t-\tau), \quad t, \tau \in \mathbb{R}$$

- $E \equiv 0$ for $t < \tau$ & $E, E', \ldots, E^{(p-2)}$ are continuous at $t = \tau$ (i.e., all are 0 at $t = \tau^+$); $a_p E^{(p-1)}$ has a unit jump at $t = \tau$, so $E^{(p-1)}(\tau^+, \tau) = \dfrac{1}{a_p(\tau)}$

- the jump condition is obtained by integrating $(*)$ from τ^- to τ^+

$\Longrightarrow$ for $t > \tau$, $E(t, \tau)$ coincides w/ the sol. $u_\tau(t)$ of the IVP

$$(**) \quad \begin{cases} Lu_\tau(t) = 0, \quad u_\tau(\tau) = u_\tau'(\tau) = \cdots = u_\tau^{(p-2)}(\tau) = 0, \\ \qquad u_\tau^{(p-1)}(\tau) = \dfrac{1}{a_p(\tau)} \end{cases}$$

- $(**)$ has a unique sol. (existence & uniqueness for linear IVPs),

we let $E(t, \tau) = H(t-\tau)\, u_\tau(t)$, which we need to verify satisfies $(\ast\ast\ast\ast)$ for $x = t$, $\xi = \tau$

$$\hookrightarrow \langle E, L^*\phi \rangle = \int_\tau^\infty u_\tau(t)\, L^*\phi \, dt \underset{\substack{\uparrow \\ \text{Green's formula}}}{=} \int_\tau^\infty \phi\, Lu_\tau(t)\, dt - \underbrace{J(u_\tau, \phi)\Big]_{t=\tau}^{t=\infty}}_{\substack{\text{boundary terms} \\ \text{in I by P.} \\ \text{(drops the formality} \\ \text{when } J = 0)}}$$

- $Lu_\tau = 0 \Rightarrow$ the integral vanishes; $\phi \equiv 0$ outside a bounded interval shows that $J \equiv 0$ is an upper limit

(over)

23a (cont)

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- at the lower limit τ, we need to look more closely at
$$J(u,v) = \sum_{m=1}^{p} \sum_{j+k=m-1} (-1)^k D^k (a_m v) D^j u$$

  - all terms w/ derivatives of u_τ of order $\leq (p-2)$ must vanish by (**)

$\Downarrow$

we have a single term w/ $j=p-1, k=0, m=p$

$\therefore$ at $t=\tau$, $J(u_\tau, \phi) = u_\tau^{(p-1)}(\tau) a_p(\tau) \phi(\tau) = \phi(\tau)$,

which gives $\langle E, L^* \phi \rangle = \phi(\tau)$, as required

- note: if the coeffs in (*) are all const., then $u_\tau(t) = v(t-\tau)$, where $v(t)$ is a sol. of the IVP w/ $\tau = 0$

(note: there may be some time for more material or review at this point)
