The Geometry of Flows

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Introducing Geometry

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- Pythagorus’ theorem,
- Circle theorems,
- Trigonometry.
A little history

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- There is a clay tablet from around 1800BC listing a collection of triples $a, b, c$ such that $a^2 = b^2 + c^2$.
- Euclid’s Elements written around 300BC contains many results on circles, cones and cylinders that are well known to you.
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Here are some faces associated to results you will know...
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Flows and the Geometry of Surfaces

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Towards 19th Century mathematics

By the 19th Century advances in Physics and Engineering in fields including Electromagnetism and fluid dynamics required mathematical tools for to study them systematically.
Where now?

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**Poincaré–Hopf**

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You can identify a shape by studying the ‘fluid flows’ that run on the surface of that shape.

We’ll start building up a some of the ideas that will go into this, including the notion of a flow in fairly abstract terms, before looking at what I mean by a ‘surface’ and a little bit of a related subject called *topology* before we finally get to the result!
Contour plots are used to represent a (scalar) quantity varying over 2-dimensional space. This is often shown as height, but it need not be literally so; it might show temperature, pressure, etc. Remember that curves in a contour plot show collections of points at the same level: if I travel around a contour I move neither 'up' nor 'down'.

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Warm-up: Contour maps

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Some Contour Maps
We can see if we can construct contour maps with certain properties.

Exercises
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We can see if we can construct contour maps with certain properties.

Can you construct...

- a contour plot with precisely two points such that if a ball was placed there it would not roll?
- a contour plot with three such points?
- a contour plot with infinitely many such points?

More on stationary points

- Can I make a stationary point such that if I move a little I always roll away?
- ... or always roll back?
- ... or roll away in some directions and back in others?
Stationary points

As we saw in the second question, there are three main types of stationary point.
Flow diagrams

We can represent our contour map in a different way by plotting its flow. Rather than drawing curves of points at the same height, we draw curves which show how a point will move in this ‘landscape’.
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- Weather patterns
- Electric/magnetic fields.
- Others?
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A flow diagram is simply a collection of curves which shows the trajectory of a small ball placed at that position, usually decorated with arrows to show the direction of the flow. Just as with contour maps we have a number of stationary points where a ball placed there will not move. Here is an example, taken from a demonstration of movement in an electric field.
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Can we do this for our own plots we’ve drawn? Remember to keep contours and flow lines at right angles!
Contours from flows?

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The Vortex

Hurricane Isabel gave us a striking example of a vortex. A vortex is simply a region where the flow ‘goes around’ in closed loops around a fixed point. Can you draw its flow diagram? Discussion: can we draw a contour map which produces this flow diagram?
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Index of a stationary point

Imagine yourself standing in a flow near a stationary point holding a needle which always points in the direction of the flow. Walk in a circle clockwise around the stationary point and count the number of times the needle spins round clockwise (this might be a negative number!)
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Index of a stationary point

Now let’s see if we can calculate the indices of the following stationary points.
Break
An Introduction to Topology

We now make a short diversion into another important area of modern mathematics, *topology*.
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**Topology**

Topology is the abstract study of shape, which is very much an active field of research today. To a topologist shapes are ‘the same’ if they can be deformed into each other by any ‘smooth’ process (no cutting or tearing), if you open a book these deformations are called ‘homotopies’.

*Fig. 2*
Euler Number

We need one important idea from topology, that of the *Euler number* of a shape. I won’t present a technical definition, but we’ll get a feel for the idea through some properties and examples (this is a common research practice).
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**Rule 1: Euler number of a polygonal shape**

We’ve already seen some examples of shapes made up of polygons (tetrahedron, cube, ...). In those cases we saw that the number:

\[ V - E + F \]

Was the same for all the Platonic solids. In fact given *any* shape made up of polygonal faces we say the number \( V - E + F \) is the Euler number.
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\[
e \left( \begin{array}{c}
\text{triangle} \\
\text{with one shaded}
\end{array} \right) = 1 \
e \left( \begin{array}{c}
\text{triangle} \\
\text{with no shaded}
\end{array} \right) = 0
\]
Euler numbers

We’ll use two more rules to compute Euler numbers.

Rule 2: Topological invariance
If two shapes can be deformed without tearing into each other, they have the same Euler number.

Rule 3: Cut and Paste
If I cut a shape into two pieces, which were attached along a shape, then the Euler number of the whole can be computed as:
$$e(\text{Left}) + e(\text{Right}) - e(\text{Middle})$$
Euler numbers

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Rule 3: Cut and Paste

If I cut a shape *Whole* into two pieces *Left* and *Right* which were attached along a shape *Middle*, then the Euler number of *Whole* can be computed as

\[ e(\text{Left}) + e(\text{Right}) - e(\text{Middle}) \]

\[
e^{\big(\text{\includegraphics[width=0.1\textwidth]{circle.png}}\big)} = 0 \\
\]
\[
e^{\big(\text{\includegraphics[width=0.1\textwidth]{torus.png}}\big)} = 2
\]
Exercises

We’ll use our three rules to compute some Euler numbers. Remember our three rules.

- If my shape is made up of polygons I can use $V - E + F$.
- If two shapes can be deformed into each other they have the same Euler number.
- If I cut a shape into two I can compute the Euler number of the original shape by adding the euler number of the two pieces and subtract the shape they are glued along.
Flows on a sphere

We can now tie everything together. What do fluid flows have to do with Euler numbers? On the face of it they come from different areas of geometry, flows are really geometric, and you can have many very different flows on the same surface. Consider the following flow on a sphere:
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We can now tie everything together. What do fluid flows have to do with Euler numbers? On the face of it they come from different areas of geometry, flows are really geometric, and you can have many very different flows on the same surface. Consider the following flow on a sphere:

On this sphere we have a flow shown with 4 vortices and 2 saddle points. We worked out the indices of these earlier, what is the sum of the indices of the stationary points?
As well as the sphere shown above your sheet contains a number of examples of flows to think about.
Exercises

As well as the sphere shown above your sheet contains a number of examples of flows to think about.

We might start to guess a remarkable result.

The Poincaré–Hopf theorem

The sum of the indices of the stationary points of a smooth surface is the Euler number of the surface.
The kind of work we have been doing is an essential component of Mathematical research, you play around with some new ideas, try some examples and notice patterns or connections between different areas. The final stage is to prove that what you think is the case is actually true. This can sometimes be very easy (even for important results) or sometimes require a great many new ideas and technical work.
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Proof of Poincaré–Hopf

This proof has two main parts, one rather more difficult than the other.

- The easier part is to construct a flow on a closed surface for which the sum of the indices is the Euler number.
- The harder part is to show that whatever flow you choose you get the same answer.
The End