

C3.1 Algebraic Topology

Please be aware there are

likely typos in these notes:

comments/corrections are welcome!

Course Book

- **Hatcher, Algebraic Topology** — Chp. 2 & 3

This is also freely available from the author's website.

Expectations

- You are expected to read chapters 2 & 3 of Hatcher
- You should read the technical remarks about orientation signs in these notes: we will likely not have time for those in lectures.
- This course will not discuss intersection numbers rigorously. The notes often mention these in order to develop your intuition.

The books by Bott & Tu and Guillemin & Pollack discuss these ideas rigorously

Other references

- Ulrike Tillmann's C3.1 notes — see course page
- The discussion of orientations in my B3.2 Geometry of Surfaces notes may be helpful.

Other books

Massey, A basic course in Algebraic Topology

(Not "Algebraic Topology: An Introduction" ← does not treat homology)

James W. Vick, Homology Theory

MORE BASIC but full of ideas:

Fulton, Algebraic Topology: a first course.

MORE ADVANCED:

May, A concise course in Algebraic Topology

Davis & Kirk, Lecture Notes in Algebraic Topology

Bredon, Topology and Geometry

Classics by **Spanier**, **Dold**, also see references in May's book

Bott & Tu, Differential forms in Algebraic Topology

Guillemin & Pollack, Differential Topology

CONTENTS

0. OVERVIEW OF THE COURSE

Motivation, category theory, functors H_* and H^* : some computations why functors are useful: Invariance of dimension, Brouwer fixed pt thm

1. ALGEBRAIC PRELIMINARIES: CHAIN COMPLEXES

Graded abelian groups

chain complexes, chain maps, subcomplex, quotient complex

chain map induces map on homology

exact sequence, snake lemma: SES induces LES on H_* , naturality of LES

5-Lemma, SES splits \Leftrightarrow direct sum

2. Δ -COMPLEXES AND SIMPLICIAL HOMOLOGY

Δ^n , n -simplices, Δ -complex (structure), simplicial cx, triangulation

simplicial chain complex, $H_*(S^n)$, $H_*(T^2)$, remark about orientations

$H_*^{\Delta}(\sqcup \text{conn.comp.}) \cong \oplus H_*^{\Delta}(\text{conn.comp.})$, $H_0^{\Delta}(X) \cong \mathbb{Z}^{\# \text{conn.comp}}$

3. SINGULAR HOMOLOGY

Motivation, singular chain cx

naturality / functoriality, $H_*(\text{point})$

4. CHAIN HOMOLOGY AND HOMOTOPY INVARIANCE

chain homotopy, prism operator

homotopic maps $f \simeq g$ (relative A), homotopy equivalent spaces $X \simeq Y$

contractible space, homotopic maps induce equal maps on homology

homotopy equivalence induces iso on H_* , $H_*(\mathbb{R}^n) = H_*(D^n) = H_*(pt)$

pairs of spaces, relative homology $H_*(X, A)$, LES in H_* for pair

reduced homology $\tilde{H}_*(X)$, LES for \tilde{H}_* , $H_*(D^n, S^{n-1}) \cong \tilde{H}_{*-1}(S^{n-1})$

naturality of LES for pairs

5. EXCISION THEOREM AND QUOTIENTS

retractions, deformation retractions, excision thm, quotients,

good pairs $\Rightarrow H^*(X/A) \cong \tilde{H}^*(X/A)$, generator of $H_n(S^n) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$

another remark about orientations

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6. MAYER-VIETRIIS SEQUENCE

MV LES, $H_*(S^n)$

wedge sum $X \vee Y$, cone CX , suspension ΣX , connected sum $\#X$

7. DEGREE OF MAPS OF SPHERES

degree, application to tangent vector-fields on sphere, hairy ball theorem
local degree, proof of fundamental thm of algebra

8. CELLULAR HOMOLOGY

CW complexes, cellular complex, rank $H_n^{CW} \leq \#n$ -cells
 $H_*^{CW}(D^1 \times D^1)$, $H_*^{CW}(\mathbb{R}P^n)$, $H_*^{CW}(S^n)$, $H_*^{CW}(\Sigma g)$

Δ -CW \Rightarrow CW CW, $H_*^{CW}(X) \cong H_*^{\Delta}(X) \cong H_*^{\Delta}(X)$, Axioms for homology

9. COHOMOLOGY

cochains, cohomology, $H^*(X)$, $H_{CW}^*(X)$, $H_{\Delta}^*(X)$, $H^*(\mathbb{R}P^3)$
functoriality, homotopy invariance, cochain homotopy, dual of a SES
excision, LES, Mayer-Vietoris for H^* , axioms for cohomology

10. CUP PRODUCT

Cup product, $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory
examples: $H^*(T^2)$, $H^*(\Sigma_2)$, remarks about intersection theory

11. KÜNNETH FORMULA AND PRODUCT SPACES

Tensor products of R -mods, tensor product of chain cxes,
algebraic Künneth thm, product spaces $X \times Y$, Euler characteristic χ
CW-CW for product space, Künneth thm, $H^*(S^n \times S^m)$, $H^*(T^n)$

12. UNIVERSAL COEFFICIENTS THEOREM

Universal coeff. thm, Background on Ext groups and free resolutions
(Co)homology with coefficients in a ring/field/module, $H^*(\mathbb{R}P^2; \mathbb{Z}/2)$

Univ. coeff. thm for PID R , Duality $H^*(X; \mathbb{F}) \cong H_*(X; \mathbb{F})$ over fields

Structure thm for f.g. mods M over PID R , $\text{Ext}_R^1(M; R)$, torsion shift H_* to H^{*+1}

13. MANIFOLDS: POINCARÉ-LEFSCHETZ DUALITY

Local orientation, orientation, fundamental class, degree, P. duality, L. duality,
Locally finite homology H_*^{lf} , cohomology with compact supports H_c^* , Cap product and P.D.,
Alexander duality, knot complements, Jordan curve thm

O. OVERVIEW OF THE COURSE

Motivation

Space X associate \longrightarrow

Algebraic object $A(X)$
like numbers, groups, rings, ...

Isomorphism of spaces $X \cong Y \implies$ Isomorphism $A(X) \cong A(Y)$

PAY-OFF Can tell spaces apart:

compute $A(X), A(Y) \rightsquigarrow$ if $A(X) \not\cong A(Y)$ then $X \neq Y$

Examples

1) Set $X \longrightarrow A(X) = \#X \in \mathbb{N} \cup \{\infty\}$
same size

(bijection $X \rightarrow Y$) \implies

2) Vector space $X \longrightarrow A(X) = \dim X \in \mathbb{N} \cup \{\infty\}$
(linear iso $X \rightarrow Y$) \implies same dim

3) Topological Space $X \longrightarrow \# \pi_0(X) = \# \text{path components} \in \mathbb{N} \cup \{\infty\}$
 $\longrightarrow \# \text{Connected components}$

$\chi(X) = \text{Euler characteristic} \in \mathbb{Z}$

$\longleftarrow \text{Loops} = C^0(S^1, X)$

Function $X \times \mathbb{Z}X \longrightarrow \mathbb{Z} \cup \{\infty\}$

$(P, \gamma) \longmapsto w(\gamma; P)$

Winding number of γ around P .

(Homeomorphism $X \rightarrow Y$) $\longrightarrow A(X) = A(Y)$



CONVENTION for this course:

Space = topological space

Map of spaces = continuous map = cts map

For spaces, " \cong " means homeomorphism

"id" = identity map

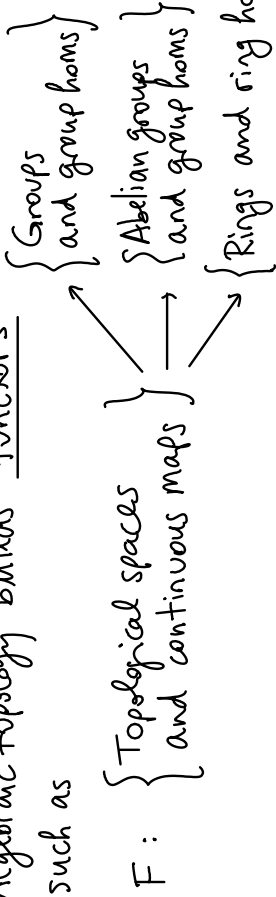
All diagrams commute unless we say otherwise, e.g.

$A \xrightarrow{\alpha} B$ means $\delta \downarrow \delta \downarrow \beta \circ \alpha = \delta \circ \beta$

Category theory is the best language to phrase all this

Algebraic topology builds functors

such as



We will not use much category theory, just basic terminology:

Def A category C consists of the data:

Ob(C) = a collection of objects

Hom(A, B) = a set of morphisms between any $A, B \in \text{Ob } C$ ("arrows")

- with composition rule $\text{Hom}(B, C) \times \text{Hom}(A, B) \rightarrow \text{Hom}(A, C)$
 $A \xrightarrow{f} B \xrightarrow{g} C$

- with identity morphs $\text{id}_A \in \text{Hom}(A, A)$ s.t. $f \circ \text{id}_A = \text{id}_B \circ f = f$

$\forall (f: A \rightarrow B) \in \text{Hom}(A, B)$

Example Sets = {sets with all maps between sets}

Top = {topological spaces with continuous maps}

Gps = {groups with group homs}

Def A (covariant) functor $F: C_1 \rightarrow C_2$ is the data:

- an assignment $(A \in \text{Ob } C_1) \mapsto (F(A) \in \text{Ob } C_2)$

- an assignment $(A \xrightarrow{f} B) \mapsto (F(A) \xrightarrow{F(f)} F(B))$

$\text{Hom}_{C_1}(A, B) \xrightarrow{F} \text{Hom}_{C_2}(F(A), F(B))$

Compatible with identities and compositions.

$$F(\text{id}_A) = \text{id}_{F(A)} \quad F(g \circ f) = F(g) \circ F(f)$$

A contravariant functor is defined similarly except it reverses the

direction of arrows: $(F(A) \xleftarrow{F(f)} F(B)) \in \text{Hom}(F(B), F(A))$

(so $F(g \circ f) = F(f) \circ F(g)$ reverses order of compositions)

Examples

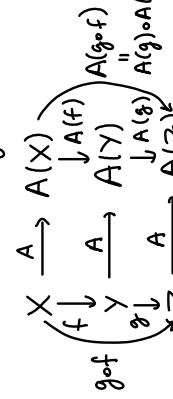
1) $F: \text{Top} \rightarrow \text{Sets}, A \mapsto A, f \mapsto f$ "forget the topology and continuity"

2) $F: \text{Sets} \rightarrow \text{Gps}, A \mapsto$ free abelian group generated by A

$$Z\langle A \rangle = \left\{ \sum_{\text{finite}} n_i \cdot a_i : a_i \in A, n_i \in \mathbb{Z} \right\}$$

$$(A \xrightarrow{f} B) \mapsto (F(f): Z\langle A \rangle \rightarrow Z\langle B \rangle)$$

When we say a construction is natural we mean functorial:



A: (a category of spaces) \rightarrow (a cat. of algebraic objects)

The algebraic objects we assigned

are assigned compatibly with maps of spaces,

and the compatibility maps $A(f)$ are also

compatible w.r.t. composition.

So we made compatible choices in constructing A.

Not to be confused with natural transformations of functors (later) which is about relating two such constructions A_1, A_2 in a compatible way

Example of a functor in algebraic topology (see B.3.5 Topology and Groups course)

$$\Pi_1(X, p) = \text{Fundamental group} = \left\{ \gamma \in \mathcal{L}X \mid \gamma(1) = p \right\} / \sim$$

Continuous deformations of loops based at p

Group multiplication: concatenate loops $\gamma_1 * \gamma_2$ (each travelling twice as fast)

Examples $\Pi_1(\mathbb{R}^n) = 0$ (for basepoint $= 0 \in \mathbb{R}^n$)

$\Pi_1(S^1) \cong \mathbb{Z}$ (deform: $h: S^1 \times [0, 1] \rightarrow \mathbb{R}^n, h(t, s) = (1-s)\gamma(t)$)

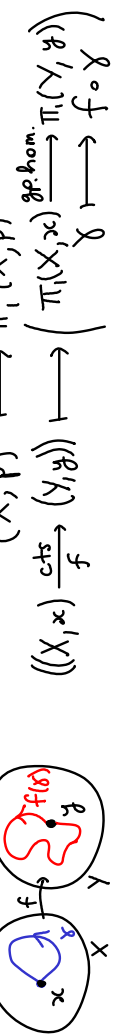
$\Pi_1(S^n) \cong 0$ (total # times wind around circle $n \geq 2$ (not obvious))

$\Pi_1(\text{torus}) \cong \mathbb{Z}^2$ (those loops generate Π_1)

Based Top = {Topological spaces with choice of base point, and continuous basepoint-preserving maps} $\xrightarrow{\Pi_1} \text{Gps}$

$$(X, p) \mapsto \Pi_1(X, p)$$

$$((X, x) \xrightarrow{f} (Y, y)) \mapsto \left(\Pi_1(X, x) \xrightarrow{f_{\text{phom.}}} \Pi_1(Y, y) \right)$$



Lemma Functors map isomorphisms to isomorphisms (iso. means \exists inverse w.r.t. composition)
 Pf $A \xrightarrow{f} B \xrightarrow{g} A \Rightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FA$, similarly for $B \xrightarrow{g} A \xrightarrow{f} B$. \square

Def **Natural transformation** $\alpha: F \rightarrow G$ between functors $C \xrightarrow{F} D \xrightarrow{G}$
 is an association $(A \in \text{Ob } C) \mapsto (\alpha_A: F(A) \rightarrow G(A)) \in \text{Hom}_D(F(A), G(A))$

such that $(A \xrightarrow{f} B) \Rightarrow \begin{matrix} F(A) \xrightarrow{\alpha_A} G(A) \\ \downarrow F(f) \quad \downarrow G(f) \\ F(B) \xrightarrow{\alpha_B} G(B) \end{matrix}$ (commutes)

It is called a **natural isomorphism** if each α_A is an isomorphism in C_2

Example of a natural transformation in algebraic topology

Let $H_1(X, P) = \text{abelianisation of } \pi_1(X, P)$ (want to identify $ab=ba$ so quotient by $\langle aba^{-1}b^{-1} \rangle$)
 \Rightarrow natural trans. $(\text{Based Top } \pi_1 \rightarrow \text{Gps}) \xrightarrow{\alpha} (\text{Based Top } H_1 \rightarrow \text{Gps})$ **Commutators**
 which associates $(X, P) \mapsto (\alpha_{(X,P)}: \pi_1(X, P) \xrightarrow{\text{quotient}} H_1(X, P))$

Cultural link higher homotopy groups $\pi_n(X, P) = \left\{ S^n \xrightarrow{\text{cts}} X \right\} / \left\{ \text{basept } \uparrow P \right\} / \text{deform}$
 FACT abelian for $n \geq 2$, but hard: e.g. $\pi_k(S^n)$ not all known.

We will not study these in this course.
 We will study simpler invariants called HOMOLOGY groups $H_n(X)$

FACT (Hurewicz) \exists natural transformation $\pi_n \rightarrow H_n$ which will make sense at the end of course:
 $f: S^n \xrightarrow{\text{cts}} X$ gives rise to a class $f_*[S^n] \in H_n(X)$.

Exercise to practice these notions from category theory:

- Summarise your undergraduate linear algebra as follows:
- \exists functor $F: \left\{ \begin{matrix} \mathbb{R}^n \text{ for } n \in \mathbb{N} \\ \text{Hom}(\mathbb{R}^m, \mathbb{R}^n) = \left\{ \begin{matrix} m \times n \\ \text{matrices} \end{matrix} \right\} \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} \text{finite dimensional real vector spaces} \\ \text{and linear maps} \end{matrix} \right\}$
 Mat $\xrightarrow{F} \text{Vect}$
 - A choice of basis for each vector space V determines a functor $G: \text{Vect} \rightarrow \text{Mat}$
 - Construct natural isomorphisms $G \circ F \xrightarrow{\alpha} \text{Id}_{\text{Mat}}$, $F \circ G \xrightarrow{\beta} \text{Id}_{\text{Vect}}$
- When functors satisfying such natural isos exist, the categories are called **equivalent** (not isomorphic). So Mat, Vect are equivalent categories.

Aim of the course: build a functor

HOMOLOGY $H_*: \text{Top} \rightarrow \text{Graded abelian groups}$
 $(X \rightarrow Y) \mapsto (H_*(X) \rightarrow H_*(Y))$
 (grading preserving hom)

and a contravariant functor

COHOMOLOGY $H^*: \text{Top} \rightarrow \text{Graded rings}$
 $(X \rightarrow Y) \mapsto (H^*(X) \leftarrow H^*(Y))$

Rough idea:

H_*X is generated by "nice" subspaces $C \subseteq X$ which have no boundary: $\partial C = \emptyset$, modulo identify C_1, C_2 if $C_1 \cup C_2$ arises as a boundary ∂B .
 Call such C_1, C_2 **homologous**.

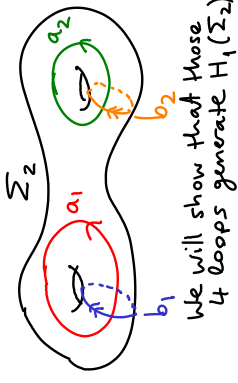


- FACTS
- $H_0(X) \cong \bigoplus_{\text{Top } X} \mathbb{Z} \leftarrow \pi_0 X = \{\text{path-connected components}\}$ ← generated by a point in each path-comp. (two points connected by a path are homologous)
 - $X = \sqcup X_i$; path-components $\Rightarrow H_*(X) \cong \bigoplus H_*(X_i)$
 - $\chi(X) = \sum_{d \geq 0} (-1)^d \text{rank } H_d(X)$ max # \mathbb{Z} -linearly independent elements

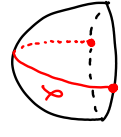
Euler characteristic

Example: compact surfaces

$H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$
 orientable surface genus g
 $\chi = 2 - 2g$



We will show that those 4 loops generate $H_1(\Sigma_2)$



$N_1 = \mathbb{R}P^2 = S^2 / \pm \text{Id}$

Notice γ is a loop. It generates $H_1(N_1) = \mathbb{Z}_2$ (notice " 2γ " is homologous to 0:)

Examples of homology calculations

$$H_*(\mathbb{R}^n) \cong H_*(D^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$$

$$H_*(S^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$$

$\{x \in \mathbb{R}^{n+1} : \|x\|=1\}$ n-dim sphere

$$H_*(X) = \begin{cases} 0 & \text{for } * < 0 \\ 0 & \text{for } * > n \\ \mathbb{Z} & \text{for } * = n \text{ for connected orientable compact manifold} \\ 0 & \text{for } * = n \text{ for non-orientable} \\ 0 & \text{for } * = n \text{ for non-compact} \end{cases}$$

boundary point has an open nbhd homeo to open nbhd of $0 \in \mathbb{R}^n$: $\{x \in \mathbb{R}^n : x_n \geq 0\}$

Rmk M compact connected n-mfd

$$\Rightarrow H_{n-1}(M) \cong \begin{cases} \mathbb{Z}^k & \text{some } k \geq 0 \text{ if orientable} \\ \mathbb{Z}^k \oplus \mathbb{Z}/2 & \text{non-orientable} \end{cases}$$

$$H_*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & \text{odd } * = 1, 3, 5, \dots < n \\ \mathbb{Z} & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

$\mathbb{R}P^n$ orientable $\Leftrightarrow n$ odd (e.g. $\mathbb{R}P^1 \cong S^1$)

(equivalently: spaced real lines through 0 in \mathbb{R}^{n+1})

$$H_*(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{even } * = 0, 2, 4, \dots, 2n \\ 0 & \text{else} \end{cases}$$

e.g. $\mathbb{C}P^1 \cong S^2$

space of complex lines through $0 \in \mathbb{C}^{n+1}$

$$\cong (\mathbb{C}^{n+1} \setminus 0) / \mathbb{C}^* \text{-rescaling} \\ = \{ [z_0, z_1, \dots, z_n] : z_j \in \mathbb{C} \text{ not all } 0 \} / \text{for } \lambda \in \mathbb{C}^*$$



stereographic projection

Examples of cohomology calculations

$$H^0(X) = \prod_{\pi_0 X} \mathbb{Z} \leftarrow \text{if } \pi_0 X \text{ finite, then } \cong \bigoplus_{\pi_0 X} \mathbb{Z} \cong H_0 X$$

but if infinite then not: here allow only finite sums

$$H^*(X) \cong \prod H^*(X_i) \leftarrow X_i \text{ path-components of } X$$

FACT If $H_n(X)$ finitely generated abelian gp, so

$$H_n(X) \cong \mathbb{Z}^r \oplus T_n \leftarrow T_n = \text{torsion elements} \\ \text{elements of finite order}$$

$$\text{Then } H^n(X) \cong \mathbb{Z}^r \oplus T_{n-1}$$

$\Rightarrow H^*(\Sigma_g), H^*(\mathbb{R}^n), H^*(D^n), H^*(S^n), H^*(\mathbb{C}P^n)$ same as for H_* , but:

$$H^*(N_k) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^{h-1} & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \quad H^*(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \quad H^*(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * \text{ even} = 2, 4, \dots \leq n \\ 0 & * = n \text{ if } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

and H^n (non-orientable compact n-mfd) $\cong \mathbb{Z}/2$.

\Rightarrow The interesting feature is the ring structure:

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x] / x^{n+1}$$

$\mathbb{Z}[x]$ = polynomials in x with \mathbb{Z} -coefficients

$$H^*(S^n) \cong \mathbb{Z}[x] / x^2$$

grading: $|x|=2$

$$H^*(T^n) \cong \wedge[x_1, \dots, x_n]$$

$|x_i|=1$

$S^1 \times \dots \times S^1$ n-torus

exterior algebra generated by symbols $x_i \wedge \dots \wedge x_k$ with $i_1 < \dots < i_k$

product given by \wedge using relations $x_i \wedge x_j = -x_j \wedge x_i$ and $x_i \wedge x_i = 0$.

$$H^*(\mathbb{R}P^{2n}) \cong \mathbb{Z}[x] / (2x, x^{n+1})$$

$$H^*(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}[x, y] / (2x, x^{n+1}, y^2, xy)$$

where $|x|=2$

$|y|=2n+1$

$$H^*(\Sigma_g) \cong \wedge[a_1, \dots, a_g, b_1, \dots, b_g]$$

$\langle a_i, b_i \rangle$ for $i \neq j$; $a_i, b_i = -a_j, b_j$

$|a_i|=|b_i|=1$ exterior alg. instead of poly. alg. since $a_i, b_i = -b_i, a_i$

Why more information? connected sum: remove a ball in each, glue along ∂ ball

$S^2 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ have same $H_* = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & * = 2 \\ 0 & * = 4 \end{cases}$

but the rings H^* are not iso, hence $S^2 \times S^2 \not\cong \mathbb{C}P^2 \# \mathbb{C}P^2$.

Example of why such functors are useful

Suppose $\exists F_*: \text{Top} \rightarrow \text{Gps}$ functors s.t.

① $F_*(S^n) \neq 0 \iff * = n$ and ② $F_*(D^n) = 0$ all $*$

Rmk we'll build such an F_* : $\text{reduced homology } \tilde{H}_*$
 s.t. $\tilde{H}_* = H_*$ for $* \neq 0$, and $\tilde{H}_0 \cong \mathbb{Z}^{\#(\text{path-components})-1}$

Theorem Invariance of dimension

(Brower 1910)
$$\begin{matrix} S^n \cong S^m & \iff & n=m \\ \mathbb{R}^n \cong \mathbb{R}^m & \iff & n=m \end{matrix}$$

by ①

Pf Lemma $\Rightarrow F_n(S^n \cong S^m)$ is iso $F_n(S^n) \cong F_n(S^m) \cong F_n(S^m)$ of gps.

If $\mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m$, then can extend $\times 0$ if $n \neq m$ ✓

φ to the one-point compactifications: $S^n \cong \mathbb{R}^n \cup \{\infty\} \xrightarrow{\cong} \mathbb{R}^m \cup \{\infty\} \cong S^m, \infty \mapsto \infty. \square$
 ("Alexandroff extension") $\xrightarrow{\text{stereographic projection } (x_0, \dots, x_n) \mapsto \frac{(x_1, \dots, x_n)}{1-x_0}}$

Rmk new open neighbourhoods at ∞ are $\{\infty\} \cup (\mathbb{R}^n \setminus C)$ where C is (closed B) compact.
 The extended map is cts since $\varphi^{-1}(C)$ is (closed B) compact since φ^{-1} is homeo.

Theorem Brouwer fixed point thm by ① & ②

$f: D^n \rightarrow D^n$ continuous $\Rightarrow f$ has a fixed point ($f(p) = p$ some p)

Proof Suppose not. Let $r(x) = (\text{ray from } f(x) \text{ to } x) \cap \partial D^n$

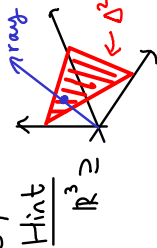
notice: $\cdot r: D^n \rightarrow \partial D^n = S^{n-1}$ continuous

$\cdot r|_{\partial D^n} = \text{id}_{S^{n-1}}$

$S^{n-1} = \partial D^n \xrightarrow{\text{inclusion } i} D^n \xrightarrow{r} S^{n-1}$
 $\cdot r \circ i = \text{id}$

apply $F_{n-1} \Rightarrow F_{n-1}(r) \circ F_{n-1}(i) = \text{id} \Rightarrow F_{n-1}(i)$ injective $F_{n-1}(S^{n-1}) \rightarrow F_{n-1}(D^n) \xrightarrow{\cong} 0$

Example $A = n \times n$ matrix, $A_{ij} > 0$ real $\Rightarrow \exists$ real value $\lambda > 0$ with real vector (v_1, \dots, v_n) with $v_i > 0$



Hint $\mathbb{R}^3 \cong X = \{\text{rays in positive octant}\} \xrightarrow{\text{notice } AX \subseteq X}$

notice $AX \subseteq X$
 notice $X \cong \Delta^n = \{x \in \text{octant} : \sum x_i = 1\} \cong D^n$
 $\text{ray} \mapsto \text{ray} \cap \Delta^n$

1. ALGEBRAIC PRELIMINARIES: CHAIN COMPLEXES

Graded abelian groups

Def A \mathbb{Z} -graded abelian group C is an abelian group together

with a direct sum decomposition

$$C = \bigoplus_{n \in \mathbb{Z}} C_n$$

abelian group

Convention: always grade by \mathbb{Z} unless say otherwise. $\leftarrow \binom{n}{n} = 0$ (for $n < 0$)

Example $C = \mathbb{Z}[x] =$ integer polynomials in x , $C_n = \mathbb{Z} \cdot x^n \leftarrow$ so grading by degree

A graded ab. gp. A is a graded subgp of C if \cdot subgp
 $\cdot A_n \subseteq C_n$.

A homomorphism $h: C \rightarrow D$ of gr.ab.gps is hom of gps s.t.

$$h(C_n) \subseteq D_n$$

A hom of degree k is hom with

$$h(C_n) \subseteq D_{n+k}$$

Shift by k: \mathbb{Z} -gr.ab.gp. $C[k]$ with

$$C[k]_n = C_{k+n}$$

Notice:

$$C[k]_0 = C_k$$

is now in degree zero, so shifted down by k

\Rightarrow Can view gr. hom of deg k as a gr. hom $h: C \rightarrow D[k]$

recall f.g. means \exists surjection $\mathbb{Z}^m \rightarrow G$ for some m

Abelian groups which are finitely generated

FACT Finitely generated abelian groups are classified:

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1^{n_1} \oplus \dots \oplus \mathbb{Z}/p_k^{n_k}$$

free part \rightarrow called rank G \leftarrow torsion part

$n_i \neq 0 \in \mathbb{N}$
 p_i primes (possibly not distinct)

Compare finite dimensional vector spaces/field \mathbb{F} : $V \cong \mathbb{F}^r$ $r = \dim V$

"homeomorphisms preserve dimension"

Non-trivial result because there are space-filling curves.

e.g. Peano (1890)

\exists cts surjection $[0,1] \rightarrow [0,1]^2$

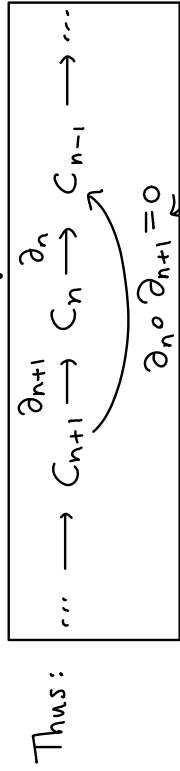
interval square

The theorem implies this is not injective.

(cts, bij, compact \rightarrow Hausdorff) \Rightarrow homeo

Chain complexes

Def A chain complex (C_*, ∂_*) is a gr. ab. gp. C together with a hom ∂ of degree -1 such that $\partial \circ \partial = 0$.



n-chains = elements of C_n

hence $\text{Im } \partial_{n+1} \subseteq \text{Ker } \partial_n$

$B_n \subseteq Z_n$ n-boundaries n-cycles

Now consider "cycles modulo boundaries":

Def The homology of (C_*, ∂_*) is the gr. ab. gp.

$$H_n(C_*, \partial_*) = \frac{\text{Ker } \partial_n}{\text{Im } \partial_{n+1}}$$

often abbreviate by $H_*(C)$

Two cycles are called homologous if they differ by a boundary

Def A chain map $h : (C_*, \partial_*) \rightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a graded hom such that $h \circ \partial_* = \tilde{\partial}_* \circ h$

Example A chain subcomplex $C_* \subseteq \tilde{C}_*$ is a graded subgp with $\partial_* = \text{restriction of } \tilde{\partial}_* \text{ to } C_*$ (so require $\tilde{\partial}_*(C_*) \subseteq C_*$)
 So the inclusion $\text{incl} : (C_*, \partial_*) \rightarrow (\tilde{C}_*, \tilde{\partial}_*)$ is a chain map.
 Also get quotient complex $\tilde{C}_*/C_* \leftarrow (\text{so cosets } [\tilde{c}] = \{\tilde{c} + c : c \in C_*\})$
 with $\tilde{\partial}_*[\tilde{c}] = [\tilde{\partial}_*\tilde{c}]$ (well-defined: $\tilde{\partial}_*(C_*) = \partial_*(C_*) \subseteq C_*$)

Lemma

A chain map induces a hom on homology

$$H_*(h) = h_* : H_*(C_*, \partial_*) \longrightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$$

$$[x] \longmapsto [h(x)]$$

Proof

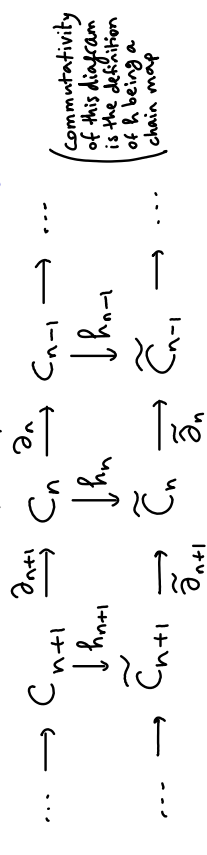
$\text{Ker } \partial_n \xrightarrow{\text{hom}} \text{Ker } \tilde{\partial}_n$ since $\tilde{\partial}_n(h(x)) = h(\partial_n x) = 0$

Need $\text{Im } \partial_n \longrightarrow \text{Im } \tilde{\partial}_n$ to get well-defined hom

$$H_n(C) = \text{Ker } \partial_n / \text{Im } \partial_{n+1} \longrightarrow \text{Ker } \tilde{\partial}_n / \text{Im } \tilde{\partial}_{n+1} = H_n(\tilde{C})$$

Proof: $h(b) = h(\partial c) = \tilde{\partial} h(c) = \tilde{\partial} h(c) \in \text{Im } (\tilde{\partial})$. \square

The last step was a very simple example of a proof by "diagram chasing"



(Commutativity of this diagram is the definition of h being a chain map)

$$\begin{array}{ccc} c & \xrightarrow{\partial} & \partial c = b \\ \downarrow h & & \downarrow h \\ hc & \xrightarrow{\tilde{\partial}} & \tilde{\partial}(hc) = h\partial c = h(b) \end{array} \quad \square$$

Curiosity (Non-examinable) If C_n free abelian gp $\forall n$, then every graded hom $H_*(C) \rightarrow H_*(\tilde{C})$ arises from a chain map. [see Dold, Proposition II.4.6]

Def (C_*, ∂_*) is exact (or acyclic) if $H_*(C) = 0$

so $\text{Im } \partial_{n+1} = \text{Ker } \partial_n$

In general exact sequence means "Im(previous map) = Ker(next map)"

Example If $0 \rightarrow A \xrightarrow{i} B \rightarrow 0$ is exact then $A \cong B$ via i .

A short exact sequence (SES) is an exact sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0$

5-Lemma

$$\begin{array}{c}
 A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \\
 \cong \downarrow \alpha \cong \downarrow \beta \quad \downarrow \gamma \quad \cong \downarrow \delta \cong \downarrow \epsilon \\
 A' \rightarrow B' \rightarrow C' \rightarrow D' \rightarrow E'
 \end{array}$$

exact rows $\implies \gamma$ also iso.

Pf exercise (diagram chase) \square

Splitting Lemma

Cor $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ SES of abelian gps

If $B \xrightarrow{\beta} C \xrightarrow{\gamma} 0$ s.t. $\beta \circ \gamma = \text{id}_C$ then the SES splits: $B \cong A \oplus C$

Pf $0 \rightarrow A \rightarrow A \oplus C \rightarrow C \rightarrow 0$
 $\parallel \downarrow \alpha + \gamma \parallel \parallel$
 $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$

converse: if $B \cong A \oplus C$
 then define $\gamma: C \rightarrow B \xrightarrow{\beta} C$

Exercise If $A \xrightarrow{\alpha} B \xrightarrow{\mu \circ \beta} A \oplus C$ then it splits: $B \cong A \oplus C$
 (and equivalent to existence of γ above)

Exercise If C is a free abelian group ($C \cong \bigoplus_{i \in I} \mathbb{Z}$) then the SES splits.

Rank A free \nRightarrow splits, e.g. $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$

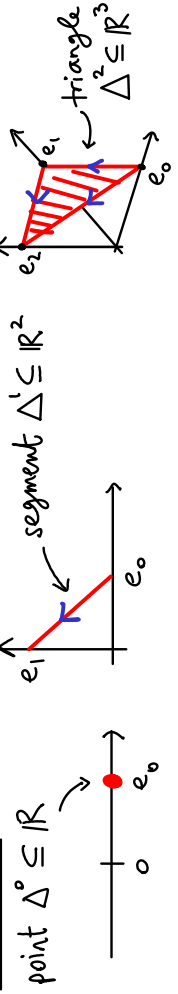
Cultural Rank Splitting Lemma generalises the rank-nullity theorem from linear algebra: $V \xrightarrow{\beta} W$ linear map of vector spaces $\implies \text{Im} \beta \oplus \text{Ker} \beta \cong V$
 Pf $0 \rightarrow \text{Ker} \beta \xrightarrow{\text{incl}} V \xrightarrow{\beta} \text{Im} \beta \rightarrow 0$ is SES, and splits since $\text{Im} \beta$ free.

2. Δ -COMPLEXES AND SIMPLICIAL HOMOLOGY

standard n-simplex $\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum t_i = 1 \right\}$

standard basis of \mathbb{R}^{n+1}
 e_0, \dots, e_n ($e_0 = (1, 0, \dots, 0)$, ...)

Examples



Def For $\{v_0, \dots, v_n\} \subseteq \mathbb{R}^{n+k}$ s.t. any $k \geq 0$

v_1, \dots, v_n \mathbb{R} -linearly independent

$[v_0, \dots, v_n] = n$ -Simplex spanned by v_0, \dots, v_n

= convex hull of v_0, \dots, v_n

= $\{ \sum t_i v_i : t_i \geq 0 \text{ and } \sum t_i = 1 \}$

= Image of \mathbb{R} -linear homeo $\sigma: \Delta^n \rightarrow \mathbb{R}^{n+k}$
 $\sigma(e_i) = v_i$

canonical homeomorphism (homeo onto the image, not onto \mathbb{R}^{n+k})

Will often blur the distinction between map σ and its image,

$$\sigma \equiv [\sigma e_0, \dots, \sigma e_n]$$

but the ordering of the v_i will be important (so the map σ is more precise)

We encode this extra data by orienting the edges $v_i \rightarrow v_j$ if $i < j$

Def d-dimensional faces $[v_{i_0}, \dots, v_{i_d}]$ for $i_0 < \dots < i_d$

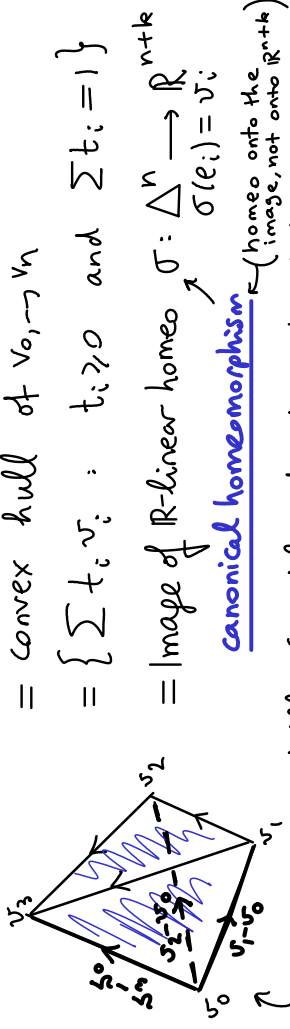
Example 0-dim faces are the vertices v_0, \dots, v_n

facets = $(n-1)$ -dimensional faces

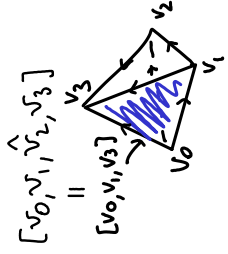
= $[v_0, \dots, \hat{v}_i, \dots, v_n]$ where we omit v_i

= $\{ \sum t_i v_i \in [v_0, \dots, v_n] : t_i = 0 \}$

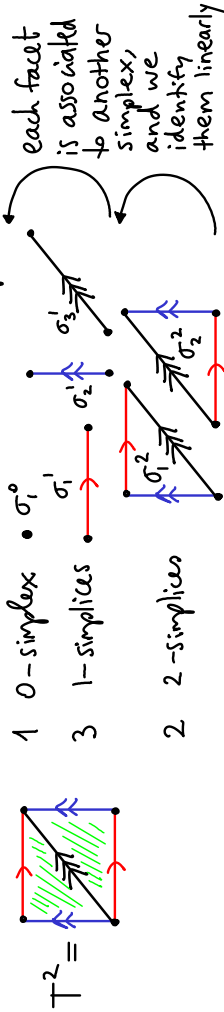
= Image $\sigma|_{\Delta_i^{n-1}} : \Delta_i^{n-1} \rightarrow \mathbb{R}^{n+k}$
 $\{ t \in \Delta^n : t_i = 0 \}$



(Solid prism: includes inside)



Example Can build a torus out of simplices:



$T^2 =$ quotient space $\sqcup \sigma_i^n$ / canonical homeos associated to the facets

for example identify facet σ_1^2 with σ_2^2 via linear homeo (orientation-preserving)

- indexing set I_n , for each $n \in \mathbb{N}$
- choice of n -simplex σ_α^n (not necessarily standard) for each $\alpha \in I_n$
- gluing data: for each $\alpha \in I_n$, $0 \leq i \leq n$, associate some $(\beta, i) \in I_{n-1}$
- consistency condition (see later)

The Δ -complex is the quotient space

$$X = \bigsqcup_{\substack{\alpha \in I_n \\ n \in \mathbb{N}}} \sigma_\alpha^n / \sim$$

i -th facet of σ_α^n is identified with $\sigma_{\beta(\alpha, i)}^{n-1}$ via the order-preserving canonical linear homeo

(quotient topology: $U \subseteq X$ is open $\Leftrightarrow U$ intersects σ_α^n in an open set, $\forall \alpha, n$)

A Δ -complex structure on a top. space Y is a homeo from a Δ -cx $X \cong Y$.

Explicit description of the facet identification

$$\left\{ \sum s_j v_j \right\} = [w_0, \dots, w_{n-1}] \longrightarrow [\sigma_0, \dots, \sigma_n] = \left\{ \sum t_j v_j \right\}$$

$$\uparrow \sigma_{\beta(\alpha, i)}^{n-1} \quad \uparrow \sigma_\alpha^n \quad \uparrow \sigma_\alpha^n \Delta_i^{n-1} = [\sigma_0, \dots, \sigma_{i-1}, \sigma_i, \dots, \sigma_n]$$

$$\Delta^{n-1} \longrightarrow \Delta_i^{n-1} \subseteq \Delta^n$$

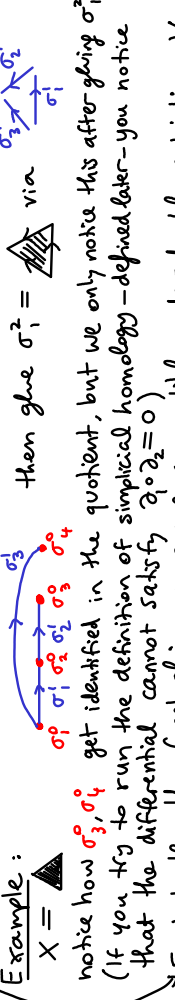
$$(\sigma_0, \dots, \sigma_{n-1}) \mapsto (\sigma_0, \dots, \sigma_{i-1}, \sigma_i, \dots, \sigma_{n-1})$$

Non-example

This decomposition for T^2 is not a Δ -complex.

Consistency condition

We want to additionally ensure that each point of X lies in the interior of exactly one σ_α^n , because we want to avoid unexpected identifications.



Example: $X = \Delta$ then glue $\sigma_1^2 = \Delta$ via $\sigma_1^1 \times \sigma_2^1$

notice how σ_3^1, σ_4^1 get identified in the quotient, but we only notice this after gluing σ_1^2 (if you try to run the definition of simplicial homology-defined later - you notice that the differential cannot satisfy $\partial^2 \circ \partial = 0$)

Equivalently: the facet gluing maps are compatible under double restriction: $\forall i, j$

$$[\sigma_0, \dots, \sigma_n] \xrightarrow{\text{facet}} [\sigma_0, \dots, \sigma_{n-1}] \xrightarrow{\text{identity}} [\sigma_0, \dots, \sigma_{n-1}] \cong [\sigma_0, \dots, \sigma_{n-1}] \xrightarrow{\text{facet}} [\sigma_0, \dots, \sigma_{n-2}]$$

$$[\sigma_0, \dots, \sigma_n] \xrightarrow{\text{identity}} [\sigma_0, \dots, \sigma_{n-1}] \xrightarrow{\text{identity}} [\sigma_0, \dots, \sigma_{n-1}] \cong [\sigma_0, \dots, \sigma_{n-1}] \xrightarrow{\text{identity}} [\sigma_0, \dots, \sigma_{n-2}]$$

this ensures that $[\sigma_0, \dots, \sigma_n]$ is identified with the same $[\sigma_0, \dots, \sigma_{n-2}]$ whether we first restrict to $t_i = 0$ (omit v_i) or first restrict to $t_j = 0$ (omit v_j).

Another equivalent condition: can define the k -th skeleton of Δ -cx X , $X^k =$ quotient space you get by gluing all simplices of dimensions $\leq k$. Consistency is the condition that the boundary of each σ_α^n should map continuously into X^{n-1} (in the above Example consider the vertex $\Delta = \partial \sigma_1^2$) (more precisely, the "topological realization" of a simple complex)

Rmk (see part A) A simplicial complex is a Δ -complex in which

each d -dim face is uniquely determined by d distinct vertices.

A homeo from such a complex to X is a triangulation of X .

Non-example both 2-simplices have vertices v, v, v



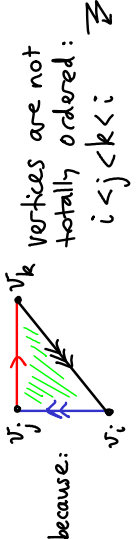
Harder exercise (Non-examinable) If you apply barycentric subdivision twice to a Δ -cx then you get a simplicial complex. (Hint: After the first subdivision, any given simplex has distinct vertices)

Simplicial chain complex

Def For a Δ -complex X , let $X_n =$ set of n -simplices of X

$$C_n(X) = \text{free abelian group generated by the set } X_n$$

$$= \left\{ \sum_{\alpha \in I_n} c_\alpha \cdot \sigma_\alpha^n : c_\alpha \in \mathbb{Z} \text{ and only finitely many } c_\alpha \neq 0 \right\}$$



differential: $\partial_n \sigma_\alpha = \sum (-1)^i \cdot \sigma_{\beta(\alpha, i)}$

so: $\partial_n [v_0, \dots, v_n] = \sum (-1)^i \cdot [v_0, \dots, \hat{v}_i, \dots, v_n]$

and extend linearly

will show $\partial \circ \partial = 0$, so get simplicial homology: $H_*^\Delta(X) = H_*(C_*^\Delta, \partial_*)$

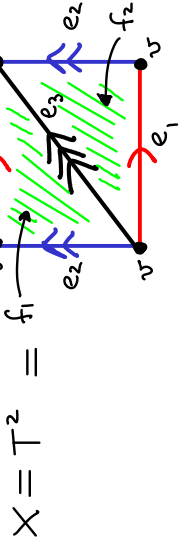
Examples
 $\partial_1 \begin{pmatrix} v_0 \\ \rightarrow \\ v_1 \end{pmatrix} = \begin{pmatrix} \bullet \\ -v_0 \\ +v_1 \end{pmatrix}$
 $\partial_2 \begin{pmatrix} v_0 \\ \nearrow \\ v_1 \\ \searrow \\ v_2 \end{pmatrix} = \begin{pmatrix} \bullet \\ +v_1 \\ -v_2 \end{pmatrix}$
 $\partial_1 \circ \partial_2$ (this) $= + (v_2 - v_1) - (v_2 - v_0) + (v_1 - v_0) = 0$
 $\partial \circ \partial = 0$ fails for Δ (not Δ -complex), try!

Lemma $\partial \circ \partial = 0$
 Pf $\partial_{n-1} (\partial_n [v_0, \dots, v_n]) = \sum (-1)^i \partial_{n-1} [v_0, \dots, \hat{v}_i, \dots, v_n]$
 $= \sum (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$
 $+ \sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$
 $= 0 \square$
 antisymmetric if swap v_i, v_j

Example $S^1 = \text{circle}$
 $\Delta\text{-cx}: X_0 = 1 \text{ 0-simplex} \cdot e_0^i = e_{\beta(1,0)} = e_{\beta(1,1)}$
 $X_1 = 1 \text{ 1-simplex} \rightarrow e_1^i$
 $0 \rightarrow C_1^\Delta \rightarrow C_0^\Delta \rightarrow 0$
 $\parallel \quad \parallel$
 $\mathbb{Z}e \quad \mathbb{Z}v$
 $e \mapsto v - v = 0$
 $\Rightarrow H_*^\Delta(S^1) = \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{else} \end{cases}$

Example $\Delta\text{-cx}$ structure on S^n :
 $S^n = \Delta^n \cup \Delta^n / \text{glue along } \partial \Delta^n$
 call this Δ_1 this Δ_0
 One can deduce: $H_* S^n = \begin{cases} \mathbb{Z} \langle \text{point} \rangle \cong \mathbb{Z} & * = 0 \\ \mathbb{Z} \langle \Delta_1 - \Delta_0 \rangle \cong \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$
 but messy! pick any vertex

Example



$$0 \rightarrow C_2^\Delta \xrightarrow{\parallel} C_1^\Delta \xrightarrow{\parallel} C_0^\Delta \rightarrow 0$$

$$\mathbb{Z}f_1 + \mathbb{Z}f_2 \xrightarrow{\parallel} \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 \xrightarrow{\parallel} \mathbb{Z}v$$

$$f_1 \mapsto e_1 - e_3 + e_2$$

$$f_2 \mapsto e_2 - e_3 + e_1$$

$$e_1, e_2, e_3 \mapsto v - v = 0$$

$$H_*^\Delta(T^2) = \begin{cases} \mathbb{Z}v & * = 0 \\ (\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3) / \mathbb{Z}(e_1 - e_3 + e_2) & * = 1 \\ \mathbb{Z}(f_1, -f_2) & * = 2 \\ 0 & \text{else} \end{cases}$$

freely generated by e_1, e_2

Alternative useful method using a trick from algebra:
 Smith normal form of ∂_2 :
 $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \xrightarrow{\text{row}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{col}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
 hence: $H_2 = \ker \partial_2 \cong \mathbb{Z}$
 $H_1 = \text{coker } \partial_2 \cong \mathbb{Z}^2$
 so after \mathbb{Z} -isos of C_2, C_1 , we get $\mathbb{Z}^2 \rightarrow \mathbb{Z}^3 \xrightarrow{(a, b)} (a, 0, 0)$

Remark about orientations (see also my B3.2 Geometry of Surfaces notes)
 For a vector space an orientation is a choice of basis modulo linear endomorphisms of $\det > 0$

Example \mathbb{R}^2

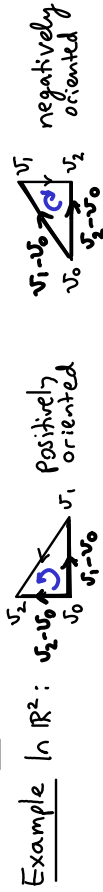
$$\begin{matrix} e_2 & \uparrow & \text{right-hand} & \text{orientation} & \text{left-hand} & \text{orientation} \\ \uparrow & \text{positive} & \text{orientation} & \text{(negative)} & \text{orientation} & \text{(negative)} \\ \mathbb{R}^2 & \xrightarrow{\det < 0} & \mathbb{R}^2 & \xrightarrow{\det < 0} & \mathbb{R}^2 & \xrightarrow{\det < 0} \end{matrix}$$

Fact $GL(n, \mathbb{R})$ has 2 path-components $\langle A : \det A > 0 \rangle$ so can always continuously deform a basis to another within same orientation

Canonical orientation on \mathbb{R}^n : e_1, \dots, e_n standard basis \leftarrow "positive orientation"

Example $[v_0, \dots, v_n]$ simplex $\Rightarrow v_1 - v_0, \dots, v_n - v_0$ is a basis of vector subspace $V = \{ \sum a_i v_i : \sum a_i = 0 \} \subseteq \mathbb{R}^{n+1}$
 hence a choice of orientation of V , and each transposition of vertices v_0, \dots, v_n switches the orientation class.
 (if swap v_i, v_j consider the reflection in the hyperplane $(e_i = e_j)$ in V)

If $v_0, v_1, v_2 \in \mathbb{R}^n$ then $V = \mathbb{R}^n$ so simplex's orientation can be compared with \mathbb{R}^n -orient.



• No canonical choice of orientation for an abstract vector space.
Need choose basis v_1, \dots, v_n then declare another basis positively oriented if the change of basis matrix has $\det > 0$.

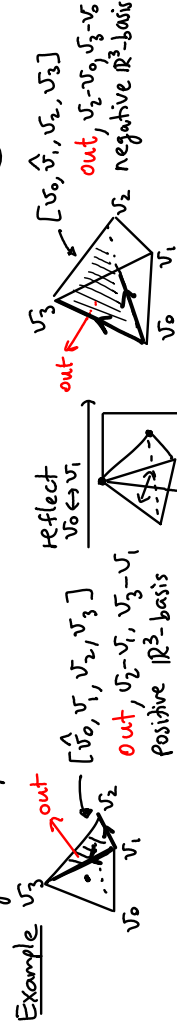
• For hyperplane $H \subseteq \mathbb{R}^n$ with choice of normal can declare orientation of basis w_1, \dots, w_{n-1} of H positive if $\text{normal}, w_1, \dots, w_{n-1}$ is positive \mathbb{R}^n -basis
convention "outward normal first"



UPSHOT For an n -simplex $[v_0, \dots, v_n]$ in \mathbb{R}^n , each facet lies in a hyperplane and have canonical choice of normal: outward normal. Hence facets are canonically oriented.



Any reflection of \mathbb{R}^n will swap orientation: after $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ get clockwise



UPSHOT $(-1)^i$ in $(-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$ in definition of simplicial ∂ secretly keeps track of whether the orientation of the simplex agrees or not with the orientation induced geometrically by the above conventions. From this point of view, the equation $\partial \circ \partial = 0$ holds because a codimension 2 face γ of a simplex σ arises as the facet of exactly two facets f_1, f_2 of σ , and the geometric orientations of f_1, f_2 induce opposite geometric orientations on γ (therefore if we keep track of orientation signs we count $+\gamma - \gamma = 0$).
(checking that they are opposite requires some thought, one approach is to say that we can deform f_1, f_2 until they make a flat angle, and) Picture: get $+\gamma$ from df_1 , then their outward normals will be opposite.

Lemma $H_*^\Delta(X) \cong \bigoplus H_*^\Delta(X_i)$ where X_i are the path-components of X .
Pf $\bigoplus C_*^\Delta(X_i) \rightarrow C_*^\Delta(X)$, $\bigoplus c_i \mapsto \sum c_i$ since Δ^k path-com.
is chain isomorphism since any simplex $\sigma: \Delta^k \rightarrow X$ has path-connected image, so $\subseteq X_i$ some i . \square

Rmk X top space \Rightarrow path com. component \subseteq connected component since path-com. \Rightarrow connected. For Δ -cx, these are same (since connected + locally path-com. \Rightarrow path-connected).

However for top space they need not be the same, e.g. Topologist's sine curve
 $\{(x, \sin \frac{1}{x}) : x \in (0, 1]\} \cup \{0\} \times [0, 1] \subseteq \mathbb{R}^2$
2 path-com. components \xrightarrow{A} connected, not path-connected, not locally path-connected

Lemma $X \Delta$ -cx, X connected $\Rightarrow \forall$ vertices v_1, v_2, \exists path consisting of edges from v_1 to v_2
Proof Pick a vertex v_0 . Let $A = \{\text{simplices } \sigma \text{ in } X \text{ such that one (hence all) vertices of } \sigma \text{ can be connected to } v_0 \text{ by a path in } X \text{ consisting of edges}\}$.
Let $B = \{\text{simplices } \sigma \text{ of } X \text{ not in } A\}$. Observe that if $\sigma \in A$ then the facets of σ are in A , similarly for B . After giving simplices as prescribed in X , the set A and B define subspaces of X (using the quotient topology). Exercise: A, B are closed subspaces. Note $A \cap B = \emptyset$ (otherwise they have a vertex in common and we contradict the definition of A) $\Rightarrow X = A \sqcup B$ is a disjoint union of closed sets, so X connected forces $B = \emptyset$. \square


Theorem X has Δ -cx structure $\Rightarrow H_0^\Delta(X) \cong \bigoplus_{\text{path-com. components}} \mathbb{Z}$
Pf By lemma, $\text{wlog } X$ path-connected

- vertex $v \Rightarrow v \in C_0(X) \xrightarrow{\partial} C_{-1}(X) = 0 \Rightarrow [v] \in H_0(X)$
- Vertices $v_0, v_1 \in X \xrightarrow{\text{Lemma}} \exists$ path γ from v_0 to v_1 , consisting of edges $\Rightarrow \gamma$ is sum of 1-chains s.t. $\partial \gamma = v_1 - v_0$, so $[v_0] = [v_1] \in H_0(X)$.
- $H_0(X) = \langle [v] \rangle \subseteq \mathbb{Z}$ is injective?

$nV \leftarrow n$ Suppose $nV = \partial c$ some $c \in C_1(X)$
consider augmentation hom $\epsilon: C_0(X) \rightarrow \mathbb{Z}$, $\sum n_i \sigma_i \mapsto \sum n_i$
 \Rightarrow composition $C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z}$ is 0 as $\partial(\text{1-simplex}) = \sigma_1 - \sigma_0 \mapsto 1 - 1 = 0$. \leftarrow 0-simplices
 $\Rightarrow n = \epsilon(nV) = \epsilon \partial c = 0$. \square

Harder exercise (non-examinable) If $\delta: [0, 1] \rightarrow X$ path from vertex v_0 to v_1 , then we can homotope (=continuously deform) it into a path consisting of edges. (Hints: first show that a Δ -cx is locally path-connected. Then use the fact that the domain $[0, 1]$ of δ is compact in order to approximate δ arbitrarily well by a piecewise linear path. Finally prove the result for piecewise linear paths.)

3. SINGULAR HOMOLOGY

Motivation Not obvious that H_*^Δ is functorial: $\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y$
 then $f \circ \sigma$ typically not a simplex: $\Delta \xrightarrow{\sigma} \Delta \xrightarrow{f} X$  $\xrightarrow{\text{continuous map}}$ Y

Solution 1: only allow simplifical maps $f: X \rightarrow Y$ (so $f \circ \sigma$ simplex $\forall \sigma$)

Solution 2: show that any cts map $f: X \rightarrow Y$ can be approximated arbitrarily well by a simplicial map after having performed barycentric subdivision on X, Y enough times. Also any two such approximations induce the same map $H_*^\Delta(X) \rightarrow H_*^\Delta(Y)$

Solution 3: develop new homology $H_*(X)$ which allows any cts map $\Delta^n \rightarrow X$ will do THIS. and prove $H_*^\Delta(X) \cong H_*(X)$ for Δ -complexes X .

Def Singular n-simplex is any continuous map $\sigma: \Delta^n \rightarrow X$
 X is any top. space

Singular n-chains $C_n(X) =$ free abelian group generated by $\left\{ \sum_{\text{singular } n\text{-simplices } \sigma} c_\sigma \cdot \sigma : c_\sigma \in \mathbb{Z} \right\}$
 only finitely many $c_\sigma \neq 0$

Lemma Here $\Delta_i^{n-1} = [e_0, \dots, \hat{e}_i, \dots, e_n]$ is identified canonically with Δ^{n-1} (send $e_k \rightarrow e_{k-1}$ for $k > i$)
 (and extend linearly)

Proof $\partial_{n+1}(\partial_n \sigma) = \partial_{n+1} \left(\sum_{j < i} (-1)^j \sigma|_{\Delta_j^{n-1}} \right) = \sum_{j < i} (-1)^j \sigma|_{[e_0, \dots, \hat{e}_j, \dots, \hat{e}_i, \dots, e_n]} + \sum_{j > i} (-1)^j \sigma|_{[e_0, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n]}$
 antisymmetric if swap i, j

$= 0$. \square
 \Rightarrow singular homology: $H_*(X) = H_*(C_*, \partial_*)$

For Δ -complex X have inclusion of subcomplex $C_*^\Delta \hookrightarrow C_*$
 \Rightarrow induces $H_*^\Delta(X) \rightarrow H_*(X)$ Fact: isomorphism (proof later, see cellular $H_*^{CW} \cong H_*$)

Corollary $H_*^\Delta(X)$ is independent of choice of Δ -cx structure on X

Example $X = \text{point} \Rightarrow C_n(X) = \mathbb{Z} \cdot (\sigma_n: \Delta^n \rightarrow X \text{ the constant map})$
 $\partial \sigma_n = \sum (-1)^i \sigma_n|_{\Delta_i^{n-1}} = \sum (-1)^i \sigma_{n-1} \Rightarrow \dots \Rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \dots$
 $\begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases} \Rightarrow H_*(\text{pt}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Lemma $H_*(X) \cong \bigoplus H_*(X_i)$ where X_i are path-components of X
Pf Image of cts map $\Delta^n \rightarrow X$ is path conn. so lies in some X_i . \square

Cor $H_0(X) = \bigoplus \mathbb{Z}_{X_i}$ \leftarrow generators of $C_0(X)$

Pf By Lemma, wlog X path-connected. $\Delta^0 = \text{pt} \rightarrow X$ is cycle since $C_{-1}(X) = 0$
 Given 2 points $x, y \in X$, a path $\Delta^1 = [0, 1] \xrightarrow{\gamma} X$, $\gamma(0) = x, \gamma(1) = y$ is also a 1-chain!
 So $y - x = \partial \gamma$, so x, y are homologous. Finally if $n \cdot [x] = 0 \in H_0(X)$ then $n \cdot x = \partial c$ some $c \in C_1(X)$ generated by paths. Now run the augmentation hom-trick like we did for H_0^{Δ} : $n = \varepsilon(n \cdot x) = \varepsilon \partial c = 0$ as $\varepsilon \partial = 0$. \square

Naturality (i.e. functoriality)

Lemma $f: X \rightarrow Y$ continuous
 $\Rightarrow H_*(f) = f_*: H_*(X) \rightarrow H_*(Y)$ induced by chain map

$f_*: C_*(X) \rightarrow C_*(Y)$ and extend linearly
 $f_*(\sigma) = f \circ \sigma$
 $\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$

Pf $\partial_n(f_* \sigma) = \sum (-1)^i f \circ \sigma|_{\Delta_i^{n-1}} = f_* \left(\sum (-1)^i \sigma|_{\Delta_i^{n-1}} \right) = f_* (\partial_n \sigma) \square$

Properties 1) $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow (g \circ f)_* = g_* \circ f_*$
 2) $\text{id}_X = \text{id} \Rightarrow \text{id}_* = \text{id}$

Pf 1) $(g \circ f)_* \sigma = g \circ f \circ \sigma = g_* (f \circ \sigma) = g_* (f_* \sigma) \checkmark$
 2) $\text{id}_* \sigma = \text{id} \circ \sigma = \sigma \checkmark$

Cor $H_*: \left\{ \begin{array}{l} \text{topological spaces} \\ \& \text{cts maps} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{graded abelian groups} \\ \& \text{graded homs} \end{array} \right\}$ is a functor

Cor $X \cong Y \Rightarrow H_*(X) \cong H_*(Y)$

4. CHAIN HOMOPIES AND HOMOPIY INVARIANCE

Algebra: chain homotopies

$$f_*, g_* : (C_*, \partial_*) \rightarrow (C_*, \tilde{\partial}_*) \text{ chain maps}$$

Def f_*, g_* are chain homotopic if \exists (degree +1)

$$\text{hom } h : C_* \rightarrow \tilde{C}_*[1] \text{ s.t.}$$

$$\tilde{\partial} \circ h + h \circ \partial = f_* - g_*$$

h is called a chain homotopy

Consequence $f_* = g_* : H_*(C_*, \partial_*) \rightarrow H_*(\tilde{C}_*, \tilde{\partial}_*)$ on homology

Pf $C_{n+1} \rightarrow C_n \xrightarrow{\partial_n} C_{n-1}$ For c cycle $\in C_n$:

$$\begin{aligned} \Rightarrow f_n(c) - g_n(c) &= \tilde{\partial}_{n+1} \circ h_n(c) + h_{n-1} \circ \partial_n(c) \\ &= \underbrace{\tilde{\partial}_{n+1} \circ h_n(c)}_{\text{boundary}} + \underbrace{h_{n-1} \circ \partial_n(c)}_{=0} \end{aligned}$$

so $0 \in H_n(C_*) \cdot \square$

Harder exercise (Non-examinable)

If chain maps $f_*, g_* : C_* \rightarrow \tilde{C}_*$ induce the same map on homology and C_n, \tilde{C}_n are free abelian groups $\forall n$, then f_*, g_* are chain homotopic.

Hints Let $B_{n-1} = \text{Im } \partial_n, K_n = \text{Ker } \partial_n$. Use fact that subgroups of free groups are free to show that B_n, K_n are free. Deduce that $C_n \cong B_{n-1} \oplus K_n$ (show $0 \rightarrow K_n \xrightarrow{\partial_n} B_{n-1} \rightarrow 0$ splits) More precisely deduce: the chain complex (C_*, ∂_*) is isomorphic to $D_* = \bigoplus_n (\dots \rightarrow 0 \rightarrow B_n \xrightarrow{\partial_n} K_n \rightarrow 0 \rightarrow \dots)$ Via that iso, $f_* - g_*$ determines a chain map $\alpha : D_* \rightarrow \tilde{D}_*$ such that on homology $\alpha = 0 : H_n(\tilde{D}) \rightarrow H_n(D)$. Consider $\beta : D_n \rightarrow D_{n+1}, \beta : K_n \rightarrow B_n$ is α . (hence $\alpha(K_n) \subseteq B_n \subseteq K_n$)

Theorem

$$\text{Let } i_0 : X \rightarrow X \times I, i_1(x) = (x, 0) \text{ where } I = [0, 1]$$

$$i_1 : X \rightarrow X \times I, i_1(x) = (x, 1)$$

$$\Rightarrow i_0, i_1 : C_*(X) \rightarrow C_*(X \times I) \text{ are chain hpic.}$$

Key idea Need the "prism operator" which cuts $\Delta^n \times I$ into a sum Γ_n of

$(n+1)$ -simplices in $\Delta^n \times I$:

$$\Gamma_n = s_0 - s_1$$

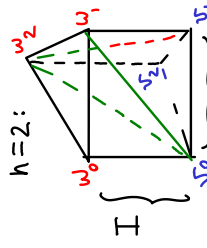
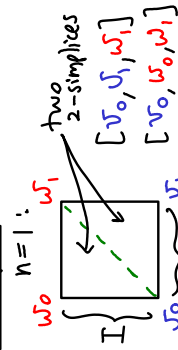


Non-examinable

Pf bottom facet $\Delta^n \times 0 = [v_0, \dots, v_n] \leftarrow v_i = e_i \times 0 \subseteq \Delta^n \times [0, 1] \subseteq \mathbb{R}^{n+1}$

top facet $\Delta^n \times 1 = [w_0, \dots, w_n] \leftarrow w_i = e_i \times 1$

Examples



Let $s_i = [v_0, \dots, v_i, w_i, \dots, w_n]$

Claim The s_i cover $\Delta^n \times [0, 1]$ and give Δ -cx structure on $\Delta^n \times I$

Pf $\sum_{k \leq i} t_k v_k + \sum_{k > i} r_k w_k = (t_0, \dots, t_{i-1}, t_i + r_i, r_{i+1}, \dots, r_n)$ $\in [0, 1]^n = I$

So given $(x_0, \dots, x_n) \in \Delta^n \times I$, equate and solve:

$$t_0 = x_0, \dots, t_{i-1} = x_{i-1}, r_i + t_i = x_i, \dots, r_n = x_n, \text{ and } \sum_{k=0}^n r_k = 1 - x_i$$

Note $x_k \geq 0, \sum x_k = 1, a \in [0, 1]$ hence $\sum_{k \neq i} x_k = 1 - x_i$

but $r_i \geq 0 \Leftrightarrow \{a \geq x_{i+1} + \dots + x_n\}$ Thus a solution exists if we pick $t_i \geq 0$

(There are multiple solutions if $a = x_{i+1} + \dots + x_n$ since then also "i+1" works since $x_{i+1} + x_{i+2} + \dots + x_n \geq a \geq x_{i+2} + \dots + x_n$. This is expected since s_i, s_{i+1} meet along a facet. Example: $n=1$ picture above: s_0, s_1 meet along facet $[v_0, w_1]$ which arises for $i=0, r_0=0$ but also $i=1, t_1=0$.)

Def $\Gamma_n = \sum (-1)^i s_i \in C_{n+1}(\Delta^n \times [0, 1]) \leftarrow$ geometrically this "represents" $\Delta^n \times I$ as a simplicial chain

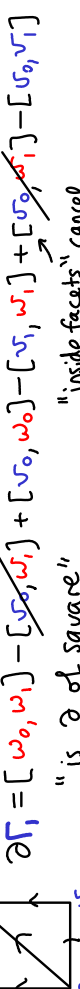
$$\Rightarrow \partial \Gamma_n = \sum_{j \leq i} (-1)^j (-1)^{i-j} [v_0, \dots, v_j, \dots, v_i, w_{i-j}, \dots, w_n] - \sum_{j > i} (-1)^j (-1)^{j-i} [v_0, \dots, v_i, w_i, \dots, w_j, \dots, w_n]$$

geometrically, this "represents" $\partial(\Delta^n \times I) = (\partial \Delta^n \times I) \cup (\Delta^n \times \partial I)$

Example $\Gamma_1 = [v_0, w_0, w_1] - [v_0, v_1, w_1]$ "is the square"

$$\partial \Gamma_1 = [w_0, w_1] - [v_0, w_1] + [v_0, w_0] - [v_1, w_1] + [v_0, v_1] - [v_0, v_1]$$

"is ∂ of square" "inside facets" cancel



Prism operator $P : C_n(X) \rightarrow C_{n+1}(X \times [0,1])$

$$P(\sigma) = (\sigma \times \text{id})_* (\bar{\Gamma}_n)$$

$\sigma : \Delta^n \rightarrow X$
 $\sigma \times \text{id} : \Delta^n \times [0,1] \rightarrow X \times [0,1]$
 $(\sigma \times \text{id})(x,t) = (\sigma(x), t)$

this abbreviated notation means the map
 $(t_0, \dots, t_n) \mapsto (\sigma(t_0 e_0 + \dots + t_j e_j + \dots + t_n e_n), t_0 + \dots + t_n)$
 $\in X \times I$

$$\begin{aligned} \partial P(\sigma) &= \partial(\sigma \times \text{id})_* (\bar{\Gamma}_n) \\ &= (\sigma \times \text{id})_* (\partial \bar{\Gamma}_n) \\ &= \sum_{j=0}^n (-1)^j (\sigma \times \text{id})_* [i_0 \sigma e_0, \dots, i_j \sigma e_j, \dots, i_n \sigma e_n] \\ &\quad + \sum_{j=0}^n (-1)^{j+1} [i_0 \sigma e_0, \dots, i_0 \sigma e_0, \dots, i_j \sigma e_j, \dots, i_n \sigma e_n] \end{aligned}$$

Example to clarify $n=2, k=0, i=1$
 $\Gamma = [v_0, v_1, v_2] = [v_0, v_1, w_1]$
 $(\sigma \times \text{id})_* \Gamma = (\sigma \times \text{id})_* [v_0, v_1, w_1]$
maps $t_0 v_0 + t_1 v_1 + t_2 w_1 = (t_0, t_1, t_2)$
to $(\sigma(t_0, t_1, t_2), t_2) = (\sigma(t_0 e_0 + t_1 e_1 + t_2 e_2), t_2)$

Homotopy invariance

Def $f_0, f_1 : X \rightarrow Y, f_0 \simeq f_1$ homotopic if \exists continuous map

$$F : X \times [0,1] \rightarrow Y \text{ called homotopy s.t. } \begin{cases} f_0 = F \circ i_0 \\ f_1 = F \circ i_1 \end{cases}$$

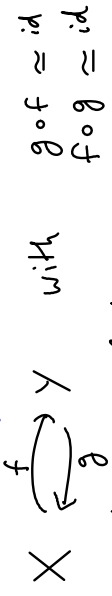
Idea Think of this as a continuous family of maps

$$f_t = F(\cdot, t) : X \rightarrow Y \text{ from } f_0 \text{ to } f_1.$$

Exercise \simeq is an equivalence relation.

Homotopic relative to $A \subseteq X$ if $F(a,t) = f_0(a) = f_1(a)$ all $a \in A$ all t .
 write "f $\simeq g$ rel A"

Def $X \simeq Y$ homotopy equivalent spaces if \exists maps



Rmk homeo \Rightarrow hpy equivalent

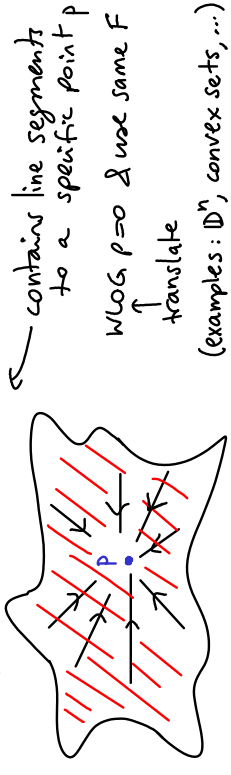
Def X contractible if $X \simeq \text{pt}$

equivalently $(X \xrightarrow{\text{id}} X) \simeq (X \xrightarrow{\text{const}} \text{point} \in X)$

Example. $\mathbb{R}^n \simeq \text{pt}$

$F(x,t) = tx$ then $f_0 \equiv 0, f_1 = \text{id}$.

(star-shaped subsets of $\mathbb{R}^n \simeq \text{pt}$)



WLOG $p=0$ & use same F
 translate
 (examples: \mathbb{D}^n , convex sets, ...)

Theorem $f_0 \simeq f_1 \Rightarrow f_{0*} = f_{1*}$

Pf $f_{1*} - f_{0*} = F_* i_{1*} - F_* i_{0*} = F_* (i_{1*} - i_{0*})$
 previous \nearrow Thm $\Rightarrow F_* (\partial P + P \partial) = \partial_*(F_* P) + (F_* P)_* \partial$
 F_* chain map $\Rightarrow F_* P$ is chain hpy from f_{0*} to f_{1*} \square

(where $F = \text{homotopy}$, $i_{0,1}$ as in previous Thm)

Cor $X \simeq Y \Rightarrow H_* X \cong H_* Y$

Pf $f_* g_* = \text{id}_*$, $g_* f_* = \text{id}_*$ \square

Example X contractible $\Rightarrow H_* X \cong H_*(\text{pt}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Cultural Rmk (Whitehead's theorem) For nice topological spaces \leftarrow (CW complexes - see later in) if X, Y are simply connected and $\exists f : X \rightarrow Y$ inducing isomorphisms on H_* then $X \simeq Y$ are homotopy equivalent.

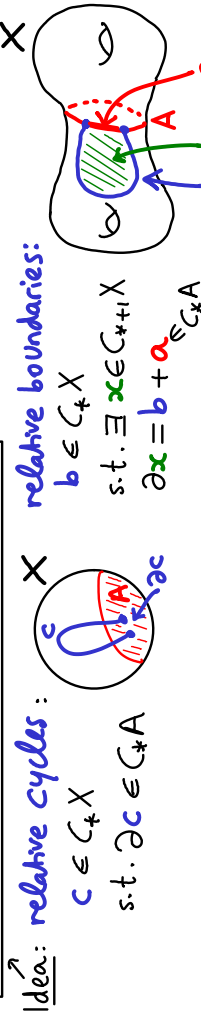
Relative homology

Def (X, A) pair of spaces if $A \subseteq X$ topological subspace

$\Rightarrow \hat{i} = \text{incl}: A \hookrightarrow X$ induces a subcx $\hat{i}_*: C_*A \rightarrow C_*X$

$\Rightarrow C_*X/C_*A$ quotient chain cx (recall $\partial[x] = [\partial x]$)

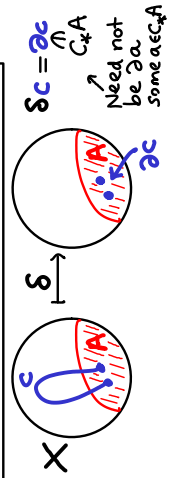
$$H_*(X, A) = H_*(C_*X/C_*A)$$



$$\Rightarrow 0 \rightarrow C_*A \rightarrow C_*X \rightarrow C_*X/C_*A \rightarrow 0 \text{ SES}$$

$$\text{Cor} \dots \rightarrow H_n(A) \xrightarrow{\hat{i}_*} H_n(X) \rightarrow H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{\hat{i}_*} \dots$$

LES for the pair



Reduced homology

$$\tilde{H}_*X = \ker(H_*X \rightarrow H_*(pt))$$

\leftarrow induced by $X \rightarrow pt$

For $X \neq \emptyset$, $\tilde{H}_*X = H_*$ of **augmented chain complex**:

$$\dots \rightarrow C_n(X) \rightarrow \dots \rightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

augmentation $\epsilon(\sum \eta_i \cdot p_i) = \sum \eta_i$ (points $\in X$)

can view $C_{-1}(X) = \mathbb{Z}$ (map $\phi \rightarrow X$) where allow the empty simplex \emptyset

Example $\tilde{H}_*(\mathbb{R}^n) \cong \tilde{H}_*(\mathbb{D}^n) \cong \tilde{H}_*(pt) = 0$

Check: $H_*X = \tilde{H}_*X$ $*$ $\neq 0$, and $H_0X \cong \tilde{H}_0X \oplus \mathbb{Z}$ for $X \neq \emptyset$

\cdot $f: X \rightarrow Y \Rightarrow f_*: \tilde{H}_*X \rightarrow \tilde{H}_*Y$

Lemma (X, A) pair $\Rightarrow \exists$ LES

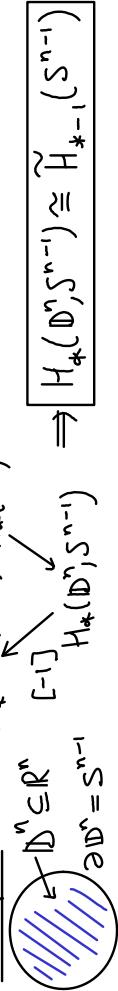
$$\dots \rightarrow \tilde{H}_n(A) \xrightarrow{\hat{i}_*} \tilde{H}_n(X) \rightarrow H_n(X, A) \rightarrow \tilde{H}_{n-1}(A) \xrightarrow{\hat{i}_*} \dots \rightarrow H_0(X, A) \rightarrow 0$$

Pf use augmented ch. cx. and $\mathbb{Z}/\mathbb{Z} = 0 \square$

Cor $H_*(X, pt) \cong \tilde{H}_*(X)$ for $X \neq \emptyset$

Pf $\tilde{H}_*(pt) = 0. \square$

Example LES: $\tilde{H}_*(S^{n-1}) \rightarrow \tilde{H}_*(\mathbb{D}^n) = 0$



Naturality of the LES for pairs

Def A map of pairs of spaces $(X, A) \xrightarrow{f} (Y, B)$

means $f: X \rightarrow Y$ and $f(A) \subseteq B$.

$$\text{Lemma} \dots \rightarrow H_*A \rightarrow H_*X \rightarrow H_*(X, A) \rightarrow H_{*+1}A \rightarrow \dots$$

$$f_* \downarrow \quad f_* \downarrow \quad f_* \downarrow$$

$$\dots \rightarrow H_*B \rightarrow H_*Y \rightarrow H_*(Y, B) \rightarrow H_{*+1}B \rightarrow \dots$$

and similarly for \tilde{H}_* .

Pf $0 \rightarrow C_*A \rightarrow C_*X \rightarrow C_*X/C_*A \rightarrow 0 \Rightarrow$ claim follows by naturality of LES induced by SES of chain cxes. \square

equivalently $r^2 = r$ then define $A = \text{im}(r)$

5. EXCISION THEOREM AND QUOTIENTS

(X, A) pair

Def $r: X \rightarrow X$ retraction onto A if $\begin{cases} r(X) = A \\ r|_A = \text{id}_A \end{cases}$



Example $X = S^2 \vee S^2 =$ two spheres glued at one point v
 $r: X \rightarrow A$ map second sphere to v (wedge sum)

Example In Pf of Brouwer fixed pt thm we built a retraction r by contradiction

Cor r retraction $\Rightarrow r_*: H_*X \rightarrow H_*A$ surjective

incl: $H_*A \rightarrow H_*X$ injective

Pf $A \xrightarrow{\text{incl}} X \xrightarrow{r} A$ now use H_* functorial \square

Def $r: X \rightarrow X$ deformation retraction onto A if $\begin{cases} r \text{ retraction} \\ r = id_X \text{ rel } A \end{cases}$

Example $X = \text{Möbius Strip}$
 $A = \text{equator}$

Lemma r def. retr. $\Rightarrow A \xrightarrow{incl} X$ is a homotopy equivalence.

$\cdot incl_*$ and r_* are isos on H_* , so $H_*A \cong H_*X$

Pf $A \xrightarrow{incl} X$ $incl \circ r = r \circ id_X$, $r \circ incl = r|_A = id_A$ \square

Example $S^n \setminus pt$ def. retracts to $D^n \cong$ lower hemisphere.

$S^n \setminus \{2 \text{ points}\} \cong D^n \setminus pt \cong D^n \setminus O \cong S^{n-1}$

$S^n \setminus \{3 \text{ points}\} \xrightarrow{\text{def. retr.}} S^{n-1} \vee S^{n-1}$

(Warning: check these separately, not true that $X \approx Y$ implies $X \setminus pt \approx Y \setminus pt$.
 (e.g. consider $X = \mathbb{R}^2, Y = \mathbb{R}^2, X \setminus y = pt, Y \setminus pt = S^1, Y \setminus pt = 2 \text{ points}$)

Excision theorem
 $E \subseteq A \subseteq X$ subspaces $\Rightarrow (X \setminus E, A \setminus E) \rightarrow (X, A)$ induces iso
 with $\bar{E} \subseteq A$

Proof later.

Example $X = S^1 \vee S^1 = \infty \supseteq A = E = O \cong S^1$
 $\Rightarrow H_1(X, A) \cong H_1(C, \partial C) \cong H_1(D^1, \partial D^1) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$
 (exc. thm.) \leftarrow hpy invce \leftarrow 2 points

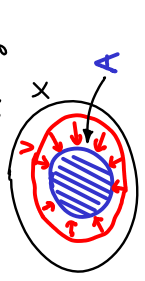
Example Invariance of dimension from chapter 0 also holds if replace $\mathbb{R}^n, \mathbb{R}^m$ by non-empty open sets $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ because for $p \in U$:
 $H_*(U, U-p) \cong H_*(\mathbb{R}^n, \mathbb{R}^n-p) \cong \tilde{H}_{*-1}(\mathbb{R}^n-p) \cong \tilde{H}_{*-1}(S^{n-1})$
 (statement becomes: $U \cong V \Rightarrow n=m$)

Rephrasing of Excision Thm
 $X = A \cup B \Rightarrow H_*(X, A) \cong H_*(B, A \cap B)$
 $(A, B \subseteq X \text{ subspaces})$

Pf Take $E = X \setminus B$ so $X \setminus E = B$ and $A \cap B = A \setminus E$. \square ($\Rightarrow \bar{E} \subseteq X \setminus B \subseteq A$)
 Idea why excision holds: $C_*(A) + C_*(B) \rightarrow C_*(X)$ is a homotopy equivalence and $C_*(A) \cap C_*(B) = C_*(A \cap B)$. Idea: can subdivide chains in X many times, and small enough chains lie either in A or in B (or in both).

Good pairs and quotients For (X, A) pair:

- Quotient $X/A = X/\sim$ \leftarrow equivalence relation $x \sim y \Leftrightarrow \begin{cases} x, y \in A \\ \text{or} \\ x=y \end{cases}$
- (X, A) good pair if $\begin{cases} A \neq \emptyset \\ A \text{ closed} \end{cases}$
 A deformation retract of nbhd V of A



Example $X = S^1 \vee S^1 = \infty \supseteq V = \infty \supseteq A = O \cong S^1$
 $X/A \cong O$ \leftarrow (all points of A are identified with the node)

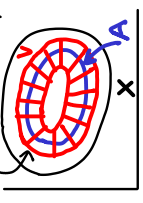
Non-example Topologist's sine curve
 $\{(x, \sin \frac{1}{x}) : x \in (0, 1]\} \cup \{0\} \times [0, 1] \subseteq \mathbb{R}^2$
 A not a good pair.

(connected, not path-connected, not locally connected, not locally path-connected)

Cultural link Smooth submanifold \subseteq smooth manifold is a good pair. \leftarrow (by tubular neighbourhood theorem)

Cor (X, A) good $\Rightarrow (X, A) \rightarrow (X/A, pt)$ induces iso
 $H_*(X, A) \rightarrow H_*(X/A, pt) = \tilde{H}_*(X/A)$

Pf good $\Rightarrow \exists$ nbhd V of A , and $A \xrightarrow{incl} V$
 LES for pair $H_n(X, A) \xrightarrow{\cong} H_n(X, V) \xrightarrow{\cong} H_n(X/A, V/A) \xrightarrow{\cong} H_n(X/A, pt)$ (see below Remark)
 \leftarrow excision \leftarrow "id" = identity
 \leftarrow 5-Lemma \leftarrow since $A=V, A \subseteq V, A \subseteq A$
 \leftarrow quot. \leftarrow call this point p Hence all arrows are isos. \square



Remark $X/A \rightarrow X \setminus A \setminus p, x \mapsto [x] = \{x\}$ is essentially an identity map but we need to check it is a homeomorphism. The composite $X/A \xrightarrow{ctr} X \xrightarrow{ctr} X/A$ is cts and lands in subspc $X/A \setminus p$, but also need $X/A \rightarrow X/A \setminus p$ to send open sets to open sets. Since A is closed, open subsets $U \subseteq X/A$ are open in X and yield an open set $U \setminus A \subseteq X/A$ not containing p , hence "id" $(U) = U \setminus A \subseteq X/A \setminus p$ open.

Example $D^n \supseteq S^{n-1}$ good: $D^n \setminus S^{n-1} \xrightarrow{quot.} S^{n-1}$ quotient \rightarrow points of $A=S^{n-1}$ identified
 $\Rightarrow H_*(D^n, S^{n-1}) \cong \tilde{H}_*(D^n/S^{n-1}) \cong \tilde{H}_*(S^n) \xrightarrow{Cor} \tilde{H}_*(S^n) \cong \mathbb{Z}$

Exercise Check that the iso in the Cor is natural (for a map $(X, A) \rightarrow (Y, B)$ of good pairs get comm. diagram...)

Recall we proved $H_*(D^n, S^{n-1}) \cong \widetilde{H}_{*-1}(S^{n-1})$ (from LES & $\widetilde{H}_*(D^n) \cong 0$)
 $\Rightarrow \widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1}) \cong \dots \cong \widetilde{H}_{k-n}(S^0) \cong \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$
 inductively, using Example 2 points

Generator of $H_n(S^n) \cong \widetilde{H}_n(D^n/S^{n-1}) \cong H_n(D^n, S^{n-1}) \cong \mathbb{Z}$

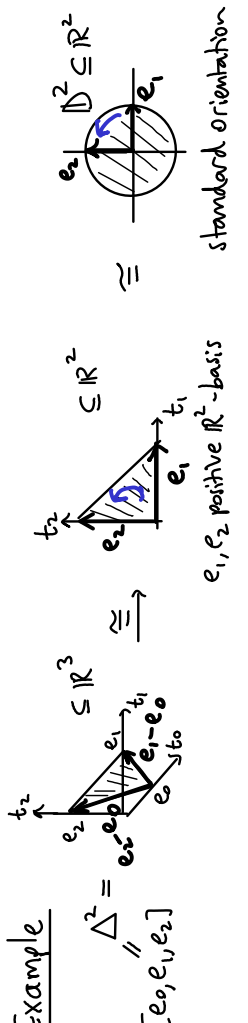
Observe \exists homeo $e^n: D^n \cong D^n$ (homework) inducing Δ -cx structure on S^{n-1} :
 $\partial D^n \cong \partial D^n = S^{n-1}$
 Example $D^2 \cong \Delta^2 \xrightarrow{\partial} -\Delta^+ + \Delta^- \cong S^1$
 stretch cktly outwards from barycentre (Δ^n)

Upshot ($n \geq 2$)
 $H_n(D^n, S^{n-1}) = \mathbb{Z} \cdot e^n$
 $H_{n-1}(S^{n-1}) = \mathbb{Z} \cdot \partial e^n$
 $\widetilde{H}_n(D^n/S^{n-1}) = \mathbb{Z} \cdot [e^n]$
 for $n \geq 1$, so $n \geq 2$ by Cor $[e^n]$ really lives in $H_n(D^n, S^{n-1}) \cong H_n(D^{n+1}/S^n, S^n)$

Exercise Recall another Δ -cx structure on S^n :
 $S^n = \Delta^n \cup \Delta^n / \text{glue along } \partial \Delta^n$
 call this Δ_1 this Δ_0
 then $H_n(S^n) = \mathbb{Z} \cdot (\Delta_1 - \Delta_0)$ because $\Delta_1 - \Delta_0$ is a cycle and $H_n(S^n) \cong H_n(S^n, \partial \Delta_0) \cong H_n(\Delta_1, \partial \Delta_1) \cong H_n(D^n, \partial D^n)$
 (by LES for (S^n, Δ_0) using $n \geq 1$)

Another remark about orientations
 Fact {homeos $\Delta^n \rightarrow D^n$ } has 2 path-components
 Above we chose a path-component by constructing e^n .
 If τ is any reflection in \mathbb{R}^{n+1} then $e^n \circ \tau$ is in the other path-component
 $H_n(S^n) \cong H_n(D^n, S^{n-1}) \xrightarrow{\cong} \mathbb{Z}$
 $e^n \circ \tau \mapsto +1$
 $e^n \circ \tau \mapsto -1$
 e.g. swap 2 coordinates in Δ^n

We will see later in the course that this corresponds to a choice of orientation of D^n and S^n .
 Our choice is consistent with the inclusion $D^n \subseteq \mathbb{R}^n$ (with the positive (canonical) orientation of \mathbb{R}^n) and the inclusion $(\Delta^n \subseteq \mathbb{R}^{n+1}) \cong \{(t_1, \dots, t_n) \in \mathbb{R}^n : t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$
 $(t_0, \dots, t_n) \mapsto (t_1, \dots, t_n)$
 $t_i \geq 0, \sum t_i = 1$



Our choice is also consistent with the "normal first" Convention for orienting hyperplanes with a given choice of normal:
 $\Delta^n \subseteq$ hyperplane $\{(t_0, \dots, t_n) : \sum t_j = 1\} \subseteq \mathbb{R}^{n+1}$ normal $(1, 1, \dots, 1)$ (so pointing to ∞ in positive quadrant)



Consistent also with the geometric boundary orientation (outward normal first)
 Geometric $\Delta^2 = [e_0, e_1, e_2]$

 $S^1 = \partial D^2 \subseteq \mathbb{R}^2$ standard orientation

Compare $\partial \Delta = +[e_0, e_1, e_2] - [e_0, \hat{e}_1, e_2] + [e_0, e_1, \hat{e}_2]$
 This $-[e_0, e_2]$ is not equal to singular chain $[e_2, e_0]$ since they are different maps and we take free abelian group generated by maps.
 But $[e_0, e_2] + [e_2, e_0]$ is homologous to 0 (Homework).

Locality (or "small simplices theorem")

$\mathcal{U} = \{ \text{subspaces } U_i \subseteq X \}$ whose interior cover X :
 $X = \bigcup U_i$

Def $C_*^u(X) \subseteq C_*(X)$ subcomplex generated by n -simplices σ with $\sigma(\Delta^n) \subseteq U_i$ some i

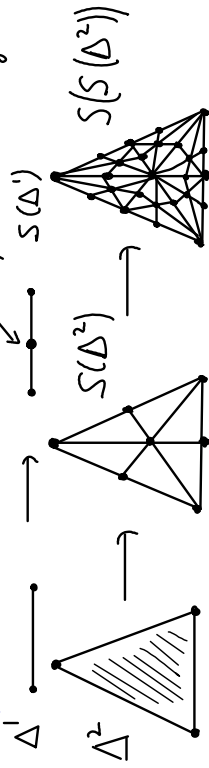
$$\boxed{H_* (C_*^u(X)) \cong H_* (C_*(X)) = H_* X}$$

Theorem

barycentre of $[v_0, \dots, v_n]$ is $\frac{1}{n+1}(v_0 + \dots + v_n)$

Sketch Pf ① Barycentric subdivision barycentre divides edge in 2

Non-examinable



\Rightarrow chain map $S: C_*(X) \rightarrow C_*(X)$ and $S(C_*^u) \subseteq C_*^u$

Construction of " $\sigma \circ S$ " is inductive:

On linear simplices (them for maps σ you restrict to...)

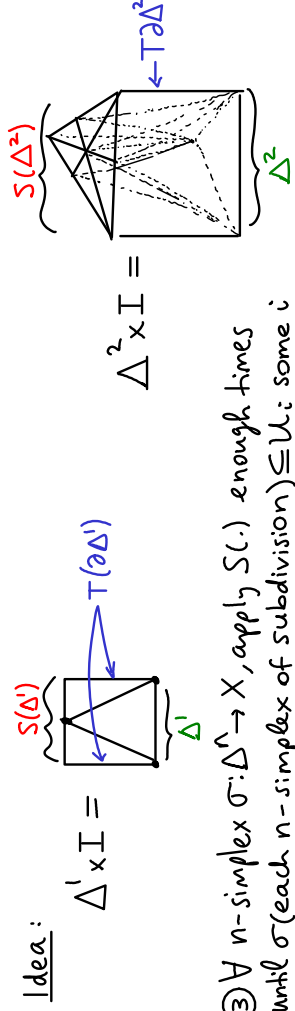
- $S[e_0] = [e_0]$
- $S[e_0, e_1] = [b, e_1] - [b, e_0]$ (geometrically $\vec{e}_0 \rightarrow \vec{b} \rightarrow \vec{e}_1$)
- $S[e_0, e_1, e_2] = "[b, S \partial[e_0, e_1, e_2]]"$
 $= "[b, S[e_1, e_2]] - [b, S[e_0, e_2]] + [b, S[e_0, e_1]]"$
 $= ([b, b_{12}, e_2] - [b, b_{02}, e_2]) - ([b, b_{02}, e_1] - [b, b_{01}, e_1]) - [b, b_{01}, e_0]$

so for $\sigma: \Delta^2 \rightarrow X$ you take $S(\sigma) = \sigma \circ S$

② S chain hpic to id:

$$\left. \begin{aligned} T: C_n(X) &\rightarrow C_{n+1}(X) \\ T(\sigma): \Delta^n \times I &\xrightarrow{\text{project}} \Delta^n \xrightarrow{\sigma} X \end{aligned} \right\} \Rightarrow \boxed{S_*: H_*(X) \xrightarrow{\text{id}} H_*(X)}$$

exercise: $\partial T + T \partial = S - \text{id}$

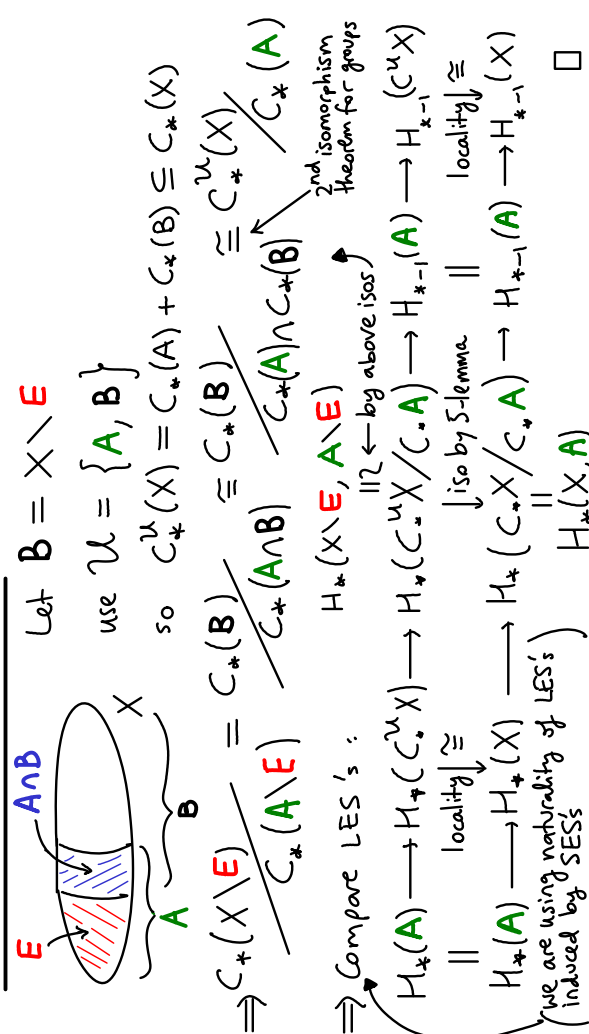


③ $\forall n$ -simplex $\sigma: \Delta^n \rightarrow X$, apply $S(\cdot)$ enough times until $\sigma(\text{each } n\text{-simplex of subdivision}) \subseteq U_i$ some i

\forall cycle $c, \exists n$ s.t. $S^n(c) \in C_*^u(X)$ cycle $\Rightarrow H_*^u(c) \rightarrow H_*(X)$ surjective $[S^n(c)] \mapsto S_*^n[c] = [c]$ by ②

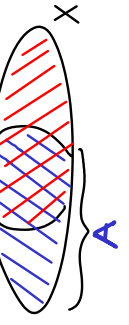
\forall bdry $c = \partial b, \exists n$ s.t. $S^n(b) \in C_*^u(X)$ claim: $H_*^u(c) \rightarrow H_*(X)$ injective suppose $[c] \mapsto 0$ then $c = \partial b$ for $b \in C_*^u(X)$ now $S^n c, S^n b \in C_*^u(X)$ for large $n \Rightarrow \partial S^n b = S^n \partial b = S^n c$ in $C_*^u(X) \Rightarrow [c] \in S_*^n[c] = [S^n c] = [\partial S^n b] = 0$ in $H_*^u(X) \checkmark \square$

Proof of excision theorem



6. MAYER-VIETORIS SEQUENCE ← Key computational tool

$X = A \cup B$ s.t. $X = A \cup B^o$



any subspaces

MV Theorem \exists LES:

$$\dots \rightarrow H_*(A \cap B) \xrightarrow{i_*} H_*(A) \oplus H_*(B) \rightarrow H_*(X) \rightarrow H_{*-1}(A \cap B) \xrightarrow{i_*} \dots$$

& same holds for \tilde{H}_* provided $A \cap B \neq \emptyset$.

Pf SES $0 \rightarrow C_*(A \cap B) \rightarrow C_*(A) \oplus C_*(B) \rightarrow C_*^u(X) \rightarrow 0$

$\sigma \mapsto (\sigma, -\sigma)$

$(\alpha, \beta) \mapsto \alpha + \beta$

\Rightarrow induces the LES (using locality $H_*^u X \cong H_* X$). D

Exercise connecting map is $\delta: H_*(X) \rightarrow H_{*-1}(A \cap B)$

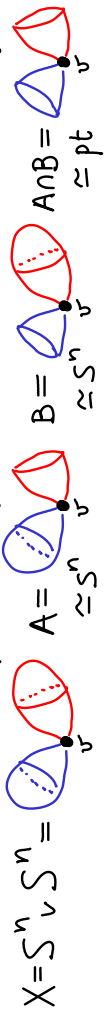
$[\alpha + \beta] \mapsto [\partial\alpha] = -[\partial\beta]$



$\dots \rightarrow H_2(\text{pt}) \oplus H_2(\text{pt}) \rightarrow H_2(S^2) \rightarrow H_1(S^1) \rightarrow H_1(\text{pt}) \oplus H_1(\text{pt}) \rightarrow \dots$

Exercise Compute $H_* S^n = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ using MV

Example wedge sum of X, Y with basepoints x, y



$0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\cong} H_n(X) \rightarrow 0 \rightarrow \dots \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow H_0(X) \rightarrow 0$

Similarly $[H_*(X \vee Y) \cong H_*(X) \oplus H_*(Y)]$ for $* \neq 0$ if \exists contractible nbhds of $x \in X, y \in Y$.

Cones and suspensions

$CX = (X \times [0,1]) / \sim_{s=t=1}$

$\Sigma X = (X \times [0,1]) / \sim_{s,t=0}$

Example $CS^n \cong D^{n+1}, \Sigma S^n \cong S^{n+1}$

Lemma $\tilde{H}_*(\Sigma X) \cong \tilde{H}_{*-1}(X)$



now apply MV. \square

Rmk $\phi \neq A \subseteq X \Rightarrow \tilde{H}_*(X \setminus A) \stackrel{LES}{\cong} H_*(X \setminus A, CA) \stackrel{exc.}{\cong} H_*(X, A)$

Connected sum

M, N n -manifolds $\Rightarrow M \# N = (M \setminus \text{open } n\text{-ball}) \cup (N \setminus \text{open } n\text{-ball})$



Fact compact connected orientable surfaces are homeo to S^2 or $T^2 \# \dots \# T^2$ and " \prime " non-orientable ones: $\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$. $g = \#$ copies called Σ_g

Exercise (Homework) For M, N compact connected n -mfd's:

By MV, $H_*(M \# N) \cong H_*(M) \oplus H_*(N)$ for $1 \leq * \leq n-2$

If M or N orientable: $* = n-1$ also works

If both non-orientable: $* = n-1$ one of $\mathbb{Z}/2$ summands becomes \mathbb{Z}

Cor 1) $\chi(M \# N) = \chi(M) + \chi(N) - \begin{cases} 2 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

2) $H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z}^{2g} & * = 0 \\ \mathbb{Z} & * = 1 \\ 0 & * = 2 \end{cases} \cong \chi(S^n)$

$H_0(M \# N) \cong \mathbb{Z}$ since connected
fact: $H_n(M \# N)$ is \mathbb{Z} or 0
 \uparrow else if M, N both orientable (see later in course)

7. DEGREE OF MAPS OF SPHERES

ASSUME $n \geq 1$ IN THIS CHAPTER

$$f: S^n \rightarrow S^n \implies f_n: H_n S^n \xrightarrow{\cong} H_n S^n \xrightarrow{\cong} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{\text{deg}(f)} \mathbb{Z}$$

$\implies f_*: \tilde{H}_* S^n \rightarrow \tilde{H}_* S^n$ is $\text{deg}(f) \cdot \text{id}$

- Properties
- $\text{deg}(\text{id}) = 1$
 - $\text{deg}(f \circ g) = \text{deg } f \cdot \text{deg } g$
 - $f \simeq g \implies \text{deg } f = \text{deg } g$
 - $f \simeq \text{const} \implies \text{deg } f = 0$
 - f homeomorphism $\implies \text{deg } f = \pm 1$

sign depends on whether f is orientation-preserving or reversing

Pf $\text{id}_* = \text{id}$, $(f \circ g)_* = f_* \circ g_*$, $f \simeq g \implies f_* = g_*$, $\text{const}_* = 0$, f homeo $\implies f_n$ iso. \square
 Examples since $S^n \xrightarrow{\text{pt}} S^n$ factors so $H_n S^n \xrightarrow{H_n(\text{pt})} H_n S^n$

- $S^n = \Delta^n \times 1 \cup \Delta^n \times 0 \leftarrow \text{call this } \Delta_0$
 $(b, 1) \sim (b, 0)$ if $b \in \partial \Delta$
 recall $H_n S^n = \mathbb{Z} \cdot (\Delta_1, -\Delta_0)$

reflection: $r: S^n \rightarrow S^n$, $r(x, t) = (x, 1-t)$

so $\Delta_0 \leftrightarrow \Delta_1$ swapped by r , so $r_*(\Delta_1, -\Delta_0) = -(\Delta_1, -\Delta_0)$
 $\implies \text{deg}(r) = -1$

2) antipodal map $-\text{id}: S^n \rightarrow S^n$ viewing $S^n \subseteq \mathbb{R}^{n+1}$

$$\text{deg}(-\text{id}) = (-1)^{n+1}$$

Pf $-\text{id} = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} \circ \dots \circ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ composition of $n+1$ reflections each homotopic to r . \square

3) $A \in O(n) \implies A: S^{n-1} \rightarrow S^{n-1} \implies \text{deg } A = \det A = \pm 1$

Pf fact $SO(n)$ is path-connected so $A \in SO(n)$ is $\simeq \text{id}$ so $\text{deg } A = \det A = +1$
 The other path-component of $O(n)$ is $r \circ SO(n)$ where r is any reflection. \square

4) f not surjective $\implies \text{deg } f = 0$

Pf If $y \notin \text{Im } f \implies H_n(S^n) \xrightarrow{f_*} H_n(S^n) \rightarrow H_n(\mathbb{R}^n) = 0$

Application to vector fields on S^n

$v: S^n \rightarrow \mathbb{R}^{n+1}$ tangent vector field on S^n
 so $v(x) \perp x$



Cor Hairy ball theorem

\exists nowhere zero v.f. on $S^n \iff n$ odd

(case $n=2$: "you cannot comb a ball of hair without creating a tuft")

Pf Suppose $v(x) \neq 0 \forall x$

$$\implies \text{hpy } F: S^n \times [0, 1] \rightarrow S^n$$

$$F(x, t) = \cos(\pi t)x + \sin(\pi t)v(x)$$

$$\implies F_0 = \text{id}, F_1 = -\text{id}$$

$$\implies 1 = \text{deg } F_0 = \text{deg } F_1 = (-1)^{n+1}$$

$\implies n$ odd

For n odd $\exists v(x) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}) \in \mathbb{R}^{2k}$. \square

Cultural Remark

Adams in 1962 proved using alg. topology:

(max # pointwise linearly independent vector fields on $S^n = 2^b + a - 1$ where $n+1 = 2^{a+b}$ (odd number), $0 \leq b \leq 3$, $a, b \in \mathbb{N}$, $n \geq 1$. \leftarrow get 0 if n even $\implies \text{cor } \checkmark$)

Local degree

$$f: S^n \rightarrow S^n$$

$$x \mapsto y = f(x)$$

\star Suppose points $x \neq y$ near x do not map to y :

$$\exists \text{ nbhds } x \in U, y \in V \text{ s.t. } (U, U \setminus x) \xrightarrow{f} (V, V \setminus y)$$

call this $f|_x$

$$\implies (f|_x)_*: H_n(U, U \setminus x) \xrightarrow{f_*} H_n(V, V \setminus y)$$

$$\xrightarrow{\cong} H_n(S^n, S^n \setminus x) \xrightarrow{\cong} H_n(S^n, S^n \setminus y) \xrightarrow{\cong} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{\text{deg } f} \mathbb{Z}$$

excise $S^n \setminus U$ $\xrightarrow{\cong} H_n(S^n, S^n \setminus x)$ $\xrightarrow{\cong} H_n(S^n, S^n \setminus y) \xrightarrow{\cong} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{\text{deg } f} \mathbb{Z}$

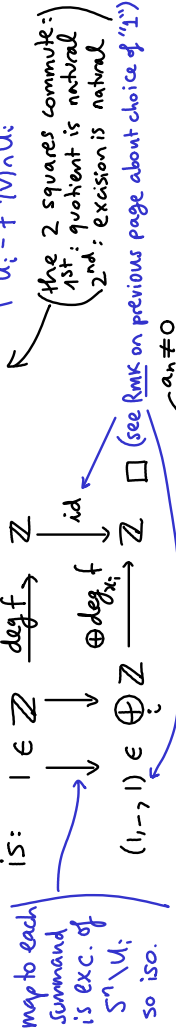
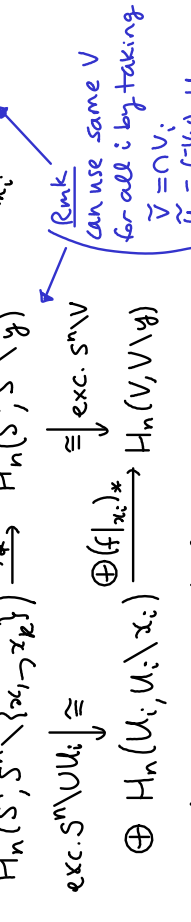
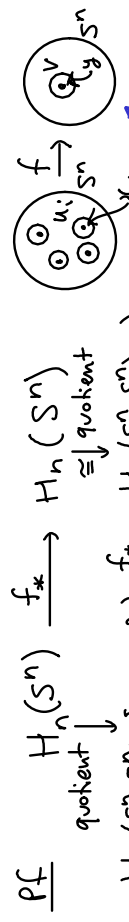
will use this again later:

$$H_n(S^n) \cong H_n(S^n, S^n \setminus \text{pt})$$

RMK By left vertical arrows: $H_n(U, U \setminus x) \cong \mathbb{Z}$ again we made a choice of iso by picking a generator "1" by combining excision, quotient and choice of $H_n(S^n) \cong \mathbb{Z}$.

Lemma $f: S^n \rightarrow S^n, f^{-1}(y) = \{x_1, \dots, x_k\} \quad (n \geq 1)$

$$\Rightarrow \deg f = \sum \deg x_i f$$



Example $p: \mathbb{C} \rightarrow \mathbb{C}$ polynomial $a_n z^n + a_{n-1} z^{n-1} + \dots + a_0, a_i \in \mathbb{C}$

$\Rightarrow f: S^2 = \mathbb{C}P^1 \rightarrow \mathbb{C}P^1 = S^2$ (where view $\mathbb{C}P^1 = \mathbb{C} \cup \infty \cong S^2$)

$\Rightarrow \text{hpy } F(z, t) = a_n z^n + t(a_{n-1} z^{n-1} + \dots + a_0)$

$F_0 = a_n z^n$ and $F_1 = f$

$\Rightarrow \deg f = \deg(a_n z^n) \stackrel{\text{Lemma}}{=} \sum_{k=0}^{n-1} \deg a_k z^k \leftarrow w_k = e_n$

$= n$

$= \text{degree of the poly } p.$

Cor (Fundamental Thm of Algebra) $n \geq 1 \Rightarrow p$ has a root

PF $p^{-1}(0) = \emptyset \Rightarrow f^{-1}(0) = \emptyset \Rightarrow \deg f = 0 \neq n$

Cultural Rmk For smooth $f: S^n \rightarrow S^n$

$\deg f =$ (the number of preimages of a generic point)

(i.e. almost any point works)

Example $S^2 \rightarrow S^2$

$S^2 \setminus \text{North pole} \cong \mathbb{C}$

map $\mathbb{C} \rightarrow \mathbb{C}, z \mapsto z^d$ and North \mapsto North

$\Rightarrow \deg = d = \#$ preimages of a point

except if pick North/South pole

$(z=0) \leftarrow$ North pole

$(z=\infty) \leftarrow$ South pole

Why "CW"? "W" = weak topology, "C" = closure-finite property: each D_α only hits finitely many interiors of other cells. By contradiction, suppose false. Then by construction it means that the image $M = \varphi_\alpha(D_\alpha) \subseteq X^{n-1}$ contains some point $x_\alpha \in \text{int}(D_\alpha^c) \cap M$, for infinitely many α .

Pick a closed neighborhood C_α of x_α inside an open nbhd U_α of x_α .

so $x_\alpha \in C_\alpha \subseteq U_\alpha \subseteq \text{int}(D_\alpha^c)$. Notice M is compact since $\text{cl } M$ is compact and ∂D_α^c is compact. Pick any open cover $W_j \subseteq X^{n-1}$ of M . Then $W_j \cap (X^{n-1} \setminus C_\alpha)$, U_α is also an open cover of M , and we cannot find a finite subcover since U_α is the only open set of cover containing x_α .

8. CELLULAR HOMOLOGY

Def CW complex X is sequence $\phi = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$ s.t. X^0 is any set with discrete topology. closed n -discs labelled by some index set I_n



$\Rightarrow X = \bigcup_{n \geq 0} X^n$ top space with weak topology:

$U \subseteq X$ open $\Leftrightarrow U \cap X^n \subseteq X^n$ open $\forall n$.

$(\Leftrightarrow U \cap D_\alpha^n \subseteq D_\alpha^n$ open $\forall n, \alpha$)

Call X n-dimensional if $X = X^n$ and this is the least such n .

Example $S^n = (D^0 \sqcup D^1) / (\partial D^0 \sim \partial D^1)$ attach S^2

Example $X = \mathbb{R}P^2 = (D^0 \sqcup D^1) / (\partial D^0 \sim \partial D^1)$ attach S^2

$X^0 = \bullet = D^0$

$X^1 = \bullet \cup \text{circle} = S^1 = (D^0 \sqcup D^1) / (\partial D^0 \sim \partial D^1)$

$X^2 = (\bullet \cup \text{circle}) / (\text{wrap } \partial \text{ of } \text{circle twice around } \bullet)$

$= (X^1 \sqcup D^2) / (\partial D^2 \sim S^1 \xrightarrow{\varphi_2} S^1, \varphi_2(z) = z^2)$

Fact If we homotope φ_α , we get a homotopy equivalent space

Example If we use another degree 2 map φ_2 above, get $X \simeq \mathbb{R}P^2$.

Cultural Rmk Every CW-complex X is homotopy equivalent to a simplicial complex Y (so in particular a Δ -complex). [If X finite/ n -dim then can ensure Y is finite/ n -dim]

X is partitioned as a set by interiors of n-cells

$$X^n = X^{n-1} \cup \bigcup_{\alpha \in I_n} \text{int}(D_\alpha^n) \cup \bigcup_{\alpha \in I_{n-1}} \text{int}(D_\alpha^{n-1}) \cup \dots \cup \bigcup_{\alpha \in I_0} \text{int}(D_\alpha^0) = \mathbb{D}^0$$

(so $e_0 = \bullet$)

Why "CW"? "W" = weak topology, "C" = closure-finite property: each D_α only hits finitely many interiors of other cells. By contradiction, suppose false. Then by construction it means that the image $M = \varphi_\alpha(D_\alpha) \subseteq X^{n-1}$ contains some point $x_\alpha \in \text{int}(D_\alpha^c) \cap M$, for infinitely many α .

Pick a closed neighborhood C_α of x_α inside an open nbhd U_α of x_α .

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Examples $\mathbb{R}P^k = S^k / (\mathbb{Z}/2\text{-action by } \pm \text{id})$ inductively

$X^k = \mathbb{R}P^k$ inductively

$X^n = X^{n-1} \cup e^n$ with $\varphi: S^{n-1} \rightarrow X^{n-1} = \mathbb{R}P^{n-1} = S^{n-1} / \pm \text{id}$

Complex projective space

$\mathbb{C}P^n = (S^{2n+1} \subseteq \mathbb{C}^{n+1}) / (S^1\text{-action by } \lambda \cdot \text{Id})$

$X^0 = X^1 = \mathbb{C}P^0 = \text{pt}$
 $X^2 = X^3 = X^0 \cup e^2 = \mathbb{C}P^1$ $\varphi: S^1 \rightarrow \text{pt}$
 $X^4 = X^5 = X^2 \cup e^4 = \mathbb{C}P^2$ $\varphi: S^3 \rightarrow \mathbb{C}P^1 = S^3 / (S^1\text{-action})$
 $X^{2n} = X^{2n-2} \cup e^{2n} = \mathbb{C}P^n$ $\varphi: S^{2n-1} \rightarrow X^{2n-2} = \mathbb{C}P^{n-1}$

In coordinates: $\mathbb{C}P^n = \{ [z_0 : \dots : z_n] : \text{not all } z_i \in \mathbb{C} \text{ are } 0 \}$ and $[z] \sim [\lambda z]$ $\forall \lambda \in \mathbb{C}^*$
 Can rescale so that $\sum |z_i|^2 = 1$ so $z \in S^{2n-1}$ and left with rescaling by $\lambda \in S^1 \subseteq \mathbb{C}^*$
 $\mathbb{C}P^{n-1} \cong X^{2n-2} = \{ [z_0 : \dots : z_{n-1} : 0] \} \subseteq \mathbb{C}P^n = X^{2n}$ and \leftarrow notice this is 0 if $w_0 = \dots = w_{n-1} = \sqrt{1 - \sum |w_j|^2}$ $\forall w_j \in S^1$

Observe: For X CW complex, for $n \geq 1$: $(X^n, X^{n-1}) = (X^0, \emptyset)$

- (X^n, X^{n-1}) is a good pair \leftarrow (since \exists nbhd of $\partial \mathbb{D}^n$ that deformation retracts to $\partial \mathbb{D}^n$)
- $X^n / X^{n-1} \cong \bigvee_{\alpha \in I_n} S^n$ \leftarrow S^n identified to a point

Def Cellular complex for X a CW cx,

$C_n^{CW}(X) = H_n(X^n, X^{n-1}) \cong H_n(X^n / X^{n-1})$
 $=$ free abelian gp gen. by the n -cells e_α^n

since $\Delta^n \cong \mathbb{D}^n \rightarrow (e^n \subseteq X^n) \rightarrow \mathbb{D}_\alpha^n / \partial \mathbb{D}_\alpha^n = S_\alpha^n$ generate as usual we use the standard orientations of $\Delta^n, \mathbb{D}^n, S^n$.

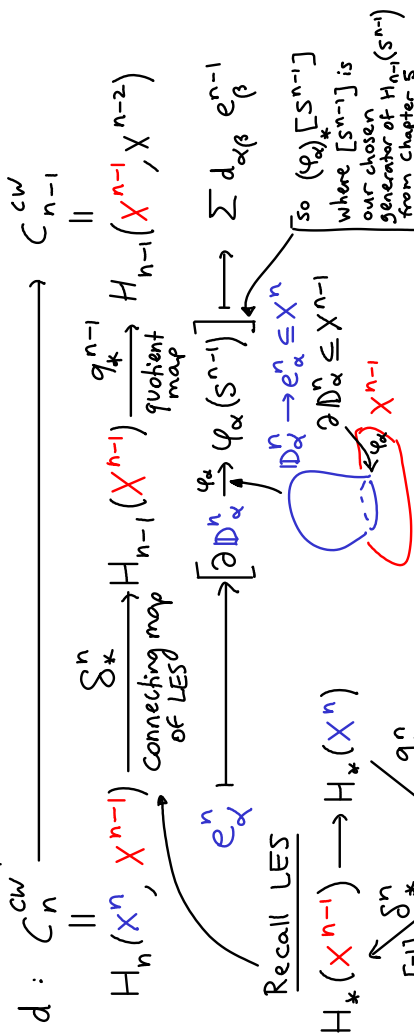
Will build cellular differential $d: C_*^{CW} \rightarrow C_*^{CW}$, prove $d \circ d = 0$

\Rightarrow get $H_*^{CW}(X) = H_*(C_*^{CW}(X), d)$

Example $C_k^{CW}(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} \cdot e^k & \text{for } k=0,2,4,\dots,2n \\ 0 & \text{else} \end{cases}$ hence $d=0$ so $H_*^{CW}(\mathbb{C}P^n) = C_*^{CW}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & 0 \leq * \leq 2n \text{ even} \\ 0 & \text{else} \end{cases}$

Cellular differential:

$d e_\alpha^n = \sum_{\beta \in I_{n-1}} d_{\alpha\beta}^n \cdot e_\beta^{n-1}$ now describe the coefficients $d_{\alpha\beta}^n \in \mathbb{Z}$ and why that is a finite sum.



Recall LES $H_*(X^{n-1}) \rightarrow H_*(X^n) \xrightarrow{q_*} H_*(X^n, X^{n-1}) \rightarrow 0$
 $\Delta^n \cong \mathbb{D}^n, S^n \cong \mathbb{D}^n / \partial \mathbb{D}^n$ compatibly with orientations.
 here it is important that we chose identifications quotient by $\bigvee_{I_{n-1}} S^{n-1}$

Therefore: $d_{\alpha\beta}^n = \text{deg}(S^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1} / X^{n-2} \cong \bigvee_{I_{n-1}} S^{n-1} \rightarrow S^{n-1})$
 $\cong \mathbb{D}_\beta^n / \partial \mathbb{D}_\beta^{n-1}$

Rmk Only finitely many $d_{\alpha\beta}^n \neq 0$ (for fixed α) because φ_α, q are continuous and S^{n-1} compact, so get a compact image in $\bigvee_{\beta} S^{n-1}$, therefore cannot surject onto ∞ many S_{β}^{n-1} .

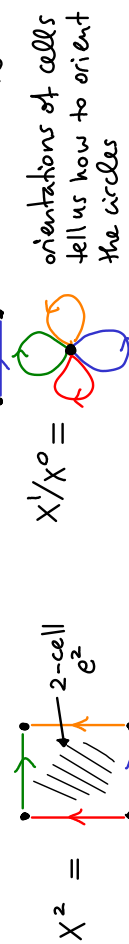
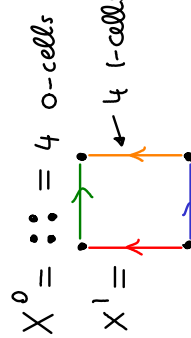
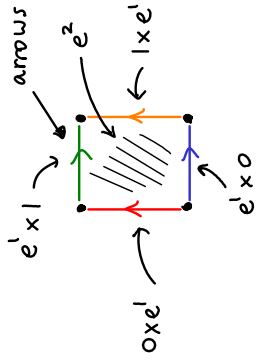
Lemma $d \circ d = 0$
 PF $d_n = q_{n-1}^{n-1} \circ S_n^{n-1} \circ \delta_n^{n-2} \circ \delta_{n-1}^{n-2} \circ \delta_{n-1}^{n-1} \circ q_{n-1}^{n-1} \circ \delta_n^{n-1}$
 (indeed see proof of "closure-finite": it shows φ_α only hits finitely many interiors of \mathbb{D}_β^{n-1})

Cor $\text{rank } H_n^{CW}(X) \leq \# \text{ n-cells}$

PF #n-cells = rank $C_n^{CW}(X) \geq \text{rank ker } d_n^{CW} \geq \text{rank } H_n^{CW}(X) = 0$

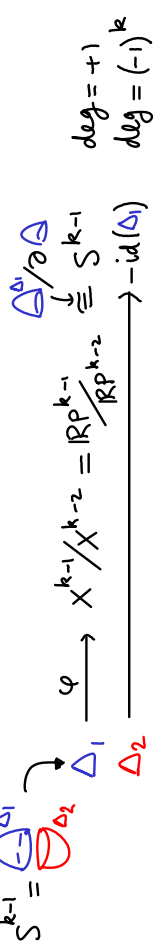
Example $I \times I$ $I = [0,1]$ $\mathbb{D}^1 = [-1,1]$

arrows here tell us how we map $[-1,1] \rightarrow$ edge (so orientation)



$e^2 : \mathbb{D}^2 \cong \square \rightarrow X^1$
 $\partial e^2 : S^1 \cong \square \rightarrow X^1/X^0 =$
 $\Rightarrow \partial e^2 = +e^1 x 0 + 1x e^1 - e^1 x 1 - 0x e^1$
 $(= (\partial e^1) x e^1 - e^1 x (\partial e^1) \leftarrow \text{we come back to this later})$

Example $\mathbb{R}P^n$ recall: 1 cell in each dim, $\varphi: S^k \rightarrow X^k = \mathbb{R}P^k$
 $x \mapsto [x]$

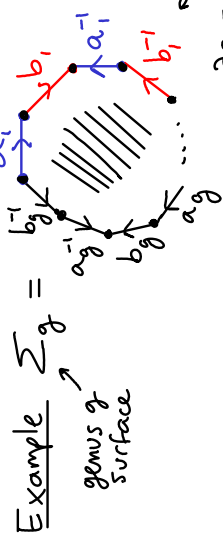


$\Rightarrow d\alpha_\beta = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$
 $0 \rightarrow \mathbb{Z} \xrightarrow{k=n} \mathbb{Z} \rightarrow \dots \rightarrow \mathbb{Z} \xrightarrow{k=1} \mathbb{Z} \rightarrow 0$
 $\begin{matrix} \swarrow & \searrow \\ 2 & 0 \end{matrix}$ if n even / odd resp.

$H_*^{CW}(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & * = 0, \text{ also } * = n \text{ if odd} \\ \mathbb{Z}/2 & * = 1, 3, 5, \dots < n \\ 0 & \text{else} \end{cases}$

Example S^n : $C_*^{CW}(S^n): n > 2: 0 \rightarrow \mathbb{Z} \cdot \mathbb{D}^n \xrightarrow{0} 0 \rightarrow \dots \rightarrow 0 \rightarrow \mathbb{Z} \cdot \mathbb{D}^0 \rightarrow 0$
 $n=1: 0 \rightarrow \mathbb{Z} \cdot \mathbb{D}^1 \xrightarrow{0} \mathbb{Z} \cdot \mathbb{D}^0 \rightarrow 0$

$\Rightarrow H_*^{CW}(S^n) \cong \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else} \end{cases}$
 $H_1(S^1, \mathbb{Z}) \cong H_0(S^1, \mathbb{Z}) \cong H_0(\mathbb{R}^2, \mathbb{Z})$
 $(\Delta^1 \cong [0,1] \rightarrow S^1) \xrightarrow{\text{orientation } 1} \partial \Delta^1$
 if you work with degrees, need to remember orientations: $\partial \mathbb{D} \cong \partial [0,1] = [1] - [0] \rightarrow \text{point}$
 so degree = $+1 = 0$



boundary identifications
 $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$
 Notice all vertices are identified, call vertex v
 $\partial a_i = v - v = 0$
 $\partial b_i = v - v = 0$

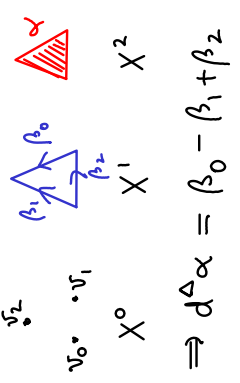
$0 \rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial=0} \mathbb{Z} \rightarrow 0 \Rightarrow H_*(\Sigma_g) \cong \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^{2g} & * = 1 \\ 0 & \text{else} \end{cases}$
 $\mathbb{D} \mapsto -a_1 - b_1 + a_1 + b_1 + \dots - a_g - b_g + a_g + b_g = 0$
 (signs: compare edge orientation with anticlockwise orientation of $\partial \mathbb{D}$)

Example Non-orientable surface $N_k: \mathbb{D} \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0 \Rightarrow H_*(N_k) \cong \begin{cases} \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2 & * = 0 \\ 0 & * = 1 \\ \text{else} & \text{else} \end{cases}$
 (since $\mathbb{D} \mapsto -a_1 - a_1 - a_2 - a_2 - \dots - a_k - a_k$)
 (use the standard basis for \mathbb{Z}^k except replace $(0, \dots, 0, 1)$ by $(1, 1, \dots, 1)$.)

Lemma $X \Delta$ -cx structure \Rightarrow induces CW-cx structure on X and $(C_*^{CW}(X), d^{CW}) \cong (C_*^\Delta(X), d^\Delta)$
 $\Rightarrow H_*^{CW}(X) \cong H_*^\Delta(X)$

Pf $X^n = \cup n$ -simplices of X and degrees are ± 1 depending on orient
 so can identify d^{CW} and d^Δ . \square

Example $X = \text{triangle} = \Delta^2$



Theorem X CW cx (or Δ -cx) $\implies H_*^{CW}(X) \cong H_*(X)$

$\implies H_*^{\Delta}, H_*^{CW}$ independent of choice of CW-cx/ Δ -cx structure.

Pf ① $H_*(X^n, X^{n-1}) \cong \tilde{H}_*(X^n/X^{n-1}) \cong \tilde{H}_*(\bigvee_{\alpha} S^n) \cong \bigoplus_{\alpha} \tilde{H}_*(S^n)$
 lives in degree n
 $= 0 \iff * \neq n$

LES for $(X^n, X^{n-1}) \implies H_*(X^{n-1}) \rightarrow H_*(X^n) \rightarrow H_*(X^n/X^{n-1})$ iso for $* < n-1$
 $* > n$

② for $* < n$: $H_*(X^n) \cong H_*(X^{n+1}) \cong H_*(X^{n+2}) \cong \dots \cong H_*(X)$

by compactness each sing-chain lands in X^N , some N , by the same proof as in the "closure-finite" property (start of Chapter)

③ for $* > n$: $H_*(X^n) \cong H_*(X^{n-1}) \cong H_*(X^{n-2}) \cong \dots \cong H_*(X^{n-1}) = 0$

④ LES: $\dots \rightarrow H_n(X^{n-1}) \rightarrow H_n(X^n) \xrightarrow{q_n} H_n(X^{n+1}) \rightarrow \dots$
 $\stackrel{0}{=} \text{by } ③$

$\implies q_n^*$ injective $\forall n$

⑤ LES: $\dots \xrightarrow{q_{n+1}^*} H_{n+1}(X^{n+1}, X^n) \xrightarrow{\delta_{n+1}^{n+1}} H_n(X^n) \xrightarrow{q_n^*} H_n(X^{n+1}) \rightarrow 0$
 hence surjective with $\text{Ker} = \text{im } \delta_{n+1}^{n+1}$

UPSHOT

④ $\text{Ker } \delta_n^{n+1} = \text{im } q_n^n$

$H_n^{CW}(X) = \frac{\text{Ker } d_{n+1}^{CW}}{\text{Im } d_{n+1}^{CW}} = \frac{\text{Ker } q_{n+1}^{n+1} \circ \delta_n^n}{\text{im } q_n^n \circ \delta_{n+1}^{n+1}}$
 exactness of LES $\dots \rightarrow H_n(X^{n-1}) \rightarrow H_n(X^n) \xrightarrow{q_n^n} H_n(X^{n+1}) \xrightarrow{\delta_{n+1}^{n+1}} \dots$

$= q_n^n (H_n(X^n)) / q_n^n (\text{im } \delta_{n+1}^{n+1})$

$\cong \textcircled{4} H_n(X^n) / \text{im } \delta_{n+1}^{n+1} \cong H_n(X^{n+1}) \cong \textcircled{2} H_n(X)$. \square
 \uparrow 1st iso thm

Rmk by ① H_k not affected if attach $(k+2)$ -cells or higher

by ② Inclusion $X^n \rightarrow X$ induces iso $H_*(X^n) \rightarrow H_*(X)$ for $* < n$

Cor X n -dimensional cell cx $\implies H_*(X) = 0$ for $* > n$

Axioms for homology: Eilenberg-Steenrod axioms

It was no accident that $H_*^{\Delta}, H_*^{CW}, H_*^*$ all agreed.

Def A generalised homology theory (GHT)

is a functor F : Top Pairs = (Category of pairs of spaces and maps of pairs) \rightarrow Graded Abelian Gps

with a natural transformation $\delta : F_*(X, A) \rightarrow F_{*-1}(X, \emptyset)$ satisfying:

1) homology invariance: $f \cong g \implies F(f) = F(g)$ abbreviated: $F_{*-1}(X)$

2) exactness: \exists LES $\dots \rightarrow F_*(A) \xrightarrow{f} F_*(X) \xrightarrow{F(\text{incl})} F_*(X, A) \xrightarrow{\delta} F_{*-1}(A) \rightarrow \dots$

3) additivity: $(X, A) = \sqcup (X_i, A_i)$, $\text{incl}_i : (X_i, A_i) \rightarrow (X, A)$
 $F(\text{incl} : A \rightarrow X)$ $F(\text{incl} : X_i, \emptyset) \rightarrow (X, A)$
 then $\Sigma F(\text{incl} : \oplus F(X_i, A_i)) \cong F(X, A)$

4) excision: $\bar{E} \subseteq A^{\circ} \subseteq X \implies F(X \setminus E, A \setminus E) \xrightarrow{\cong} F(X, A) \xrightarrow{\cong} F(X, A)$
 $\uparrow F(\text{incl})$

Remark (4) $\iff X = A^{\circ} \cup B^{\circ}$, $\text{incl} : (B, A \cap B) \rightarrow (X, A)$

then $F(\text{incl}) : F(B, A \cap B) \cong F(X, A)$

Pf $B = X \setminus E$, $E = X \setminus B$ noticing that $(X \setminus E)^{\circ} \cup A^{\circ} = X$

$E = A \setminus B$ noticing that $\bar{E} \subseteq \bar{A} \setminus B^{\circ} \subseteq A^{\circ} \setminus B^{\circ} \subseteq A^{\circ}$. \square
 $X = A^{\circ} \cup B^{\circ}$
 so $\partial B \subseteq A^{\circ}$

Rmk In (3), the topology on the disjoint union $\sqcup (X_i, A_i)$ is defined by: $U \subseteq \sqcup (X_i, A_i)$ open $\iff U \cap X_i \subseteq X_i$ open $\forall i$

FACT Theorem

a) $(F, \delta_F), (G, \delta_G)$ GHTs, $\alpha : F \rightarrow G$ a natural transformation commuting with δ_F, δ_G

such that $\alpha_{\text{point}} : F(\text{point}) \rightarrow G(\text{point})$ is an iso, then α is an iso.

b) If (F, δ_F) GHT satisfies (5) dimension: $F_*(\text{point}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

Then \exists natural iso $F \cong H_*$ (such an F is called a homology theory)

Rmk In (b) if require $F_0(\text{point}) = \mathbb{G}$ an abelian group (instead of \mathbb{Z}) $\implies F(X, A) \cong H_*(X, A; \mathbb{G})$ = (homology with coefficients in \mathbb{G}) \leftarrow later in course

9. COHOMOLOGY

(C_*, ∂_*) chain complex s.t. C_n free \mathbb{Z} -module $\leftarrow C_* \cong \bigoplus_{\mathbb{Z}}$

Def **n-cochains**

$$C^n = \text{Hom}(C_n, \mathbb{Z})$$

boundary map

$$\partial^n : C^n \rightarrow C^{n+1}$$

(this is the dual of ∂)

$$\partial^n(\phi) = \phi \circ \partial_{n+1}$$

Notice ∂^* is degree +1 map (not -1)

$$H^m(C_*, \partial_*) = \text{Ker} \frac{\partial^m}{\text{Im } \partial^{m-1}} \leftarrow \begin{matrix} \text{cocycles} \\ \text{coboundaries} \end{matrix}$$

Remark If we use negative grading, (C^{-*}, ∂^{-*}) is a chain complex with homology so many results from H_* carry over to H^* . So abusively write

$$H^*(C_*, \partial_*) = H_*(\text{Hom}(C_*, \mathbb{Z}), \partial^*)$$

Warning A cochain $\varphi \in C^*$ takes values $\varphi(c) \in \mathbb{Z}$ on chains $c \in C_*$. However the cohomology class $\alpha = [\varphi] \in H^*$ does not have a well-defined value on c : $[\varphi] = [\varphi + \partial^*(\psi)]$ and $(\varphi + \partial^*(\psi))(c) = \varphi(c) + \psi(\partial_* c)$. If c is a cycle, so $\partial_* c = 0$ then $\alpha(c) = \varphi(c)$ is well-defined, so \exists pairing $H^* \times H_* \rightarrow \mathbb{Z}$

Remarks about dualisation

$\text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$ generated by projection maps $\pi_i(x_1, \dots, x_n) = x_i$

$$\alpha : \mathbb{Z}^n \rightarrow \mathbb{Z}^m \Rightarrow \text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \xleftarrow{\alpha^*} \text{Hom}(\mathbb{Z}^m, \mathbb{Z}) \quad \alpha^* \phi = \phi \circ \alpha$$

$$\begin{matrix} \mathbb{Z}^n & \xleftarrow{\text{transpose}(A)} & \mathbb{Z}^m \\ \uparrow & & \uparrow \\ \text{mxn matrix} & & \end{matrix}$$

Def X space \Rightarrow **singular cohomology** $H^*(X) = H^*(C^*(X), \partial^*)$

similarly define H_{Δ}^* , H_{CW}^*

Example $\mathbb{RP}^3 : C_*^{CW}(\mathbb{RP}^3) : 0 \rightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \rightarrow 0$

dualise : $C_*^*(\mathbb{RP}^3) : 0 \leftarrow \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \rightarrow 0$

$$H^*(\mathbb{RP}^3) \cong H_{CW}^*(\mathbb{RP}^3) = \begin{cases} \mathbb{Z} & * = 0, 3 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases} \leftarrow \text{notice } H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2 \text{ has moved to grading 2.}$$

Functoriality

$f : X \rightarrow Y \Rightarrow f_* : C_* X \rightarrow C_* Y$ \leftarrow called **pull-back**

$\Rightarrow C^* X \xleftarrow{f^*} C^* Y$ is dual : $f^* \phi = \phi \circ f_*$

Lemma f^* is a **cochain map** (meaning $\partial^* \circ f^* = f^* \circ \partial^*$)

$$\Rightarrow f^* : H^* Y \rightarrow H^* X$$

Pf $(\partial^* \circ f^*)(\phi) = \partial^*(\phi \circ f_*)$

$$= (\phi \circ f_*) \circ \partial$$

$$= (\phi \circ \partial) \circ f_* \quad \text{as } f_* \text{ chain map}$$

$$= f_* \circ (\phi \circ \partial)$$

$$= f_* \circ (\partial^* \phi)$$

$$= (f_* \circ \partial^*)(\phi)$$

Properties $\cdot \text{id}^* = \text{id}$

$\cdot (f \circ g)^* = g^* \circ f^*$ notice order!

$$\Rightarrow H^* : \text{Top} \rightarrow \text{Graded AbGps}$$

Contravariant functor

Exercise $H^0(X) = \prod_{\text{Path } X} \mathbb{Z}$ where $\text{Path } X = \{\text{path-components of } X\}$

Homotopy invariance

Lemma $f_*, g_* : C_* \xrightarrow{\text{free}} C_*$ chain hpic $\Rightarrow f^* = g^* : H^* \tilde{C} \rightarrow H^* \tilde{C}$

Pf $f_* - g_* = \tilde{\partial} \circ h + h \circ \partial$ same $h : C_* \rightarrow \tilde{C}_*[1]$

$f^* - g^* = h^* \circ \tilde{\partial}^* + \partial^* \circ h^*$ for dual $h^* : \tilde{C}^* \rightarrow C^*[-1]$.
(notice degree -1, not +1) \square

Def h^* called **cochain homotopy**

Cor $f \simeq g : X \rightarrow Y \Rightarrow f^* = g^* : H^* Y \rightarrow H^* X \quad \square$

unital: $(\partial 1)(\sigma) = 1(\sigma|_{[e_0]}) - 1(\sigma|_{[e_1]}) = 1 - 1 = 0$

$(1 \cup \psi)(\sigma) = 1(\sigma|_{[e_0]}) + \psi(\sigma|_{[e_0, \dots, e_n]}) = \psi(\sigma)$ ($\psi|_{1} = \phi$ similar)

graded-comm. sketch proof: \leftarrow non-examinable

Let $r: C_n(X) \rightarrow C_n(X)$, $r(\sigma) = \epsilon_n \bar{\sigma}$ where: $\epsilon_n = (-1)^{\frac{n(n+1)}{2}}$

and $\bar{\sigma} [v_0, \dots, v_n] = \sigma [v_n, \dots, v_0]$ \leftarrow reverse order of vertices

(idea: each transposition is a reflection in hyperplane, so reverses orientation of simplex, so insert ϵ_n to compensate)

one checks: \bullet r chain map

$r^* \psi \cup r^* \psi = r^*(\psi \cup \psi)$

\leftarrow differ by $(-1)^{kl}$

\bullet $r \simeq id$ so can drop $r^* = id$ on cohomology

$(r-id = \partial \partial + \partial P$ with v_i, w_i like for prism operator)

$(P\sigma = \sum (-1)^i \epsilon_{n-i} (\sigma \circ \pi_i) [v_0, \dots, v_i, w_0, \dots, w_i])$

projection $\Delta^n \times I \xrightarrow{\pi} \Delta^n$

Naturality of cup product

Lemma $f: X \rightarrow Y \implies f^*: H^* Y \rightarrow H^* X$ hom of unital rings

Pf $f^*(\psi \cup \psi)(\sigma) = (f_* \psi) \cup (f_* \psi)(\sigma)$

$= \psi(f_* \sigma|_{[e_0, \dots, e_k]}) \cup \psi(f_* \sigma|_{[e_k, \dots, e_n]})$

$= ((\psi \circ f_*) \cup (\psi \circ f_*))(\sigma)$

$= (f^* \psi \cup f^* \psi)(\sigma)$

unital: $f^*(1) = 1 \circ f_* = 1$ \square

UPSHOT

$H^*: \text{Top} \rightarrow \{\text{Graded-commutative unital rings}\}$

contravariant functor.

Warning An (iso)morphism $H^*(Y) \rightarrow H^*(X)$ of groups will also preserve the ring structure if f^* is induced by a map of spaces $X \rightarrow Y$ (by above Lemma).

\implies Cor The excision theorem iso on cohomology is an iso of rings. However the connecting hom in MV. or LES cannot possibly be a ring hom since it raises gradings by 1 ($\implies \delta(a \cup b)$ and $\delta(a) \cup \delta(b)$ have different gradings!)

Example $H^1(T^2) \times H^1(T^2) \xrightarrow{\cup} H^2(T^2)$ is bilinear form $\mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{Z}$ with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Pf By the Useful Trick, it is enough to work with H^* instead of H^* .



Δ -complex structure for T^2 : $0 \rightarrow \mathbb{Z}^2 \xrightarrow{\text{gens: } D_1, D_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \rightarrow 0$

dualize: $0 \leftarrow \mathbb{Z}^2 \xleftarrow{\text{gens: } D_1^*, D_2^*} \mathbb{Z}^3 \xleftarrow{0} \mathbb{Z} \rightarrow 0$

for basis a, b, c \leftarrow dual basis a^*, b^*, c^* (e.g. $a^*(a)=1, a^*(b)=0, a^*(c)=0$)

$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{gens: } D_1, D_2} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{gens: } D_1^*, D_2^*} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

Claim $A \cup B = D_1^*$

Pf $\Delta^2 \xrightarrow{D_1} (A \cup B)(D_1) = A(D_1|_{[e_0, e_1]}) \cup B(D_1|_{[e_1, e_2]}) = A(\alpha) \cup B(b) = 1 \cdot 1 = 1$

$\Delta^2 \xrightarrow{D_2} (A \cup B)(D_2) = A(b) \cup B(\alpha) = 0$ \square

Graded-comm. $\implies B \cup A = -D_1^*$, $A \cup A = (-1)^{|A|} A \cup A$ so $= 0$, similarly $B \cup B = 0$.

Recall that to specify a cochain in $C_k^*(X)$ one needs to specify values on all generators of $C_k^*(X)$ so not just on generators of $H_k^*(X)$ (e.g. above A and α agree on gens a, b of $H_1^*(T^2)$ but disagree on $c \in C_1^*(T^2)$, note α is 1-cochain $\in C_1^*(T^2)$ but is not a 1-cycle. Some (but not all) k -cochains ψ can be specified by drawing a "nice" $(n-k)$ -dimensional subspace $\Sigma \subseteq X$ and defining $\psi(c) = \#(\text{times } \Sigma \text{ intersects } c)$ for all $c \in C_k^*(X)$ where one must explain with what sign \pm an intersection point is counted and one has ensured that Σ intersects the generators of $C_k^*(X)$ in a finite # points.

\leftarrow We obtained the curves α, β by "pushing off" the curves a, b respectively away from themselves. Note the endpoints of α (and β) are the same so it is a loop (hence a 1-cycle in T^2). \implies get 1-cochains $\psi_a, \psi_b \in C_1^*(T^2)$:

$\psi_a^*(c) = \# \alpha$ intersects c counted with orientation signs: $c \uparrow + 1 \rightarrow \alpha \downarrow -1$

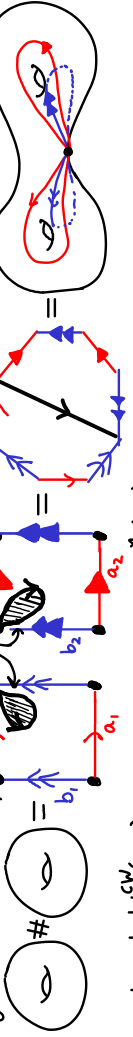
$\psi_b^*(c) = \# \beta$ intersects c counted with orientation signs: $c \uparrow + 1 \rightarrow \beta \downarrow -1$

Notice $\psi_a(\alpha) = 0, \psi_a(\beta) = 1, \psi_a(c) = 1$ so $\psi_a = B$. Similarly, $\psi_b = -A$.

Fact Since T^2 is an orientable manifold, $\psi_a \cup \psi_b = (\alpha \cdot \beta)$ vol where vol is a generator of $H^2(T^2)$. Later in the course: vol is the "Poincaré dual" of the point class, and corresponds to the dual of the oriented sum of the top simplices. Above: vol = D_1^* and $\psi_a \cup \psi_b = B \cup (-A) = A \cup B = (\alpha \cdot \beta)$ vol = vol = D_1^* .

Defining intersection numbers rigorously is tricky, even when using smooth chains. One can calculate $\psi_\Sigma(c)$ on a cycle c by first deforming c to a smooth homologous cycle \tilde{c} which is "transverse" to Σ , and then we count intersection points $\Sigma \cap \tilde{c}$ ("with orientation signs"). The fact that we consider the intersection number $a \cdot a = 0$ is because we can push a off itself.

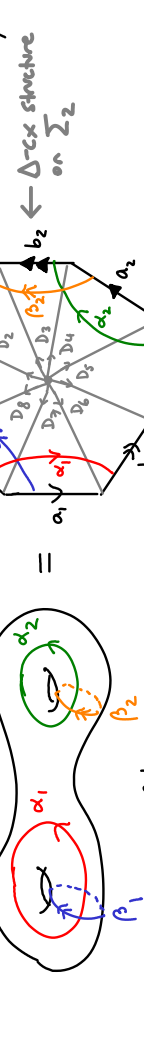
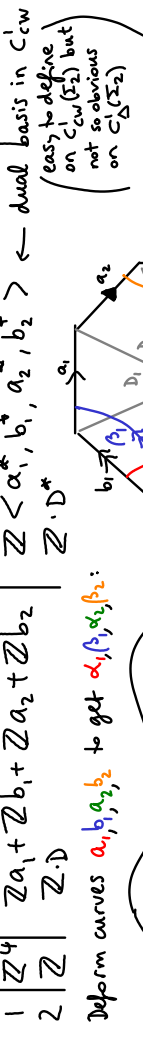
Exercise Σ_2 (genus 2 surface)



remove balls & glue handles

\mathbb{Z}	$\mathbb{Z} \text{ pt}$
\mathbb{Z}^4	$\mathbb{Z}a_1 + \mathbb{Z}b_1 + \mathbb{Z}a_2 + \mathbb{Z}b_2$
\mathbb{Z}	$\mathbb{Z} \cdot D$

Deform curves a_1, b_1, a_2, b_2 to get $\alpha_1, \beta_1, \alpha_2, \beta_2$



Then notice for $c \in C_1^{CW}(\Sigma_2)$ signed count
 $a_i^*(c) = -\#(\beta_i \text{ intersects } c)$ so can extend this to a definition of $a_i^*, b_i^* \in C_1^*(\Sigma_2)$ by allowing $c \in C_1^*(\Sigma_2)$.
 $b_i^*(c) = \#(\alpha_i \text{ intersects } c)$ Check that a_i^*, b_i^* are 1-cocycles in $C_1^*(\Sigma_2)$.
 Exercise: $a_i^* \cup b_j^* = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ (key idea: a loop has total signed intersection $\neq 0$ with the boundary of a triangle)

so same as geometric intersection numbers of corresponding curves.
 $a_i^* \cup a_j^* = b_i^* \cup b_j^* = 0$
 (orientation signs: $D = -D_1 - D_2 + D_3 + D_4 - D_5 - D_6$ using + if outer edge is oriented anticlockwise)

Cultural Rmk on general theory (Intersection Theory/Differential Topology)

M^m oriented m -manifold $\Rightarrow H_n(N) \xrightarrow{\text{incl}} H_n(M) \xleftarrow{\text{see later in course}}$
 $N^n \subseteq M^m$ oriented n -dim submfld Compact \Rightarrow generator $[N] \xrightarrow{\text{with signs}}$ $[M]$
 N, M also smooth (see Differential Geometry course) $\Rightarrow \omega_N \in H^{m-n}(M)$ counts # intersections with N

$n_1, n_2 \in \mathbb{Z}$ \Rightarrow compact oriented smooth submflds $N_1, N_2 \subseteq M$ and M connected, then the tangent spaces to N_1, N_2 at p span the tangent space to M at p .
 (can always "homotope" N_1 (or N_2) to achieve transversality, and class ω_N does not change if homotope)
 $N_1 \cup N_2 = \omega_{N_1} \cup \omega_{N_2} = \omega_{N_1 \cup N_2}$
 $N_1 \cup N_2$ is a compact orientable mfd of $\dim = n_1 + n_2 - m$
 In particular if $n_1 + n_2 = m$, and M connected, then $H^m(M) \cong \mathbb{Z}$ s.t. $\omega_M \cup \omega_{N_1} \cup \omega_{N_2} \mapsto \#(N_1 \cap N_2) \in \mathbb{Z}$.
 In non-orientable case, this all holds if work over $\mathbb{Z}/2$.
 (tang. space means the best vector space approximation at p determined by the local smooth coordinates) (if $n_1, n_2 = m$, get $N_1 \cup N_2$ is sum of \mathbb{Z} points. Sign compares oriented bases of tang. space: $T_M = T_{N_1} \oplus T_{N_2}$.)

Fact (Thom 1954)
 Not all $a \in H^j(M)$ arise as ω_N for connected compact oriented codim $= j$ smooth submfld N .
 But $\exists N \in N$ s.t. N does arise. They do arise for $H^*(M; \mathbb{Q}), H^*(M; \mathbb{R}), H^*(M; \mathbb{Z}/2)$

II. KÜNNETH FORMULA AND PRODUCT SPACES

Algebra: tensor products

R ring (comm. with 1) e.g. abelian groups = \mathbb{Z} -mods
 vector spaces/ \mathbb{F} = \mathbb{F} -mods
 Def A, B R -modules \Rightarrow Tensor product is R -module

$A \otimes_R B = \langle (a, b) : a \in A, b \in B \rangle / \text{relations of bilinearity \& rescaling}$

(or $A \otimes B$) R -mod generated write $a \otimes b$ for its class

bilinearity: $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$
 $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$

rescaling: $r \cdot (a \otimes b) = (ra) \otimes b = a \otimes (rb) \quad \forall r \in R$

So general element looks like $\sum a_k \otimes b_k$ (finite sum) \leftarrow NOT UNIQUE!
 Don't confuse with $A \times B$: e.g. $0 \otimes b = 0 \quad \forall b$

Rmk Can define $A \otimes_R B$ also by a universal property: for all R -mods C ,

$$\text{Hom}_R(A \otimes_R B; C) \xrightarrow{\text{natural}} \{R\text{-bilinear maps } A \times B \rightarrow C\}$$

Using above description of $A \otimes B: \varphi \mapsto (A \times B \rightarrow C, (a, b) \mapsto \varphi(a \otimes b))$

Example $(R = \mathbb{F}) \quad V, W$ v.s. \mathbb{F} $\Rightarrow V \otimes W$ v.s. \mathbb{F} basis $v_i \otimes w_j$
 basis v_i \leftarrow basis w_j \leftarrow $\dim_{\mathbb{F}} V \otimes W = \dim V \cdot \dim W$

Exercise V, W finite $\dim_{\mathbb{F}} \Rightarrow V^* \otimes W \cong \text{Hom}_{\mathbb{F}}(V, W)$
 Hint $f: V \rightarrow \mathbb{F}, u \in W, f \otimes w \mapsto (V \rightarrow W, v \mapsto f(v) \cdot w)$

Examples $(R = \mathbb{Z}) \cdot \mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{nm}$
 $\cdot \mathbb{Z}/n \otimes \mathbb{Z} \cong \mathbb{Z}/n \leftarrow \sum a_i \otimes b_i = (\sum a_i b_i) \otimes 1$
 $\cdot \mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0 \leftarrow 1 \otimes x = 3 \otimes x = 1 \otimes 3x = 0$
 $\cdot \mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong \mathbb{Z}/2 \leftarrow \{1 \otimes 1 = 3 \otimes 1 = 1 \otimes 3 = 0\}$
 $\cdot A \otimes B \cong B \otimes A$

Examples $A \otimes B \cong B \otimes A$
 $\cdot (\bigoplus_i A_i) \otimes (\bigoplus_j B_j) \cong \bigoplus_{i,j} (A_i \otimes B_j)$ hence now know $A \otimes B$ for any $f.g.$ R -mods A, B .
 $\cdot A \otimes R \cong A$ (so " \otimes_R does nothing")
 $\cdot A \otimes R/d \cong A/d \cdot A$
 for example $\mathbb{Z}/2 \otimes \mathbb{Z}/4 \cong (\mathbb{Z}/4)/2 \cdot \mathbb{Z}/4 \cong \mathbb{Z}/2 \leftarrow \left(\frac{\text{Rmk } (\mathbb{Z}/m)/m \cdot \mathbb{Z}/n}{\cong \mathbb{Z}/\text{gcd}(m,n)} \right)$

More generally: $\left\{ \begin{array}{l} R/I \otimes R/J \cong R/(I+J) \text{ for ideals } I, J \subseteq R. \\ A \otimes_R R/J \cong A/J \cdot A \end{array} \right.$

Warning $\otimes A$ often not an exact functor, i.e. does not preserve exact sequences
 indeed it can ruin injectivity: $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$

Fact $\cdot \otimes \mathbb{Z}$ and $\otimes \mathbb{R}$ are exact functors on \mathbb{Z} -mods
 ← More generally $\otimes \text{Frac}(R)$ is exact on R -mods
 where $\text{Frac } R$ is fraction field, and R is an integral domain
 // Localization is an exact functor!!

example A f.g. \mathbb{Z} -mod $\Rightarrow A \cong \mathbb{Z} \oplus \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_k$ some $d_i \neq 0$
 $\Rightarrow A \otimes \mathbb{Q} \cong \mathbb{Q}^r \Rightarrow r = \text{rank } A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$

Corollary Rank-nullity thm holds for \mathbb{Z} -modules if use rank instead of dim.
 Pf $0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$ exact $\Rightarrow 0 \rightarrow A \otimes \mathbb{Q} \rightarrow B \otimes \mathbb{Q} \rightarrow C \otimes \mathbb{Q} \rightarrow 0$ exact
 here $\lim_{\leftarrow} \Rightarrow \dim(C \otimes \mathbb{Q}) + \dim(A \otimes \mathbb{Q}) = \dim(B \otimes \mathbb{Q})$. \square
 rank-nullity for \mathbb{Q} -vector spaces.

Tensor product of chain cxes
 C_*, \tilde{C}_* chain cxes $\Rightarrow (C_* \otimes \tilde{C}_*)_n = \bigoplus_{i+j=n} C_i \otimes \tilde{C}_j$
 of R -mods

$\partial(x \otimes y) = (\partial x) \otimes y + (-1)^{\deg x} x \otimes \partial y$
 Think of ∂ as an operator of $\deg = -1$ acting from left
 since ∂ "jumps over x "
 get $(-1)^{\deg x} \cdot \deg x$

Exercise $\partial \circ \partial = 0$ ← would fail without sign
 recall $\mathbb{Z}_k = \ker \partial = \text{cycles}$
 $B_k = \text{im } \partial = \text{boundaries}$
 $Z_i \otimes \tilde{Z}_j \subseteq Z_{i+j}(C_* \otimes \tilde{C}_*)$ and $B_i \otimes \tilde{B}_j \subseteq B_{i+j}(C_* \otimes \tilde{C}_*)$

Cor \exists natural maps
 $H_i(C_*) \otimes H_j(\tilde{C}_*) \rightarrow H_{i+j}(C_* \otimes \tilde{C}_*)$
 $\sum [c_k] \otimes [\tilde{c}_k] \mapsto \sum [c \otimes \tilde{c}_k]$

FACT: Algebraic Künneth Thm
 $C_*, H_*(C_*)$ f.g. free R -mods (no assumption on \tilde{C}_*)
 $\Rightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(\tilde{C}_*) \cong H_n(C_* \otimes \tilde{C}_*)$ via PID (principal ideal domain)

Algebra: Euler characteristic
 C finitely generated graded abelian gr (so \mathbb{Z} -mod) | more generally: R -mod for PID R
 Def Euler characteristic $\chi(C) = \sum (-1)^i \text{rank } C_i$

Example/Motivation X finite CW-cx then take $C = C_*^{CW}(X)$ to get
 $\chi(X) = \#(0\text{-cells}) - \#(1\text{-cells}) + \#(2\text{-cells}) - \dots$

Lemma If C_* f.g. chain cx $\Rightarrow \chi(C_*) = \chi(H_*(C_*))$ ($= \sum (-1)^i \text{rank } H_i(C_*)$)

Pf Abbreviate $|C_i| = \text{rank } C_i$ ($= \dim_{\mathbb{Q}}(C_i \otimes \mathbb{Q})$)
 By previous Corollary about rank-nullity:
 $0 \rightarrow Z_i \rightarrow C_i \xrightarrow{\partial} B_{i-1} \rightarrow 0 \Rightarrow |C_i| = |Z_i| + |B_{i-1}|$
 $0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0 \Rightarrow |H_i| = |Z_i| - |B_i|$
 $\Rightarrow \chi(C_*) - \chi(H_*) = \sum (-1)^i |B_{i-1}| + \sum (-1)^i |B_i| = \sum (-1)^i (|B_{i-1}| + |B_i|) = 0$. \square

Cor X space $\Rightarrow \chi(X) := \sum (-1)^i \text{rank } H_i(X)$
 $= \sum (-1)^i \text{rank } C_i(X)$
 ← if finite rank $H_*(X)$
 ← if finite rank $C_*(X)$

So $\chi(X)$ is invariant up to hty equivalence! Example $\chi(\text{platonics}) = \chi(S^2) = 2$
 Technical Remark The CW-topology agrees with the product topology if X, Y have countably many cells, or X, Y are locally compact (see Appendix inatcher)

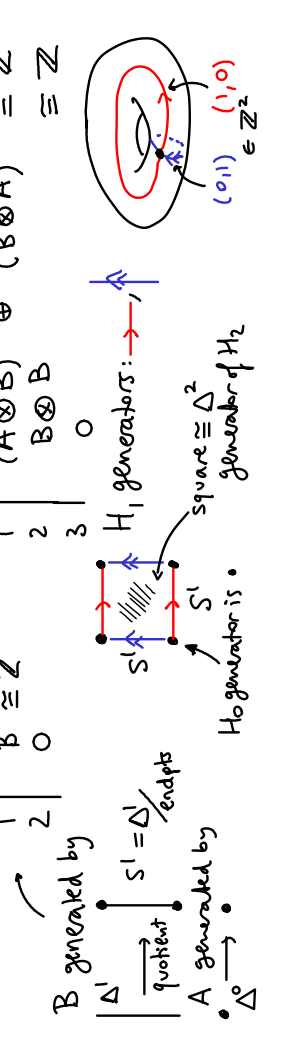
Product spaces
 X, Y CW-cxes $\Rightarrow X \times Y$ CW-cx with cells $e_\alpha \times e_\beta$ attaching maps
 $\partial(D_\alpha^i \times D_\beta^j) = (\partial D_\alpha^i) \times D_\beta^j \cup D_\alpha^i \times (\partial D_\beta^j)$
 $\downarrow \text{id}_X \times \text{id}_Y$
 $X^{i-1} \times Y^j \cup X^i \times Y^{j-1}$
 $(X \times Y)^{i+j-1}$

Cor $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$
 \forall finite CW-cxes X, Y

Pf $\sum (-1)^k \text{rank } H_k^{CW}(X \times Y) = \sum_k \sum_{i+j=k} (-1)^{i+j} \text{rank } C_i^{CW}(X) \cdot \text{rank } C_j^{CW}(Y)$
 $= \sum (-1)^k \text{rank } C_k^{CW}(X \times Y) = (d e_i^i \times e_j^j) = (-1)^i e_i^i \times (d e_j^j)$

Lemma $d(e_i^i \times e_j^j) = (d e_i^i) \times e_j^j + (-1)^i e_i^i \times (d e_j^j)$
 hence $C_*^{CW}(X \times Y) \cong C_*^{CW}(X) \otimes C_*^{CW}(Y)$
 (proof later)

Hence if $H_*(Y)$ free then by Künneth $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.
 Example $H_*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \otimes H_*(S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

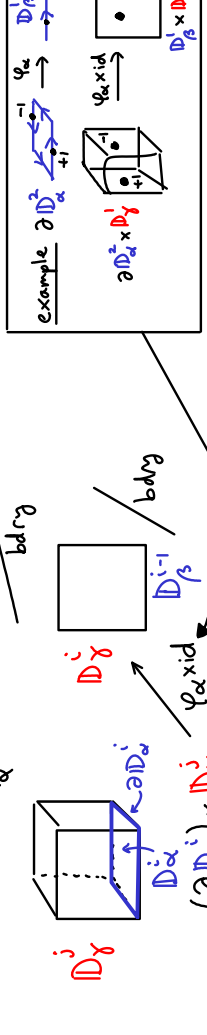


Sketch Pf. $(\partial D_\alpha^i) \times D_\beta^j \xrightarrow{\varphi_\alpha \times \text{id}} X^{i-1} \times Y^j \rightarrow X^{i-1} \times Y^j$

This proof is Non-examinable

$X^{i-1} = X^{i-2} \cup (D_\beta^{i-1} \cup \dots)$
 $Y^j = Y^{j-1} \cup (D_\beta^j \cup \dots)$
 $X^{i-1} \times Y^j = X^{i-2} \times Y^j \cup (D_\beta^{i-1} \times D_\beta^j \cup \dots)$
 $\Rightarrow \textcircled{\star} = (D_\beta^{i-1} \times D_\beta^j \cup \dots) / \text{boundaries}$
 $= D_\beta^{i-1} \times D_\beta^j / \partial(D_\beta^{i-1} \times D_\beta^j)$

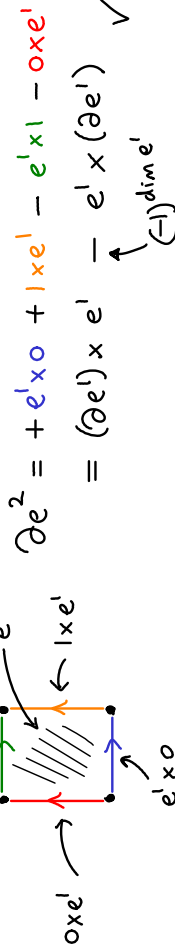
$(\partial D_\alpha^i) \times D_\beta^j \xrightarrow{\varphi_\alpha \times \text{id}} D_\beta^{i-1} \times D_\beta^j / \text{bdry}$
 $\xrightarrow{\varphi_\alpha \times \text{id}} D_\beta^{i-1} \times D_\beta^j / \text{bdry}$



By considering local degrees of "we get" degree = $d_\alpha d_\beta$ so contribution $(d_\alpha d_\beta) \times e_\beta$
 [Remark This step is actually quite tricky, see Hatcher Chapter 3.B (it requires homotopying maps to make them cellular). The cleanest argument may be to use smooth maps: one shows that continuous maps between discs can be homotoped a little to make them smooth, and that for smooth maps almost any point in the target has finitely many preimages s.t. the local maps are local homeomorphisms (so the local degrees are ± 1). If φ_α has been homotoped in this way, then $(\varphi_\alpha \times \text{id})^{-1}(y, t) = \varphi_\alpha^{-1}(y) \times \{t\}$ has the same # preimages as φ_α and local degrees ± 1 like φ_α (i.e. problem reduces to discussing orientation signs).]

Similarly $D_\alpha^i \times \partial D_\beta^j \xrightarrow{\text{id} \times \varphi_\beta} D_\alpha^i \times D_\beta^{j-1} / \text{bdry}$
 $\Rightarrow \text{degree } (-1)^i d_\alpha d_\beta$
 so get $(-1)^i e_\alpha^i \times d_\beta^j$
 The $(-1)^i$ is caused by orientations:
 could reorder factors: $D_\alpha^i \times D_\beta^j \cong D_\beta^j \times D_\alpha^i$ by $(\text{Id}_i \circ)$
 whose $\det = (-1)^{ij}$. Then $\partial D_\beta^j \times D_\alpha^i \rightarrow D_\beta^{j-1} \times D_\alpha^i / \text{bdry}$ gives degree $d_\beta d_\alpha$.
 Swap factors $D_\beta^{j-1} \times D_\alpha^i / \text{bdry} \rightarrow (\text{Id}_{j-1} \circ)$, $\det = (-1)^{(j-1)i}$. Total sign = $(-1)^{ij}$.

Example Recall after definition of H_*^{CW} we had example $I \times I$:
 arrows here tell us how we map $[-1, 1] \rightarrow \text{edge}$ (so orientation)



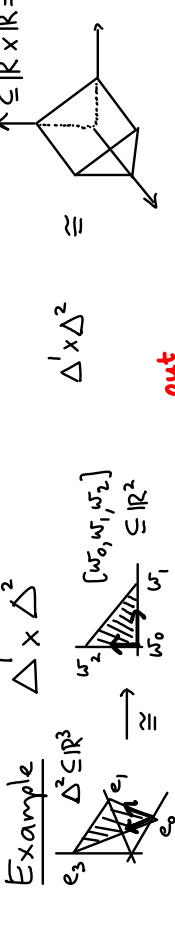
A further comment on orientation sign $(-1)^i$

$D^i \times D^j \cong \Delta^i \times \Delta^j \cong [v_0, \dots, v_i] \times [w_0, \dots, w_j]$
 viewed in $\mathbb{R}^i \times \mathbb{R}^j$
 project $\mathbb{R}^{i+j} \rightarrow \mathbb{R}^i$
 $(t_0, \dots, t_i) \mapsto (t_0, \dots, t_i)$

$\partial(D^i \times D^j) \cong \partial \Delta^i \times \Delta^j \cup \Delta^i \times \partial \Delta^j$
 $\cong \sum_k (-1)^k [v_0, \dots, \hat{v}_k, \dots, v_i] \times [w_0, \dots, w_j]$
 $\cong \sum_k (-1)^k [w_0, \dots, \hat{w}_k, \dots, w_j]$

would be correct orientation sign for basis $w_1 - w_0, \dots, w_k - w_{k-1}, \dots, w_j - w_0$ but actually we have $[w_0, \dots, w_k, \dots, w_j] \subseteq \mathbb{R}^i \times \mathbb{R}^j$
 and $(-1)^k$ is the orientation sign for the basis $v_1 - v_0, \dots, v_i - v_0, w_1 - w_0, \dots, w_j - w_0$
 for the hyperplane in \mathbb{R}^{i+j+1} containing

\Rightarrow need $(-1)^i$ to fix orientation sign.



Example $\Delta^1 \times \Delta^2 \cong \Delta^1 \times \Delta^2$
 $[v_0, v_1] \times [w_0, w_1, w_2]$
 out $w_2 - w_1$ is **outward normal**
 out $w_2 - w_1$ is **positive** \mathbb{R}^2 -basis
 out $w_2 - w_1$ is **negative** \mathbb{R}^3 -basis
 differ due to $(-1)^i, i=1$.

Projections $X \times Y \xrightarrow{p_X} X$
 $\xrightarrow{p_Y} Y$

FACT: no conditions on X

Künneth Theorem If $H_n(Y)$ finitely generated, free $\forall n$

$$H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y)$$

$$H^n(X \times Y) \cong \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)$$

eg. $Y \cong \text{finite CW cplx}$
 automatic if use field coefficients

Recall for cellular homology this on generators is: (chain level)

$$e_i \times e_j \mapsto e_i \otimes e_j$$

This is hom of rings if use following product $(a \otimes b) \cdot (\tilde{a} \otimes \tilde{b}) = (-1)^{|b||\tilde{a}|} (a \cup \tilde{a}) \otimes (b \cup \tilde{b})$

think of it as 'exchanging order of b, \tilde{a} '

An indirect proof the Thm is to write down two generalised cohomology theories $F(X, A) = H^*(X, A) \otimes H^*(Y)$ and $G(X, A) = H^*(X \times Y, A \times Y)$, and consider the natural transformation $\alpha: F \rightarrow G$ given by \otimes , notice for $X = \text{pt}$ both F, G give $H^*(Y)$.

Example $X = S^n, Y = S^m, n \neq m$

$$H_*(S^n \times S^m) \cong \begin{cases} \mathbb{Z} & * = 0, n, m, n+m \\ 0 & \text{else} \end{cases}$$

$$H_*(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & * = 0, 2n \\ \mathbb{Z}^2 & * = n \leftarrow \text{gens: } a_n^{(1)}, a_n^{(2)} \end{cases}$$

Cor $H^k(T^n) \cong \wedge^k [x_1, \dots, x_n]$

where $x_i = p_i^*(\text{gen. of } H^1(S^1))$
 $\deg x_i = 1$
 $\{x_{i_1} \wedge \dots \wedge x_{i_k} : i_1 < \dots < i_k\}$
 so $\text{rank} = \binom{n}{k}$
 product is " \wedge " using the rule $x_i \wedge x_j = -x_j \wedge x_i$ (compare graded-commutativity of cup product)

pf Künneth & induction $(T^n = T^{n-1} \times S^1)$ \square

FACT cup product equals composition

Idea: \exists natural chain maps $C_*(X) \otimes C_*(Y) \cong C_*(X \times Y)$ whose composites are chain homotopic to identity maps (Eilenberg-Zilber Thm)
 Then get $C^*(X) \otimes C^*(Y) \rightarrow \text{dual}(C_*(X) \otimes C_*(Y)) \cong \text{dual}(C_*(X \times Y)) \cong C^*(X \times Y) \xrightarrow{\alpha} C^*(X \times Y)$
 $\alpha \otimes \beta \mapsto (\sigma \otimes \tau \mapsto \alpha(\sigma) \wedge \beta(\tau))$ For $X=Y$ this is the Künneth map.

12. UNIVERSAL COEFFICIENTS THEOREM

Proof is non-examinable. For (C_*, ∂_*) chain cplx:

$$\Rightarrow 0 \rightarrow Z_* = \ker \partial_* \xrightarrow{\text{incl}} C_* \xrightarrow{\partial_*} B_{*-1} \rightarrow 0 \text{ is SES}$$

FACT: Submodules of a free \mathbb{Z} -module are free

Rmk The same holds for R-mods if R is PID

Assume C_* free \mathbb{Z} -mod

$$\begin{array}{ccccccc} 0 & \leftarrow & Z^* & \xleftarrow{\text{incl}^*} & C^* & \xleftarrow{\partial^{*-1}} & B^{*-1} \leftarrow 0 \\ 0 & \leftarrow & Z^n & \xleftarrow{\partial} & C^n & \xleftarrow{\partial} & B^{n-1} \leftarrow 0 \\ 0 & \leftarrow & Z^{n-1} & \xleftarrow{\partial} & C^{n-1} & \xleftarrow{\partial} & B^{n-2} \leftarrow 0 \end{array}$$

Connecting map $\delta: Z^{n-1} \rightarrow B^n$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \phi|_{Z^*} = \phi & \leftarrow & \partial^* \psi \\ \uparrow & & \uparrow \\ \phi|_{Z^*} = \phi & \leftarrow & \partial^* \psi \end{array}$$

$$\begin{array}{ccccccc} 0 & \leftarrow & \ker \delta^n & \leftarrow & H^n C & \xleftarrow{\partial} & B^{n-1} \xleftarrow{\delta} Z^{n-1} \\ \Rightarrow & & \Rightarrow & & \Rightarrow & & \Rightarrow \\ \ker \delta^n = \{ \phi \in Z^n : \phi(B_n) = 0 \} & \Rightarrow & \text{so: } \phi: Z_n \rightarrow Z & & & & \\ & = & \text{Hom}(H_n(C_*), \mathbb{Z}) & & & & \end{array}$$

Universal Coefficients Thm:

$$0 \rightarrow B^n / \text{Im } \delta^{n-1} \rightarrow H^n(C) \rightarrow \text{Hom}(H_n(C_*), \mathbb{Z}) \rightarrow 0 \text{ is SES}$$

and SES splits (but not naturally): $B^n / \text{Im } \delta^{n-1} \xrightarrow{\beta} H^n(C)$
 $\Rightarrow H^n(C) \cong \text{Hom}(H_n(C), \mathbb{Z}) \oplus \text{Ext}^1(H_{n-1}(C), \mathbb{Z})$

MOTIVATION: What is difference between $H^*(\text{Hom}(C_*, \mathbb{Z}))$ and $\text{Hom}(H_n(C_*), \mathbb{Z})$
 Similarly: $H_n(C_* \otimes G)$ vs. $H_n(C_*) \otimes G$.

\mathbb{Z} -module \equiv abelian gr free means: $\bigoplus_{\text{indexing set}} \mathbb{Z}$

PID = principal ideal domain = integral domain R s.t. every ideal = $R \cdot a$ some a

note: $\text{incl}^* = \text{restrict to } Z^*$ since $\text{incl}^* \phi: Z^* \rightarrow B^*$
 Rmk Although $\partial^* = 0: B^* \rightarrow B^{*+1}$ the map $\partial^*: B^{n-1} \rightarrow C^n$ need not = 0
 $\psi: B_{n-1} \rightarrow Z$
 $\Rightarrow \partial^* \psi = \psi \circ \partial: C_n \rightarrow B_{n-1} \xrightarrow{\psi} Z$

$$\delta(\phi) = \phi|_{B_*}$$

recall just pick preimages under ∂^* of a basis for B_*

see next Lemma

evaluation of a cohomology class on cycles

$S^* \circ \partial^* = \text{id}$ since $\partial \circ S = \text{id}$ ($\Rightarrow \text{id} = (\partial \circ S)^* = S^* \circ \partial^*$)

Lemma $\text{Ext}^1(H_{n-1}(C); \mathbb{Z}) \cong B^{n-1}/\text{Im } S^{n-1}$ canonically

Algebra background on Extension groups $\text{Ext}_R^i(M, R)$ ($= \text{Ext}_R^i(M, R)$)

our case

M R -module, R ring (comm. with 1)

$\Rightarrow \exists$ free resolution:

$$\dots \rightarrow P_2 \xrightarrow{\psi_2} P_1 \xrightarrow{\psi_1} P_0 \xrightarrow{\psi_0} M \rightarrow 0 \quad \text{exact, } P_i \text{ free } R\text{-mods}$$

(pick gens x_α for $M \Rightarrow P_0 = \bigoplus_\alpha R \xrightarrow{\psi_0} M, e_\alpha \mapsto x_\alpha$
 " " y_β for $\text{Ker } \psi_0 \Rightarrow P_1 = \bigoplus_\beta R \xrightarrow{\psi_1} \text{Ker } \psi_0, e_\beta \mapsto y_\beta$
 continue inductively)

Take $\text{Hom}(\cdot; R)$ and drop $\text{Hom}(M; R)$

$$0 \rightarrow \text{Hom}(P_0; R) \xrightarrow{\psi_1^*} \text{Hom}(P_1; R) \xrightarrow{\psi_2^*} \dots$$

Is cochain complex but not exact

\Rightarrow take cohomology groups:

Def $\text{Ext}^0(M; R) = \text{Ker } \psi_1^*$

$\text{Ext}^1(M; R) = \text{Ker } \psi_2^* / \text{Im } \psi_1^*$

Fact independent of choices P_i, ψ_i

Example 1 $\text{Ext}^0(M; R) \cong \text{Hom}(M, R)$

$$P_1 \xrightarrow{\psi_1} P_0 \xrightarrow{\psi_0} M$$

$\downarrow \phi$ descends: $m \mapsto \phi(\psi_0^{-1}m)$ will be defined since $\phi(\text{Ker } \psi_0) = 0$

Example 2 $\text{Ext}^1(M; R) = \left\{ \begin{array}{l} \phi : P_2 \xrightarrow{\psi_2} P_1 \rightarrow P_0 \\ \downarrow \phi \quad \downarrow \phi \quad \downarrow \phi \\ R \quad \quad R \quad \quad R \end{array} \right\} / \left\{ \begin{array}{l} \phi = \psi_0 \psi_1 : P_1 \xrightarrow{\psi_1} P_0 \\ \downarrow \phi \quad \downarrow \phi \\ R \quad \quad R \end{array} \right\}$

Rmk If R PID, then $\text{Ker}(P_0 \rightarrow M)$ is free (since submod of free mod P_0)
 \Rightarrow can pick $P_1 = \text{Ker}(P_0 \rightarrow M), P_k = 0$ for $k \geq 2 \Rightarrow \text{Ext}_R^k(M; R) = 0 \quad k \geq 2$

$H_{n-1}(C_*) \mathbb{Z}$ -mod ($R = \mathbb{Z}$)

$$0 \rightarrow B_{n-1} \hookrightarrow \mathbb{Z} \rightarrow H_{n-1}(C) \rightarrow 0$$

$$\begin{array}{ccc} \parallel & & \parallel \\ P_1 & & P_0 \\ \parallel & & \parallel \\ & & M \end{array}$$

$$0 \rightarrow \mathbb{Z}^{n-1} \rightarrow B^{n-1} \rightarrow 0$$

Proof of Lemma

By Example 2,

$$\text{Ext}^1(H_{n-1}(C_*); \mathbb{Z}) =$$

$$= \left\{ \begin{array}{l} 0 \rightarrow B_{n-1} \rightarrow \mathbb{Z}^{n-1} \\ \downarrow \phi \quad \downarrow \phi \\ \mathbb{Z} \quad \quad \mathbb{Z} \end{array} \right\} \text{ modulo those arising from restriction}$$

those arising from restriction

$$\left\{ \begin{array}{l} 0 \rightarrow B_{n-1} \rightarrow \mathbb{Z}^{n-1} \\ \downarrow \phi \quad \downarrow \phi \\ \mathbb{Z} \quad \quad \mathbb{Z} \end{array} \right\}$$

Thus $B^{n-1}/\text{Im } S^{n-1} \square$

(Co)homology with coefficients in a ring/field/module

Motivation

So far we had (C_*, ∂_*) chain cx of abelian groups } in graded sense
 $\Rightarrow H_*(C_*) = \text{Ker } \partial_* / \text{Im } \partial_*$ abelian group (since $\text{Ker } \partial, \text{Im } \partial$ are)
 We cannot use a chain cx of (non-abelian) groups, because
 $\text{Im } \partial_*$ need not be a normal subgroup of $\text{Ker } \partial_*$.

However, abelian groups can be thought of as \mathbb{Z} -modules,
 then given any abelian group G , define homology with coeffs in G

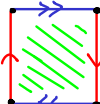
$H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$ with differential $\partial_* \otimes \text{id}$

Def X space $\Rightarrow H_*(X; G) = H_*(C_*(X) \otimes_{\mathbb{Z}} G)$

Explanation:

$C_k(X)$ free \mathbb{Z} -mod $\cong \bigoplus_{I_k} \mathbb{Z} \Rightarrow C_k(X) \otimes_{\mathbb{Z}} G \cong \bigoplus_{I_k} G$: just replace \mathbb{Z} by G (as $\mathbb{Z} \otimes \cong \cdot$)

Why care? We hope to get more/new invariants of spaces

Example $X = \mathbb{R}P^2 =$ 

$$C_*(\mathbb{R}P^2; G) = \begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline & G \oplus G \oplus G \oplus G & G \oplus G \oplus G & G \oplus G \oplus G \end{array}$$

for $G = \mathbb{Z}/2$: $0 \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial_2} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial_1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow 0$
 $\left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right) \quad \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)$

$\Rightarrow H_*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$ compare: $H_*(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & \text{else} \end{cases}$ ($G = \mathbb{Z}$ case)

Form cochain complex using $\text{Hom}_{\mathbb{Z}}(\cdot, G)$ (= group homs) in place of $\text{Hom}(\cdot, \mathbb{Z})$

$H^*(C_*; G) = H_*(\text{Hom}_{\mathbb{Z}}(C_*, G))$ with differential ∂^* : $\partial^* \phi = \phi \circ \partial_*$
 $H^*(X; G) = H_*(\text{Hom}_{\mathbb{Z}}(C_*(X), G))$ so: $H^*(C_*(X); G)$

Universal coefficients thm (Same proof using $\text{Hom}_{\mathbb{Z}}(\cdot, G)$)

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_*; G) \rightarrow H^n(C_*; G) \rightarrow \text{Hom}_{\mathbb{Z}}(H_n(C_*), G) \rightarrow 0$$

$$[\varphi] \mapsto (\varphi : H_n(C_*) \rightarrow G)$$

Example $X = \mathbb{R}P^2$, $G = \mathbb{Z}/2$, apply $\text{Hom}_{\mathbb{Z}}(\cdot, G)$ to the complex in previous example

$$0 \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial^1} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\partial^0} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \leftarrow 0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$$

compare: $H^*(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * = 1 \\ \mathbb{Z}/2 & * = 2 \\ 0 & \text{else} \end{cases}$

($G = \mathbb{Z}$ case)

Compare $\text{Hom}(H_*(\mathbb{R}P^2), \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 & * = 1 \\ 0 & * = 2 \end{cases}$

caused by $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$

Can generalise further:

$C_* =$ chain cx of ...	coefficients in:
abelian gps (\mathbb{Z} -mods)	$H_*(C_*; G) = H_*(C_* \otimes_{\mathbb{Z}} G)$
R -modules (R ring comm. with 1)	$H_*(C_*; M) = H_*(C_* \otimes_R M)$

Rmk $H_*(C; M)$ will be an R -module since $\ker \partial, \text{Im } \partial$ are (∂_* is R -linear hom by assumption)

X space $\Rightarrow C_k(X; R) = C_k(X) \otimes_{\mathbb{Z}} R \cong \bigoplus_{\mathbb{Z}} R$: just replace \mathbb{Z} by R (as $\mathbb{Z} \otimes R \cong R$)

$$\Rightarrow H_*(X; M) = H_*(C_*(X; R) \otimes_R M)$$

so just replace each \mathbb{Z} by M in $C_*(X)$

Form cochain complex using $\text{Hom}_R(\cdot, M) (= R\text{-linear homs to } M)$ in place of $\text{Hom}(\cdot, \mathbb{Z})$

$$\begin{aligned} H^*(C_*; M) &= H_*(\text{Hom}_R(C_*, M)) \\ H^*(X; M) &= H_*(\text{Hom}_R(C_*(X; R), M)) \end{aligned}$$

with differential ∂^* : $\partial^* \phi = \phi \circ \partial_*$

so: $H^*(C_*(X; R), M) \leftarrow H^*(C_*(X; R), M)$

Rmk These are R -mods. If we use $M=R$, then they are also rings via cup product

Universal Coefficient Thm For R any PID, C_* chain cx of R -mods,

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_*), M) \rightarrow H^n(C_*, M) \rightarrow \text{Hom}_R(H_n(C_*), M) \rightarrow 0$$

$\text{Ext}_{\mathbb{Z}}^1$ is SES and natural.

$B^{n-1}/\text{im } \delta^{n-1}$ working over R using homs to M

$[\varphi] \mapsto (\varphi: H_n(C_*) \rightarrow M)$

and the SES splits but the splitting is not natural.

Example $R = \mathbb{F}$ field $\Rightarrow C_*, H_*, H^*$ are vector spaces/ \mathbb{F} .

Rmk all \mathbb{F} -mods (i.e. vector spaces/ \mathbb{F}) are free \mathbb{F} -mods $\cong \bigoplus \mathbb{F} b_i$ up to iso they are determined by $\dim_{\mathbb{F}} =$ cardinality of basis.

Cor $C_* =$ chain cx of \mathbb{F} -vector spaces $\Rightarrow H^n(C_*; \mathbb{F}) \cong (H_n(C_*))^*$ dual v.s.: $\text{Hom}_{\mathbb{F}}(H_n(C_*), \mathbb{F})$

Pf Pick any basis v_i for \mathbb{F} -v.s. B_{n-1} , extend it to a basis v_i, w_j of Z_{n-1} (Cob works in ∞ dim case).

\Rightarrow can extend any \mathbb{F} -linear map $\psi: B_{n-1} \rightarrow \mathbb{F}$ to $\phi: Z_{n-1} \rightarrow \mathbb{F}$ just pick any values $\phi(w_j) \in \mathbb{F}$ e.g. $\phi(w_j) = 0$.

$\Rightarrow B^{n-1}/\text{im } \delta^{n-1} = 0$ so $H^n(C_*; \mathbb{F}) \rightarrow \text{Hom}(H_n(C_*), \mathbb{F})$ iso \square

Cor $H^n(X; \mathbb{F}) \cong H_n(X; \mathbb{F})^*$ for any field \mathbb{F} .

$$H^n(X; M) \cong H_{CW}^n(X; M) \cong H_{\Delta}^n(X; M)$$

if X is CW-cx \uparrow if X is Δ -cx

Pf Cor holds for homology and the isos are natural. \leftarrow i.e. functorial w.r.t. maps

The universal coeff. thm SES is natural. So result holds by 5-Lemma. \square

Algebra: structure thm for f.g. abelian groups and R-mods

Fact 1 A f.g. abelian gp $\Rightarrow A \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1 \oplus \dots \oplus \mathbb{Z}/p_a$

where $p_i \in \mathbb{Z}$ prime (need not be distinct) \leftarrow free part \mathbb{F}

Also r, a, p_i, n_i are unique (up to reordering) \leftarrow torsion part T

Example $\mathbb{Z}/4 = \mathbb{Z}/2 \neq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (Chinese Remainder Thm)

Fact 2 $T \cong \mathbb{Z}/d_1 \oplus \dots \oplus \mathbb{Z}/d_k$ with $d_1 | d_2 | \dots | d_k$ ($d_i \in \mathbb{N}$ unique)

Example $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3 \cdot 2^2$ $d_1=2, d_2=12$

Fact 3 M f.g. R -mod, R PID, then:

$$\begin{aligned} M &\cong \mathbb{F} \oplus T \\ \mathbb{F} &\cong R^r \\ T &\cong R/p_1 \oplus \dots \oplus R/p_a \\ &\cong R/d_1 \oplus R/d_2 \oplus \dots \oplus R/d_k \end{aligned}$$

$r \in \mathbb{N}$ unique, called rank of M

$p_i \in \mathbb{R}$ primes, p_i unique up to ordering & mult by d_i

d_i called invariant factors

unique up to multⁿ by invertible elements e.g. ± 1 if $R = \mathbb{Z}$

Rmk $T = \{m \in M : rm = 0 \text{ some } r \neq 0 \in R\} =$ torsion elements

$\mathbb{F} \cong M/T$

Torsion shift

Easy Exercise $\text{Ext}_R^*(\bigoplus_i M_i, \prod_j N_j) \cong \prod_i \text{Ext}_R^*(M_i, N_j)$ (as you would expect for $\text{Ext}^0 = \text{Hom}$ since a hom. from $\bigoplus M_i \rightarrow \dots$ means you have to specify it on each M_i)

Upshot To compute $\text{Ext}_R^1(M, R)$ for $M = R \oplus R/d \oplus \dots$ just need:

- $\text{Ext}_R^1(R, R) = 0$ since $0 \rightarrow R \xrightarrow{\text{id}} R \rightarrow 0$
- $\text{Ext}_R^1(R/d, R) \cong R/d$ since $0 \rightarrow R \xrightarrow{d} R \rightarrow R/d \rightarrow 0$ so choice of $\phi(1) \in R$ module ϕ coming from $R \xrightarrow{d} R$ so $\phi(1) = d \cdot \varphi(1) \in d \cdot R$
- $\text{Ext}_R^1(M, R) \cong \text{Torsion}(M)$
- Exercises
 - $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_m, \mathbb{Z}/m) \cong \mathbb{Z}/\text{gcd}(m, m)$
 - Gabelian gp $\Rightarrow \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, G) = 0$ and $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/d, G) \cong G/d \cdot G$
 - R any ring (comm. with 1) $x \in R$ not zero divisor $\Rightarrow \text{Ext}_R^1(R/(x), N) \cong \begin{cases} \{n \in N : x \cdot n = 0\} & * = 0 \\ N/x \cdot N & * = 1 \\ 0 & \text{else} \end{cases}$

Cor If $H_n(X; R)$ f.g. R -mod $\forall n$, R PID,

$\Rightarrow H_n(X; R) = R^n \oplus T_n$ (free & torsion parts)

$\Rightarrow H^n(X; R) \cong R^n \oplus T_{n-1}$ (torsion moves up!)

Pf $0 \rightarrow T_{n-1} \rightarrow H^n(X; R) \rightarrow \text{Hom}(R^n \oplus T_{n-1}, R) \rightarrow 0$

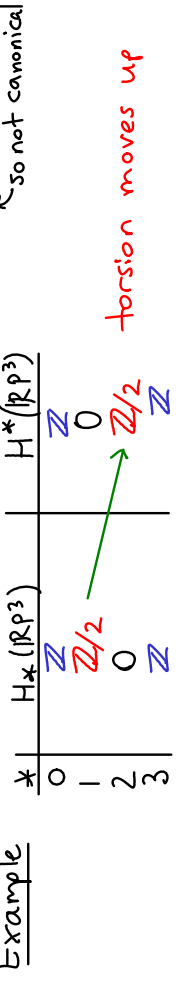
$\text{Hom}(R^n \oplus T_{n-1}, R) \cong (\text{Hom}(R, R))^n \times \text{Hom}(T_{n-1}, R)$

$R \rightarrow R \xrightarrow{\text{id}} R \xrightarrow{\text{id}} R \rightarrow 0$ \leftarrow $(R$ -linear hom. of R -modules)

proof: $T_{n-1} \xrightarrow{\varphi} R$ then $r \cdot t = 0$ some $r \neq 0 \in R$ since t torsion. Then $0 = \varphi(r \cdot t) = r \cdot \varphi(t)$ but R is an integral domain, so $\varphi(t) = 0$. \square

x determines the hom $0 \rightarrow T_{n-1} \rightarrow H^n(X; R) \rightarrow R^n \rightarrow 0$

free, so can split the SES (pick lifts of basis). \square



Universal coefficients Theorem in homology

FACT Theorem C_* chain cx of free R -mods, M R -module

\Rightarrow natural SES $0 \rightarrow H_*(C_*) \otimes_R M \rightarrow H_*(C_* \otimes_R M) \rightarrow \text{Tor}_1^R(H_{*+1}(C_*), M) \rightarrow 0$

$[C] \otimes m \mapsto [C \otimes m]$

The SES splits, but the splitting is not natural.

Torsion groups: A, B R -mods (R comm. ring with 1)

pick $\dots \rightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \rightarrow 0$ free resolution

$\Rightarrow \dots \rightarrow P_2 \otimes B \xrightarrow{\varphi_2 \otimes \text{id}} P_1 \otimes B \xrightarrow{\varphi_1 \otimes \text{id}} P_0 \otimes B \rightarrow 0$ not exact but is chain cx

take $\otimes B$ omit $A \otimes B$

$\text{Tor}_k^R(A, B) = H_k$ (this complex) \leftarrow fact independent of choices of P_i, φ_i

Rmk R PID $\Rightarrow \ker \varphi_0$ free \Rightarrow pick $\{P_i = \ker \varphi_i\}$ \Rightarrow only $\text{Tor}_0^R, \text{Tor}_1^R$ can be non-zero

Example $\text{Tor}_*^{\mathbb{Z}}(\mathbb{Z}/a, \mathbb{Z}/b) = ?$

$0 \rightarrow \mathbb{Z} \xrightarrow{a} \mathbb{Z} \xrightarrow{\varphi_0} \mathbb{Z}/a \rightarrow 0$ free resolution

take $\otimes \mathbb{Z}/b \Rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{a} \mathbb{Z} \xrightarrow{\varphi_0} \mathbb{Z}/a \rightarrow 0$ (since $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$ any G)

are $\mathbb{Z}/a \otimes \mathbb{Z}/b = (\mathbb{Z}/b)/a \cdot \mathbb{Z}/b \cong \mathbb{Z}/\langle a, b \rangle \cong \mathbb{Z}/\text{gcd}(a, b) \cong \mathbb{Z}/a \otimes \mathbb{Z}/b$

$\text{Tor}_{\mathbb{Z}}^2(\mathbb{Z}/a, \mathbb{Z}/b) = \{x \in \mathbb{Z}/b : a \cdot x = 0\} \cong \mathbb{Z}/\text{gcd}(a, b)$

Facts $\text{Tor}_0^R(A, B) = P_0 \otimes B / \text{Im}(\varphi_0 \otimes \text{id}) \cong A \otimes B$

$\text{Tor}_*^R(A, B) \cong \text{Tor}_*^R(B, A)$

Exercise $\text{Tor}_*^R(\bigoplus_i A_i, \bigoplus_j B_j) \cong \bigoplus_i \text{Tor}_*^R(A_i, B_j)$

$\text{Tor}_*^R(A, B) = 0$ for $* \geq 1$ if A or B is free (use $M \otimes_R R \cong M$)

$\text{Tor}_*^R(R/u, M) \cong \begin{cases} M/u \cdot M & * = 0 \\ 0 & \text{else} \end{cases}$

deduce $\text{Tor}_*^R(A, M)$ for f.g. R -mods A

$u \in R$ not zero divisor \leftarrow u -torsion $(M) = \{x \in M : u \cdot x = 0\} \quad * = 1$

R any ring (comm. with 1) \leftarrow 0 else

Example $H_*(\mathbb{R}P^2; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & * = 0 \\ \mathbb{Z}_2 & * = 1 \\ \mathbb{Z}_2 & * = 2 \end{cases}$

\leftarrow caused by $\text{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{R}P^2), \mathbb{Z}_2) = \text{Tor}_1(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$

Künneth Thm

R PID \Rightarrow natural SES: $0 \rightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \rightarrow H_n(C_* \otimes D_*) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}_1^R(H_i(C_*), H_j(D_*)) \rightarrow 0$

$(C_*$ free ch. cx. R -mods \rightarrow $(D_*$ any ch. cx. R -mods)

and the SES splits but the splitting is not natural. Example $R = \text{field} \Rightarrow$ that $\text{Tor}_1 = 0$

Example (take $C_* = C^*(X), D_* = C^*(Y)$): $0 \rightarrow H^*(X) \otimes H^*(Y) \rightarrow \text{Tor}_1(H^*(X), H^*(Y)) \rightarrow 0$ exact (if $C_*(X)$ has ∞ rank then $C^*(X)$ may not be free but it will be "flat" and Tor_1 holds if C_* is flat R -mod)

Exercise

$\text{Tor}_*^R(\bigoplus_i A_i, \bigoplus_j B_j) \cong \bigoplus_i \text{Tor}_*^R(A_i, B_j)$

$\text{Tor}_*^R(A, B) = 0$ for $* \geq 1$ if A or B is free (use $M \otimes_R R \cong M$)

$\text{Tor}_*^R(R/u, M) \cong \begin{cases} M/u \cdot M & * = 0 \\ 0 & \text{else} \end{cases}$

deduce $\text{Tor}_*^R(A, M)$ for f.g. R -mods A

$u \in R$ not zero divisor \leftarrow u -torsion $(M) = \{x \in M : u \cdot x = 0\} \quad * = 1$

R any ring (comm. with 1) \leftarrow 0 else

Example $H_*(\mathbb{R}P^2; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & * = 0 \\ \mathbb{Z}_2 & * = 1 \\ \mathbb{Z}_2 & * = 2 \end{cases}$

\leftarrow caused by $\text{Tor}_1^{\mathbb{Z}}(H_1(\mathbb{R}P^2), \mathbb{Z}_2) = \text{Tor}_1(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$

Local orientations and orientability

Def A Local orientation of M at $x \in M$ is a choice of generator


$$\mu_x \in H_n(M, M \setminus x) \cong H_n(D^n, D^n \setminus \{0\}) \cong \tilde{H}_n(S^n) \cong \mathbb{Z}$$

excise complement of pt $V_x \cong D^n$ choice of generator is not canonical!

(see Section 5 of these notes) $\tilde{H}_n(S^n) \cong \mathbb{Z}$

Rmk Even though we can pick the last 2 isos in a canonical way, we cannot make the first one canonical. The intuitive notion that an orientation of \mathbb{R}^n is a choice of basis for \mathbb{R}^n can be made rigorous now, since a generator for $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ can be specified by an \mathbb{R} -linear map $\Delta^n \rightarrow \mathbb{R}^n$ sending $e_i \mapsto i$ -th basis vector, provided that $0 \in \text{interior}(\text{image } \psi)$. By Gram-Schmidt, one can homotope any basis to an orthonormal one, and by successive rotations can homotope any o.n. one to another o.n. basis one basis vector at a time except possibly the last one when a reflection may be necessary. This argument shows that $GL(n)$ has 2 connected components depending on whether $\det > 0$ or < 0 (once we homotoped the matrix to an orthogonal matrix $O(n)$ we have $\det = +1$ for $SO(n)$ and $\det = -1$ for $SO(n)$ -reflection, which are the 2 connected components of $O(n)$).

Def An orientation of M is a locally consistent choice $x \mapsto \mu_x$

meaning:  $H_n(M, M \setminus V_x) \cong H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$

$\mu_x \in \mathbb{Z}$ $\mu_y \in \mathbb{Z}$

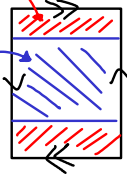
\leftarrow quotient maps

Def M orientable if \exists orientation on M

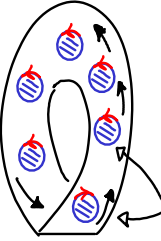
oriented if we chose an orientation

Examples $S^n, \mathbb{R}^n, \mathbb{C}P^n$, orientable surfaces Σ_g , $\mathbb{R}P^n \mathbb{K}$ -odd n

Non-example $\mathbb{R}P^2 = \text{Möbius band} \cup D^2$



by local consistency can move disc continuously and preserves orientation



discs are differently oriented \Rightarrow contradicts local consistency.



choice of μ_x is choice of orientation of boundary circle of small disc containing x

$\Rightarrow \mathbb{R}P^2$ not orientable

The fundamental class $[M]$

FACT Theorem For M closed n -mfd: $\xrightarrow{\text{natural map from the LES}}$

1) M orientable connected $\Rightarrow H_n(M) \cong H_n(M, M \setminus x) = \mathbb{Z} \cdot \mu_x$ \uparrow once we choose an orientation

$\Rightarrow \exists [M] \longleftarrow \mu_x$

once we choose an orientation $(\mu_x)_{x \in M}$ \uparrow called fundamental class

(if swap orientation: for $-\mu_x$ get $-[M]$)

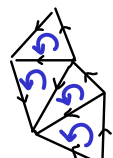
2) M not orientable connected $\Rightarrow H_n(M) = 0$
 $H_n(M; \mathbb{F}_2) = \mathbb{F}_2 \cdot [M] \cong \mathbb{F}_2$ \leftarrow (or any field of characteristic 2)

Construction of $[M]$ if M has Δ -complex structure

M compact \Rightarrow finite # n -simplices $\delta_1, \dots, \delta_N$

M oriented \Rightarrow pick orientations of $\delta_1, \dots, \delta_N$ to agree with given orientation of $M: \sigma$ for $x \in \text{Int}(\delta_i)$

$$\mathbb{Z} \cdot \mu_x = H_n(M, M \setminus x) \xrightarrow{\text{exc.}} H_n(\delta_i, \delta_i \setminus x) = \mathbb{Z} \cdot \delta_i$$



$$\Rightarrow [M] := \sum \delta_i \text{ satisfies } \partial [M] = 0 \checkmark$$

(each facet arises twice with opposite signs)

$$H_n(M) \rightarrow H_n(M, M \setminus x) \cong H_n(\delta_i, \delta_i \setminus x)$$

$$[M] \xrightarrow{\mu_x} \delta_i$$

Not difficult to see that $H_n^\Delta(M) = \mathbb{Z} \cdot [M]$, so $\Rightarrow H_n(M) \cong H_n(M, M \setminus x)$

Also $\mathbb{Z} \cong \mathbb{Z} \cdot [M]$ since $C_{n+1}(M) = 0$ ($\mathbb{A}^{(n+1)}$ -simplices since $\dim M = n$)

M non-orientable \Rightarrow each facet of δ_i appears twice in $\partial \Sigma \delta_i$

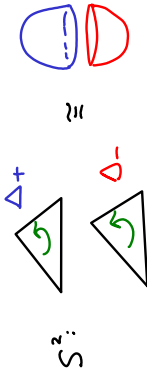
$\Rightarrow \partial \Sigma \delta_i = 0$ over \mathbb{F}_2 independently of choices of orientations of δ_i .



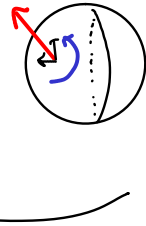
More generally:
 $[M] := \sum \pm \delta_i$
 where signs come from $H_n(M, M \setminus x) \cong H_n(\delta_i, \delta_i \setminus x)$
 $\mu_x \mapsto \pm \delta_i$
 (so compare orientation of μ_x with orientation of δ_i)

Examples

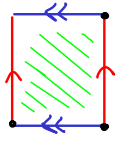
1) $S^n = \Delta^n \cup \Delta^n$
 glue bodies



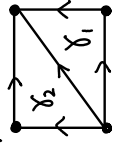
$[S^n] = \Delta_+ - \Delta_-$ if use canonical orientation we discussed
 hence $\partial[S^n] = \partial\Delta - \partial\Delta = 0$
 $D^n \subseteq \mathbb{R}^n$ canonical orientation
 $\Rightarrow S^{n-1} = \partial D^n$ using outward normal first rule



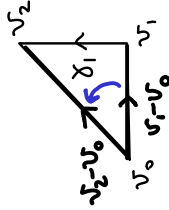
2) $T^2 =$



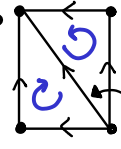
Δ -complex structure (compatibly with side identifications!)



Want orientation induced by square $\subseteq \mathbb{R}^2$



$v_1 - v_0, v_2 - v_0$ positive \mathbb{R}^2 -basis
 $\Rightarrow \delta_1$ agrees with orientation



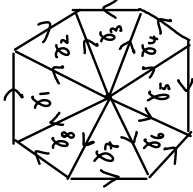
$[T^2] = +\delta_1 - \delta_2$
 \uparrow δ_2 orientation disagrees

RMK general feature: if two simplices are identified along a facet then the 2 respective outward normals are related by reflection.

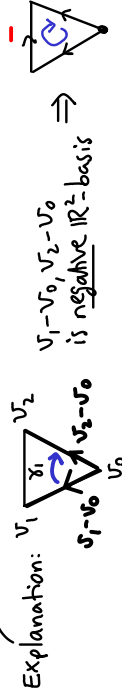
So consistency \Rightarrow either simplices are compatibly oriented and the two induced orientations on facet are opposite or not compatibly oriented but facet orientⁿ is same, then need sign like in example when build $[T^2]$

3) Recall $\Sigma_2 =$ $=$

Δ -cx structure (compatible with side identifications!):



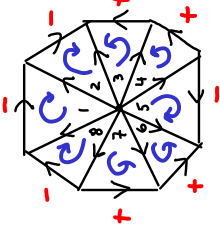
Use the orientation induced by polygon $\subseteq \mathbb{R}^2$
 $\Rightarrow [\Sigma_2] = -\delta_1 - \delta_2 + \delta_3 + \delta_4 - \delta_5 + \delta_6 + \delta_3 - \delta_2$



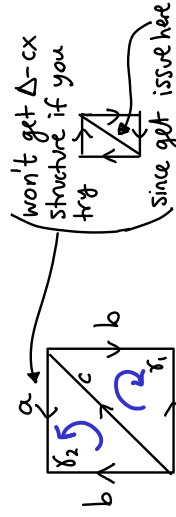
Explanation: $v_1 - v_0, v_2 - v_0$ is negative \mathbb{R}^2 -basis \Rightarrow

All simplices δ_i have $v_0 =$ centre of polygon

\Rightarrow sign $<$ $\begin{matrix} + \\ - \end{matrix}$ if overedge clockwise $\overline{\text{anti}}$



3) $\mathbb{RP}^2 =$ (non-orientable example)



won't get Δ -cx structure if you try (since get issue here)

Use the orientation induced by square $\subseteq \mathbb{R}^2$

$\Rightarrow [\mathbb{RP}^2] = -\delta_1 + \delta_2$
 $\partial[\mathbb{RP}^2] = -(b - a + c) + (a - b + c)$
 $= -2b + 2a$
 $\neq 0$ so not cycle in $C_*^{CW}(\mathbb{RP}^2)$

However, working modulo 2:

$\partial[\mathbb{RP}^2] = 0 \in C_*^{CW}(\mathbb{RP}^2; \mathbb{F}_2)$ since $2=0$ in \mathbb{F}_2
 $\Rightarrow [\mathbb{RP}^2] \in H^2(\mathbb{RP}^2; \mathbb{F}_2)$

Degree

Def M, N oriented closed connected n -mfds, $f: M \rightarrow N$

$$f_*: H_n(M) \rightarrow H_n(N)$$

$$[M] \mapsto \deg(f) \cdot [N]$$

Local degree

Lemma If $f^{-1}(y)$ finite,

$$\text{then } \deg(f) = \sum_{x \in f^{-1}(y)} \deg(f|_x)$$

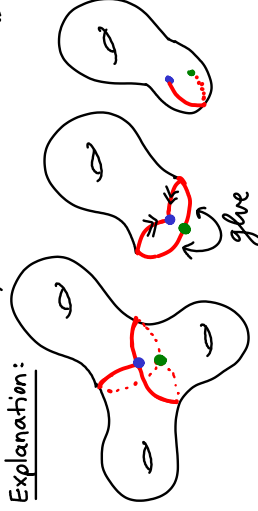
pf

$$\begin{aligned} [M] \in H_n(M) &\xrightarrow{f_*} H_n(N) \ni [N] \\ \downarrow \oplus_{x \in f^{-1}(y)} &\downarrow \text{local map} \\ \oplus_{x \in f^{-1}(y)} H_n(M, M \setminus x) &\xrightarrow{(f_x)_*} H_n(N, N \setminus y) \ni \mu_y^N \\ \downarrow \cong &\downarrow \cong \\ \oplus_{x \in f^{-1}(y)} (\sum \deg(f_x) \cdot \mu_y^N) &\xrightarrow{\cong} (\sum \deg(f_x) \cdot \mu_y^N) \end{aligned}$$

Examples

1) $S^1 \rightarrow S^1, z \mapsto z^n, [S^1] \mapsto n \cdot [S^1]$ so $\deg = n$

2) $\Sigma_3 = \Sigma_3 / \mathbb{Z}_3$ -rotation action \rightarrow torus $= \Sigma_1$



Explanation: rotation symmetry

Easy check: $\deg(\eta) = 3$ (e.g. use local degrees)

Cultural Rmk

For M, N, f smooth, the $\deg f = \#$ (preimages of a generic point of N)

Idea: $\deg f$ tells you how many times you cover N . (almost all points work)

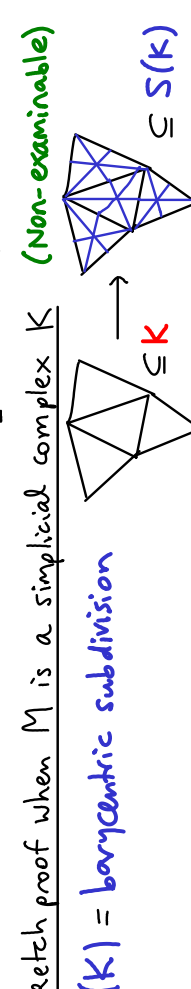
Poincaré duality

FACT Theorem For M closed n -mfd

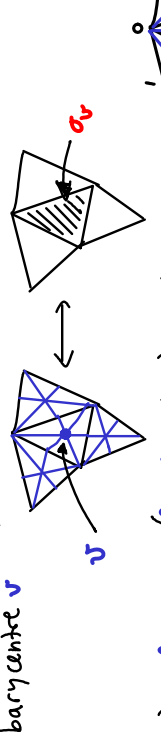
$$M \text{ oriented} \Rightarrow H^k(M) \cong H_{n-k}(M)$$

$$M \text{ non-oriented} \Rightarrow H^0(M) \cong H_n(M)$$

Sketch proof when M is a simplicial complex K (Non-examinable)



1) simplex $\sigma = \sigma_v$ of K with barycentre $v \rightarrow v^*$ vertex of $S(K)$



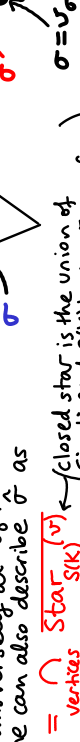
$$\text{ht}(v) = (\text{height of } v) = \dim \sigma_v$$



$$3) \sigma \text{ k-simplex of } K \rightarrow \text{dual simplex } \hat{\sigma} = \bigcup_{\tau \in S(K), v_\sigma \in \tau} \tau$$



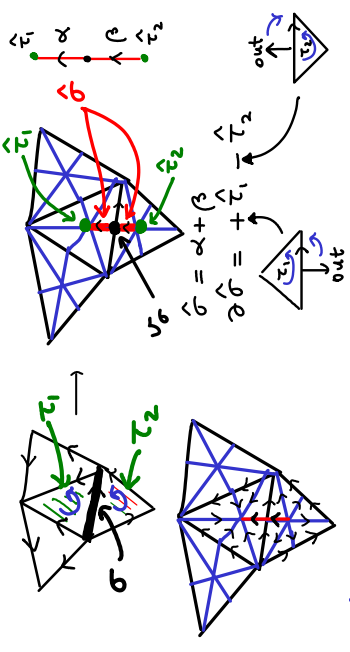
Rmk: $\bigcup_{v \in \sigma} \tau$ with $\text{ht}(v_\sigma) \max$ will give back $\hat{\sigma}$. Thus $\hat{\sigma}, \sigma$ intersect transversely at v_σ . One can also describe $\hat{\sigma}$ as $\text{Star}_{S(K)}(v)$ (closed star is the union of simplices of $S(K)$ having v as a face)



FACTS • $\dim \hat{\sigma} = n - \dim \sigma$

• dual cells $\hat{\sigma}$ give a cell decomposition of M ("polygonal" complex rather than Δ -cx)
 • need compare orientations of σ, τ (+ if σ as a facet of τ has boundary orientation)

(*) $\partial \hat{\sigma} = \sum_{\substack{\text{facet} \\ \sigma \subseteq \tau \\ \tau \in K}} \pm \hat{\tau}$



4) dual chain complex $D_{n-k} = \text{free abelian group on dual chains } \hat{\sigma}$ (since $\hat{\sigma}$ give a cell decomp of M)

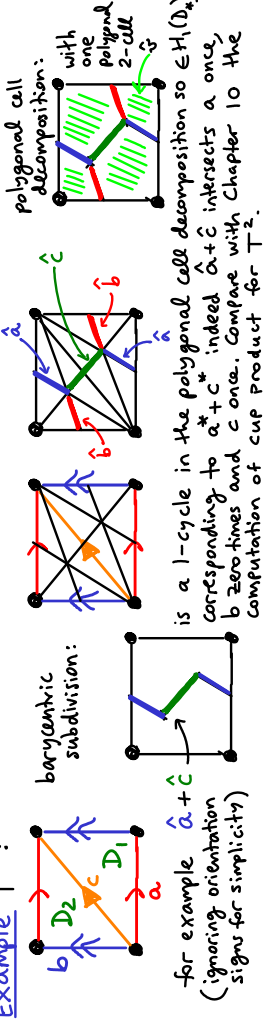
$H_*(M) \cong H_*(D_*, \partial_*)$ (since $\hat{\sigma}$ give a cell decomp of M)

5) $\varphi: D_{n-k} \rightarrow C^k(M)$ where $\sigma^*(\alpha) = \begin{cases} 0 & \text{for } k\text{-cells } \alpha \neq \hat{\sigma} \\ 1 & \text{for } \alpha = \hat{\sigma} \end{cases}$

• φ linear bijection ✓
 • chain map: $\hat{\sigma} \mapsto \sigma^*$

$\varphi(\partial \hat{\sigma}) \equiv \varphi \sum \pm \hat{\tau} = \sum \pm \tau^*$
 $\partial^* \varphi(\hat{\sigma}) = \partial^* \sigma^* = (\sigma^* \circ \partial: \tau \mapsto \sum \pm \sigma_i) \mapsto \begin{cases} \pm 1 & \text{if one of } \sigma_i = \sigma \\ 0 & \text{else} \end{cases} = \sum \pm \tau^* = \varphi(\partial \hat{\sigma})$

USMOT φ chain iso so get iso: $H_*(M) \oplus H_*(D_*, \partial_*) \xrightarrow{\varphi} H^{n-*}(M)$

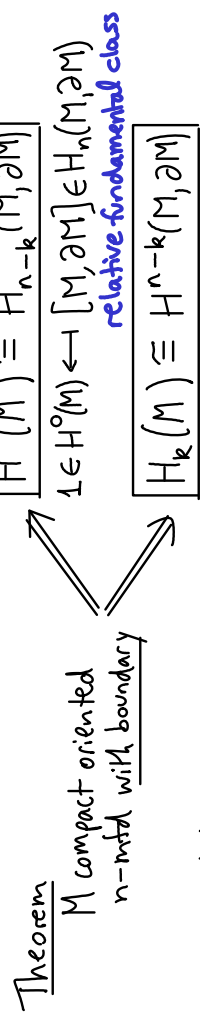


Cor χ (odd dimensional closed orientable mfd) = 0

Pf Betti numbers $b_i = \text{rank } H_i(M) \stackrel{\text{universal coeff. thm.}}{=} \text{rank } H_i(M) = \text{rank } H_{n-i}(M)$ ← Poincaré duality

$\chi(M) = b_0 - b_1 + \dots + b_{\dim M-1} - b_{\dim M}$ equal. □

(Poincaré-)Lefschetz duality



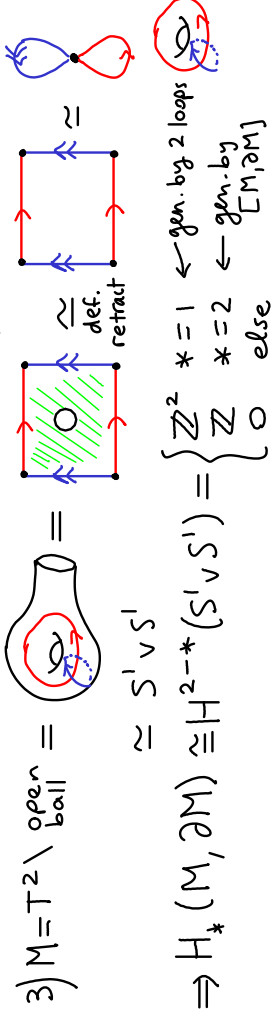
Non-oriented \Rightarrow same holds with \mathbb{F}_2 coefficients.

"Pf" basically same as Poincaré duality. □ (In simplicial case you extend a simplicial decomposition of ∂M to a simplicial decomposition of M , using a collar neighbourhood of ∂M)

Cor M compact, connected, $\partial M \neq \emptyset \Rightarrow \begin{cases} H^n(M) \cong H_0(M, \partial M) = 0 \\ H_n(M) \cong H^0(M, \partial M) = 0 \end{cases}$

Examples

- D^n $\partial D^n = S^{n-1}$ generator $D_1 - D_2$
- $A = \text{annulus} \subseteq \mathbb{R}^2 \cong S^1$ $Z \cong H^0 A \cong \mathbb{R}^2$ $Z \cong H^1 A \cong H_1(A, \partial A) \leftarrow \text{generator}$ $0 \cong H^2 A \cong H_0(A, \partial A)$ (notice $\partial D \rightarrow \partial A$)



either by universal coefficient thm since $H_0(M, \partial M) = 0$ or by hand since given $p \in M, q \in C^0(M, \partial M)$ consider $(\partial q)(p)$ for δ path from p to any $q \in \partial M$.

What happens in the non-compact case?

Locally finite homology (Borel-Moore)

$C_*^{lf}(X)$ allow infinite sums $\sum_{i \in \mathbb{Z}} n_i \sigma_i$ generators of $C_*(X)$
 s.t. given any compact subset $K \subseteq X$,
 $\#\{n_i \neq 0 : K \cap \text{supp } \sigma_i \neq \emptyset\} < \infty$.

Examples

$C_1^{lf}(\mathbb{R}) \ni \sum_{m \in \mathbb{Z}} \sigma_m, \sigma_m: I \cong [m, m+1] \subseteq \mathbb{R}$

\Rightarrow get cycle $[\mathbb{R}] \in H_1^{lf}(\mathbb{R})$

$C_0^{lf}(\mathbb{R}) \ni [\text{point}] \ni \partial(\sigma_1 + \sigma_2 + \dots)$ is a boundary:

 point

exercise $H_*^{lf}(\mathbb{R}) = \begin{cases} \mathbb{Z} & * = 1 \\ 0 & \text{else} \end{cases} (\cong H^{1-*}(\mathbb{R}))$
 e.g. in simplicial case $[M] = \sum \pm \sigma_i \in \mathbb{Z}^n$

FACT Theorem M orientable n-mfd $\Rightarrow H^*(M) \cong H_{n-*}^{lf}(M)$
 (possibly not compact)

cohomology with compact supports $H_c^*(X)$

$C_c^*(X)$: only allow cochains $\phi: C_* X \rightarrow \mathbb{Z}$ s.t. \exists compact $K \subseteq X$ with $\phi(C_*(X \setminus K)) = 0$ (vanish on chains in $X \setminus K$) depends on ϕ

Example $c \in C_*(X) \Rightarrow \phi(\alpha) = \text{signed \# intersections of } c \text{ with } \alpha$
 (geometric intersection #)

FACT $\text{Thm } M$ orientable n-mfd $\Rightarrow H_*(M) \cong H_c^{n-*}(M)$
 (possibly not compact)

Warning H_*^{lf}, H_c^* are not homotopy invariant (indeed non-trivial for \mathbb{R}^n)
 Caused because they are not functorial. They are however functorial for proper maps
 (preimages of compact sets are compact)

Fact $H_c^*(X) \cong \varinjlim H^*(X, X \setminus K)$ where compacts $K_1 \subseteq K_2$ give $H^*(M, M \setminus K_1) \rightarrow H^*(M, M \setminus K_2)$
Direct limit $\varinjlim G_i$ via maps $G_i \rightarrow G_j$ means $\sqcup G_i$ / identifying $g \in G_i$ with its images under those maps
 (The indices are partially ordered & directed: $\forall i, j, \exists k > i, j$ so can compare G_i, G_j inside G_k)
Fact \varinjlim is an exact functor. (via $G_i \rightarrow G_k, G_j \rightarrow G_k$)

Cap product and Poincaré duality revisited

X space, $k \geq l$

$n: C_k(X) \times C^l(X) \rightarrow C_{k-l}(X)$ cap product

$(\sigma: \Delta^k \rightarrow X) \cap (\phi: C_l(X) \rightarrow \mathbb{Z}) = \underbrace{\phi(\sigma|_{[\sigma_0, \dots, \sigma_l]})}_{\text{"bottom face"}} \cdot \underbrace{\sigma|_{[\sigma_{l+1}, \dots, \sigma_k]}}_{\text{"top face"} \cong \Delta^{k-l}} \in C_{k-l}(X)$
 (sometimes write $\phi \cap \sigma$)

(easy) Properties

- \cap bilinear
- $\partial(\sigma \cap \phi) = (-1)^l (\partial\sigma \cap \phi - \sigma \cap \partial\phi)$
- cycle \cap cocycle is cycle
- boundary \cap cocycle are boundaries
- cycle \cap coboundary

$\Rightarrow n: H_k(X) \times H^l(X) \rightarrow H_{k-l}(X)$ bilinear

Theorem (Poincaré duality) The map $\phi \mapsto [M] \cap \phi$ gives following isos

① For M closed oriented n-mfd $[M] \cap \cdot: H^*(M) \xrightarrow{\cong} H_{n-*}(M)$
 ② For M non-compact oriented n-mfd, $[M] \cap \cdot: H_c^*(M) \xrightarrow{\cong} H_{n-*}(M)$
 ③ $[M] \cap \cdot: H^*(M) \xrightarrow{\cong} H_{n-*}^{lf}(M)$

Sketch Pf of ③ for smooth mfd (Non-examinable)

If M smooth $\Rightarrow \exists$ "good cover" U_i of M meaning open cover s.t.
 FACT from Riemannian geometry ("convex neighbourhoods") $U_i \cong \mathbb{R}^n$
 $U_{i_1} \cap \dots \cap U_{i_k} \cong \mathbb{R}^n$ or \emptyset
 Then compute $H_c^*(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else} \end{cases}$ and ③ holds for \mathbb{R}^n .
 \Rightarrow holds $\forall U_i$
 \Rightarrow by naturality of ③ and of Mayer-Vietoris get ③ for $\cup_{i \in \text{finite}} U_i$
 \Rightarrow ③ for M , which is ①. \square (use 5-lemma)

General Pf of Poincaré duality ← **Non-examinable**

Step 1: holds for \mathbb{R}^n

Pf $H_c^k(\mathbb{R}^n) \cong H^k(D^n, S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases} \cong H_{n-k}(\mathbb{R}^n)$

(recall fact: $H_c^k(X) \cong \lim_{\leftarrow} H^k(X, \mathbb{R} \setminus K)$ can make K larger by picking $K = \text{large ball}$)
 then $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^k(K, \partial K)$

Pick Δ -cx structure for \mathbb{R}^n . So $[\mathbb{R}^n] = \sum \pm \sigma_i \leftarrow$ sum over n -simplices.
 Say \exists simplex $\sigma_0: \Delta^n \rightarrow \mathbb{R}^n$. Define $\phi: C_c^{CW}(\mathbb{R}^n) \rightarrow \mathbb{Z}, \phi(\sigma_0) = \pm 1$ (other simplices) = 0

$[\mathbb{R}^n] \cap \phi = \sum \phi(\pm \sigma_i) = \phi(\pm \sigma_0) = 1$ (pick sign in \oplus)

Step 2 holds for $A, B, A \cup B \Rightarrow$ holds for $A \cup B$

Pf Mayer-Vietoris for H_c^* , naturality, 5-Lemma \checkmark

Step 3 holds for A_i , and $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \Rightarrow$ holds for $\cup A_i$

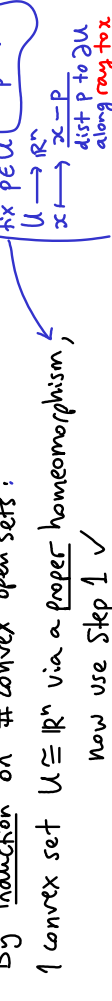
Pf By applying ling: both sides of P.D. iso commute with limits \checkmark

Step 4 holds for open subsets in \mathbb{R}^n

Pf Every such set is a union of convex open sets (e.g. balls)

By Step 3 enough to consider case of finite union.

By induction on # convex open sets:



$k+1$ convex sets: $A = \cup \{\text{first } k \text{ convex sets}\}, B = \text{last convex set} \Rightarrow$ use step 2 & inductive hypothesis

Step 5 holds for mfd M

Consider open sets in M for which it holds.

By a Zorn's Lemma argument we get a maximal open subset U where holds.

If $U \neq M$ pick $p \in M \setminus U$ and nbhd $V \cong \mathbb{R}^n$ of p . Then holds for $U, V, U \cup V$

(note $U \cup V \subseteq V \cong \mathbb{R}^n$ open, so Step 4 applies) so by Step 2 holds for $U \cup V$

Contradicts maximality. $\checkmark \square$

Recall there is a well-defined evaluation of H^* -classes on H_* :

$\langle \cdot, \cdot \rangle: H_k(M; \mathbb{R}) \otimes H^k(M; \mathbb{R}) \rightarrow \mathbb{R}$
 $\subset \otimes \alpha \mapsto \langle c, \alpha \rangle = \langle \varphi(c), \alpha \rangle$
 any representative cocycle φ for α

Easy exercise $\langle c, \alpha \cup \beta \rangle = \langle c \cap \alpha, \beta \rangle$
 any $\alpha, \beta \in H^*, c \in H_*$

Corollary of Poincaré duality

M compact oriented n -mfd, \mathbb{F} field.

$H^k(M; \mathbb{F}) \otimes H^{n-k}(M; \mathbb{F}) \xrightarrow{\otimes} H^k(M; \mathbb{F}) \otimes H^{n-k}(M; \mathbb{F}) \xrightarrow{\langle \cdot, \cdot \rangle} \langle [M], \alpha \cup \beta \rangle$
 $\alpha \otimes \beta \mapsto \langle [M], \alpha \cup \beta \rangle$
 is a non-singular bilinear form.

Pf. By exercise, $\langle [M], \alpha \cup \beta \rangle = \langle [M] \cap \alpha, \beta \rangle = \langle PD(\alpha), \beta \rangle$
 So the following diagram commutes:
 $H^k(M; \mathbb{F}) \otimes H^{n-k}(M; \mathbb{F}) \xrightarrow{PD \otimes id} H^{n-k}(M; \mathbb{F}) \otimes H^{n-k}(M; \mathbb{F}) \xrightarrow{\langle \cdot, \cdot \rangle} \langle [M], \alpha \cup \beta \rangle$
 $\downarrow \cong$ (PD is iso over \mathbb{Z} , hence iso over \mathbb{F} by universal coefficients.)
 $H^k(M; \mathbb{F}) \otimes H^{n-k}(M; \mathbb{F}) \xrightarrow{\langle \cdot, \cdot \rangle} \langle [M], \alpha \cup \beta \rangle$
 hence: $H^k(M; \mathbb{F}) \cong (H^{n-k}(M; \mathbb{F}))^*$
 definition of Poincaré duality PD

By universal coefficients, $H^*(M; \mathbb{F}) \cong \text{Hom}(H_*(M; \mathbb{F}), \mathbb{F})$ via $\beta \mapsto \langle \beta, \cdot \rangle$

Hence $\langle \cdot, \cdot \rangle$ is a non-degenerate bilinear pairing $\left(\begin{matrix} \text{using that for any } \mathbb{F}\text{-vector-space } V \\ V \otimes V^* \rightarrow \mathbb{F}, v \otimes \varphi \mapsto \varphi(v) \\ \cong \text{Hom}(V, \mathbb{F}) \end{matrix} \right)$
 Hence so is the pairing $\langle \cdot, \cdot \rangle$ in the diagram. \square
 is non-deg. pairing.

Remark For M non-orientable, the same holds for \mathbb{F} of characteristic 2, e.g. \mathbb{Z}_2

For \mathbb{Z} coefficients it can fail if $H^*(M) \neq \text{Hom}(H_*(M), \mathbb{Z})$. So we define:

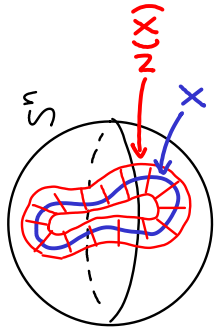
Betti group $B^k(M) = H^k(M) / \text{torsion}(H^k(M))$
 $B_k(M) = H_k(M) / \text{torsion}(H_k(M))$

By what we proved in the section on universal coefficients, $B^1(M) \cong \text{Hom}(B_0(M), \mathbb{Z})$
 whenever $H_{q-1}(M)$ is finitely generated (which we know holds for compact mfd's)
 The iso is given by $\langle \cdot, \cdot \rangle$ again: this descends to quotients since $\langle c, \alpha \rangle = 0 \in \mathbb{Z}$
 if c or α has finite order (i.e. torsion). The same proof as above yields:
 M compact oriented n -mfd $\Rightarrow B^k(M) \otimes B^{n-k}(M) \rightarrow \mathbb{Z}, \alpha \otimes \beta \mapsto \langle [M], \alpha \cup \beta \rangle$
 is non-degenerate bilinear form.

Also the Remark holds.

Example Use this to prove ex. 4(c) sheet 3. (Hint: $H^{2k}(CP^n) \cup H^{2n-2k}(CP^n) = H^{2n}(CP^n)$)

Alexander duality



(in fact, enough to assume \$X\$ is locally contractible)

\$\emptyset \neq X \subseteq S^n\$ compact subset s.t.

\$\exists\$ open neighbourhood \$N(X)\$ which deformation retracts to \$X\$

such that \$\overline{N(X)} \subseteq S^n\$ is an \$n\$-mfd with boundary.

Fact assumption about \$N(X)\$ holds if \$X\$ is a smooth submanifold by tubular neighborhood thm see Chapter 5

Theorem

$$\tilde{H}_*(X) \cong \tilde{H}^{n-*}(\overline{S^n \setminus X})$$

Pf later

Example \$X \subseteq S^3\$ knot (i.e. \$X = \text{image}(S^1 \xrightarrow{\text{homeomorphism}} S^3)\$ onto the image)

\$\Rightarrow N(X) \cong \text{solid torus} \cong S^1\$

\$\Rightarrow \tilde{H}_0(X) = 0 = \tilde{H}^2(S^3 \setminus X)\$

\$\tilde{H}_1(X) = \mathbb{Z} = \tilde{H}^1(S^3 \setminus X)\$

\$\tilde{H}_2(X) = 0 = \tilde{H}^0(S^3 \setminus X)\$

\$\uparrow\$ embedding

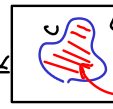
so the homology of a knot complement does not tell knots apart (always same)

Theorem (Jordan curve theorem)

\$C \cong S^1\$ closed curve in \$\mathbb{R}^2 \subseteq S^2\$

\$\Rightarrow \mathbb{R}^2 \setminus C\$ has 2 path-components (= connected components)

Similarly for \$S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1}\$.



\$\uparrow\$ Alexander duality

Pf \$S^n \cong C \subseteq \mathbb{R}^{n+1} \subseteq S^{n+1} \Rightarrow \mathbb{Z} \cong \tilde{H}_n(S^n) \cong \tilde{H}^0(S^{n+1} \setminus C)\$

\$\Rightarrow H^0(S^{n+1} \setminus C) \cong \mathbb{Z}^2\$

\$\Rightarrow S^{n+1} \setminus C\$ has 2 path components. \$\square\$

Proof Alexander duality

Abbreviate \$N = N(X)\$

(mfd of \$X\$ which is \$\cong X\$)

\$Y := S^n \setminus N \cong S^n \setminus X\$

for \$* \leq n-1\$

\$\tilde{H}^{n-*}(\overline{Y}) = H^{n-*}(\overline{Y})\$

\$\cong H_{*+1}(Y, \partial Y)\$

Lefschetz

\$\cong H_{*+1}(S^n, \overline{N})\$

exc.

\$\cong \tilde{H}_*(\overline{N})\$

LES using \$* \leq n-1\$

\$\cong \tilde{H}_*(\overline{N}) \oplus \mathbb{Z} \cong H^0(Y)\$

for \$* = n-1\$

\$\tilde{H}^0(Y) \oplus \mathbb{Z} \cong H^0(Y)\$

\$\cong H_n(Y, \partial Y)\$

Lef.

\$\cong H_n(S^n, \overline{N})\$

exc.

\$\cong \tilde{H}_{n-1}(\overline{N}) \oplus \mathbb{Z}\$

Explanation of \$\uparrow\$

LES: \$0 \rightarrow \tilde{H}_n(S^n) \rightarrow H_n(S^n, \overline{N}) \rightarrow \tilde{H}_{n-1}(\overline{N}) \rightarrow 0\$ is SES

\$\downarrow\$ quotient

\$\cong H_n(S^n, S^n \setminus \infty) \cong \mathbb{Z}\$

\$\uparrow\$ (see Cor. to Poincaré-Lefschetz) using: each (path-) connected component of the manifold \$\overline{N}\$ has non-empty boundary.



\$\uparrow\$ Hence that quotient map gives a splitting of the SES.

for \$* = n\$ \$H^{n-*}(\overline{Y}) = H^{-1}(Y) = 0\$

\$H_n(X) \cong H_n(N) \cong H^0(N, \partial N) = 0. \square\$

\$\uparrow\$ Lefschetz duality (see \$\star\$)