C3.1 Algebraic Topology

Sheet 3

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Convention: all spaces are topological spaces, maps of spaces are always continuous.

1) Construct a degree d map $S^n \to S^n$ for any $n \geq 1$.

2) Given finitely generated abelian groups $A_1, \ldots, A_n$, construct a space with
$$H_*(X) \cong \begin{cases} \mathbb{Z} & * = 0 \\ A_k & * = k \in \{1, \ldots, n\} \\ 0 & \text{else} \end{cases}$$
(Hint. CW-complex)

3) Let $f, g : S^n \longrightarrow S^n$ satisfy $f(x) \neq g(x)$, $\forall x \in S^n$.
Prove that $f \simeq -\mathrm{id} \circ g$. (Hint. consider $\frac{\varphi_t}{\|\varphi_t\|}$ where $\varphi_t = tf - (1-t)g$)
Deduce that

- if $f : S^n \to S^n$ has no fixed point then $f \simeq -\mathrm{id}$.
- if G is a group acting continuously and freely on $\underline{\underline{S^{2n}}}$ then $G = 1$ or $\mathbb{Z}/2$. (Hint. degree)

$g \neq e \in G$ has no fixed points

4) a) In the CW complex for $\mathbb{C}P^n$ from the course notes, show that the attaching maps commute with the obvious inclusions $S^{k-1} \subseteq S^k$ via $\mathbb{R}^k \equiv \mathbb{R}^k \times 0 \subseteq \mathbb{R}^{k+1}$, and $\mathbb{C}P^k \subseteq \mathbb{C}P^{k+1}$ via $\mathbb{C}^{k+1} \equiv \mathbb{C}^{k+1} \times 0 \subseteq \mathbb{C}^{k+2}$.
(You have to decide in which dimensions to consider these inclusions, and also recall $\mathbb{R}^2 \cong \mathbb{C}, (x,y) \mapsto x+iy$)

b) Explain why $\mathbb{R}P^n \cong D^n / (\pm \mathrm{id}$ action on $\partial D^n)$.
Under this identification, show that the i-th hyperplane $x_i = 0$ intersects $\mathbb{R}P^n$ in a copy of $\mathbb{R}P^{n-1}$. Show that the corresponding inclusion $\mathrm{incl}_i : \mathbb{R}P^{n-1} \longrightarrow \mathbb{R}P^n$ induces isomorphisms $H_*(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \to H_*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $* \neq n$. (Hint: "homotope it")
State and prove an analogous result for $\mathbb{C}P^n$ (using $\mathbb{Z}$).

means: replace $\mathbb{Z}$, $\mathrm{Hom}(\cdot, \mathbb{Z})$ by $\mathbb{Z}/2$, $\mathrm{Hom}(\cdot, \mathbb{Z}/2)$ in definitions of C_*, C^*

c) Use the Cultural Remark on page 57 of the notes for this exercise:
Compute the cup product to deduce
$$H^* \mathbb{C}P^n \cong \mathbb{Z}[x] / x^{n+1} \qquad |x| = 2$$
$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[y] / y^{n+1} \qquad |y| = 1$$

You may assume as known that $\mathbb{C}P^n$, $\mathbb{R}P^n$ are compact connected smooth manifolds, and that $\mathbb{C}P^n$ is orientable.

5) Let $\mathbb{CP}^\infty = \bigcup_{n\geq 0} \mathbb{CP}^n$, $S^\infty = \bigcup_{n \geq 0} S^n$, $\mathbb{RP}^\infty = \bigcup_{n\geq 0} \mathbb{RP}^n$ (using the natural inclusions from 4(a))

a) Describe a CW-complex structure on these spaces and compute H_*.

b) Compute $H_*(\mathbb{RP}^\infty; \mathbb{Z}/2)$

c) Describe the ring structure on their cohomologies (for $\mathbb{RP}^\infty$ work over $\mathbb{Z}/2$)

6) $Y = (X \cup \mathbb{D}^m) / (\text{attaching map } \varphi : \partial \mathbb{D}^m \to X)$ ← so identify $\underset{\partial\mathbb{D}^m}{\underset{\wedge}{x}} \sim \underset{X}{\underset{\wedge}{\varphi(x)}}$

$m \geq 1$

Prove:
$$H_*(Y) \cong \begin{cases} H_*(X) & * \neq m-1, m \\ H_{m-1}(X)/\operatorname{Im}\varphi_* & * = m-1 \\ H_m(X) \oplus \operatorname{Ker}\varphi_* & * = m \end{cases}$$

← (Hint. Consider $(Y, Y \setminus D)$ where $D \subseteq \mathbb{D}^m$ is a closed disc in the interior of $\mathbb{D}^m$)

7) a) Prove that if each $x_i \in X_i$ has a contractible neighbourhood, then:

$$H^*(\bigvee_i X_i) \cong \prod_i H^*(X_i) \text{ for } * \geq 1 \text{ is an iso of } \underline{\text{rings}}.$$

b) Show that $S^1 \vee S^1 \vee S^2$ and T^2 have the same homology, but different cohomology rings.

8) a) Let $X =$ **Moore space** $M(\mathbb{Z}/m, n) = \dfrac{S^n \cup \mathbb{D}^{n+1}}{\varphi : \partial \mathbb{D}^{n+1} = S^n \to S^n}$ of degree m, $n \geq 1$

Compute $H_*^{CW}(X)$ and $H^*_{CW}(X)$.

b) Let $Y = (\mathbb{CP}^2 \cup \mathbb{D}^3) / \left(\text{attaching map } \varphi : \partial \mathbb{D}^3 = S^2 \xrightarrow{\text{degree } p \text{ map}} S^2 \cong \mathbb{CP}^1 \subseteq \mathbb{CP}^2 \right)$

Compute $H^*_{CW}(Y)$.

c) For $X = M(\mathbb{Z}/p, 2)$:

Show that $H^*(Y) \cong H^*(X \vee S^4)$ as rings

but $H^*(Y; \mathbb{Z}/p) \not\cong H^*(X \vee S^4; \mathbb{Z}/p)$ ← (means: replace $\mathbb{Z}$, $\operatorname{Hom}(\cdot, \mathbb{Z})$ by $\mathbb{Z}/p$, $\operatorname{Hom}(\cdot, \mathbb{Z}/p)$ in constructions of C_* and C^*)

9) Compute directly the cup product structure on $H^*(K)$ and $H^*(K; \mathbb{Z}/2)$, where $K =$ Klein bottle.

(i.e. do not use intersection theory, only use Δ-complexes and the definition of $\cup$)

10) Let $I = [0,1]$. Build orientation-preserving homeomorphisms of pairs

$$(\mathbb{D}^n, S^{n-1}) \cong (I^n, \partial I^n) \cong (I^\ell \times I^k, \partial I^\ell \times I^k \cup I^\ell \times \partial I^k)$$
$$\cong (\mathbb{D}^\ell \times \mathbb{D}^k, S^{\ell-1} \times \mathbb{D}^k \cup \mathbb{D}^\ell \times S^{k-1})$$

where $\ell + k = n$.