C3.1 Algebraic Topology

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Sheet 4

Convention: all spaces are topological spaces, maps of spaces are always continuous.

1) a) For M an oriented closed connected n-mfd, prove that (you may use Poincaré duality and universal coefficients thms)

- $H^n M \cong \mathbb{Z}$
- $H_{n-1}(M)$ has no torsion
- $\exists$ a generator $\omega_M \in H^n(M)$ with $\omega_M([M]) = 1$.

b) For M, N oriented closed connected n-mfds, $f: M \to N$

Prove that
$$f^*: H^n(N) \longrightarrow H^n(M)$$
$$\omega_N \longmapsto \deg f \cdot \omega_M.$$

c) Let $f: S^n \to T^n = S^1 \times \cdots \times S^1$, $n \geq 2$. Prove $\deg f = 0$.

Construct a map $T^n \to S^n$ of non-zero degree.

2) Show that any matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ defines a map $f: T^n \to T^n$ on the **n-torus** $T^n = \mathbb{R}^n / \mathbb{Z}^n\text{-translation action} \cong S^1 \times \cdots \times S^1$.

Describe $f_*: H_1 T \longrightarrow H_1 T$ in terms of explicit generators.

Show that $\deg f = \det A \in \mathbb{Z}$.

Cultural Rmk Any Lie group homomorphism $\varphi: T^n \to T^n$ gives rise to such a Lie algebra homomorphism $D_1\varphi = A : \mathbb{R}^n \cong \mathrm{Lie}\, T^n \to \mathrm{Lie}\, T^n \cong \mathbb{R}^n$, $\mathbb{Z}^n \longrightarrow \mathbb{Z}^n$

3) a) For M, N compact connected orientable n-manifolds, prove that $M \# N$ is also a compact connected orientable n-mfd, and that (Hint. M.V.)

$$\boxed{H_*(M\#N) \cong H_*(M) \oplus H_*(N) \quad \text{for } 1 \leq * \leq n-1}$$

b) Formulate and prove such an isomorphism on cohomology, as a ring iso.

c) What can you say about the case $* = n$, and cup products of $H^*(M), H^*(N)$ classes that land in $H^n(M\#N)$?

d) Deduce what $H_*(\Sigma_g)$, $\chi(\Sigma_g)$ and the ring $H^*(\Sigma_g)$ are, for the genus g surface Σ_g.

4) a) Verify that: $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}; G) = 0$, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d; G) \underset{d \neq 0}{\cong} G/d\cdot G$ $\forall$ abelian group G

b) Use the universal coefficients theorem to compute $H^*(\mathbb{RP}^3; \mathbb{Q}/\mathbb{Z})$

c) Compute $H_*^{CW}(\mathbb{RP}^3; \mathbb{Q}/\mathbb{Z})$ and $H^*_{CW}(\mathbb{RP}^3; \mathbb{Q}/\mathbb{Z})$ directly.

d) We typically expect the torsion of H_* to move <u>up</u> by 1 in H^*, how come that failed in (c)?

5) Let X = **Moore space** $M(\mathbb{Z}/m, n) = S^n \cup D^{n+1} / \varphi: \partial D^{n+1} = S^n \to S^n$ of degree m

a) Show that the quotient map $X \to X/S^n \cong S^{n+1}$ is zero on $\tilde{H}_*$ but non-zero on $\tilde{H}^*$.

b) Deduce that in the universal coefficient theorem the splitting cannot be natural.

6) State and prove a locality theorem for cohomology when viewed as a <u>ring</u>. (Hint. Naturality of the universal coefficients SES)

7) Show that $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ ($\overline{\mathbb{CP}^2}$: $\mathbb{CP}^2$ with opposite orientation) have the same homology but have a different cup product on cohomology.
(Hint. Compare quadratic forms associated to the symmetric bilinear form $H^2 \times H^2 \to H^4$)
Explain why this argument does not work if we use $\mathbb{R}$-coefficients.

8) a) Let W be a compact oriented $(n+1)$-mfd with boundary $M = \partial W$. Prove that $\chi(M) = 2\chi(W)$ if n even.

b) Can $\mathbb{RP}^2$ arise as the boundary of a compact 3-manifold?

9) <u>**Borsuk–Ulam Theorem**</u> Prove that if $f: S^n \to S^n$ is an odd map $(f(-x) = -f(x))$ then $\deg f$ is odd. Deduce that if $g: S^n \to \mathbb{R}^n$ then $\exists\, x \in S^n$ with $g(x) = g(-x)$.

<u>Application</u>: show that there are two antipodal points on the Earth's surface with the same temperature and barometric pressure.

<u>Hints</u>: f induces a map $\bar{f}: \mathbb{RP}^n \to \mathbb{RP}^n$. Show that $\bar{f}: H_1\mathbb{RP}^n \to H_1\mathbb{RP}^n$ is iso (recall $H_1\mathbb{RP}^n \cong \mathbb{Z}/2$ generated by any path in S^n from a point x to $-x$), deduce that $\bar{f}^*$ is iso on $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$
To show $\deg f$ odd, suffices to show $H_n(S^n; \mathbb{Z}/2) \to H_n(S^n; \mathbb{Z}/2)$ sends $[S^n] \to [S^n]$ (Hint: universal coeff. thm)
Consider "transfer map" $C_*(\mathbb{RP}^n; \mathbb{Z}/2) \to C_*(S^n; \mathbb{Z}/2)$: singular simplex $(\sigma: \Delta^n \to \mathbb{RP}^n) \mapsto \tilde{\sigma} + a \circ \tilde{\sigma} =$ (sum of possible "lifts" of σ to S^n)
Show it is functorial w.r.t. f and then consider fund. class $[\mathbb{RP}^n]$ over $\mathbb{Z}/2$. (antipodal map)

10) A <u>**good cover**</u> of a manifold is an open cover U_i such that $U_i \cong \mathbb{R}^n$ and $U_{i_1} \cap \dots \cap U_{i_k} \cong \mathbb{R}^n$ or $\emptyset$, for all $i_1, \dots, i_k, k$.
<u>Fact/Example</u>: Smooth manifolds always admit a good cover.
Prove that any manifold M which admits a <u>finite</u> good cover has finitely generated homology groups.