B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 3

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Exercise 1. The curvatures for a torus in $\mathbb{R}^3$ and Gauss-Bonnet.

Recall the torus in $\mathbb{R}^3$ given by

$$T^2 = \{((a + b\cos\psi)\cos\theta,\ (a + b\cos\psi)\sin\theta,\ b\sin\psi) : \text{ all } \theta, \psi \in [0, 2\pi]\}$$

where $a > b > 0$ are constants. Calculate the first fundamental form, the second fundamental form, the principal directions, the principal curvatures, the mean curvature and the Gaussian curvature K. In a picture, shade the regions where K is positive, zero, and negative.

Explicitly compute the area integral $\int K\,dA$. Using the Gauss-Bonnet theorem, deduce from this the Euler characteristic of T^2.

Exercise 2. Geodesic curvature.

Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a smooth periodic map with period ℓ, i.e. $\gamma(t + \ell) = \gamma(t)$ for all $t \in \mathbb{R}$. Assume that $\gamma'(t) \neq 0$ and that the curvature $k(t) = \|\gamma''(t)\| \neq 0$ for all $t \in \mathbb{R}$. Also, assume that the closed curve $\gamma : [0, \ell] \to \mathbb{R}^3$ is parametrized by arc length.

Show that the curve $a(t) = \gamma''(t)/k(t)$ lies in the unit sphere $S^2 \subset \mathbb{R}^3$ and that $a(t)$ is perpendicular to $\gamma'(t)$. Let $b(t) = \gamma'(t) \times a(t)$. Show that $\gamma'(t)$, $a(t)$, $b(t)$ form an orthonormal basis of $\mathbb{R}^3$, and deduce that $a = b \times \gamma'$.

Deduce that $b'(t) = \tau(t)a(t)$ for some function $\tau(t)$, and that $a'(t) = -\tau b - k\gamma'$.

Let s be the arc length of $a(t)$, so $\|\frac{da}{ds}\| = 1$. By computing $\frac{da}{ds} \equiv \frac{da}{dt}\frac{dt}{ds}$, show that $(\frac{dt}{ds})^2 = \frac{1}{\tau^2+k^2}$. Show that the signed geodesic curvature of a inside S^2 is $-\frac{d}{ds}\tan^{-1}(\frac{\tau}{k})$.

[*Hint. Recall that* $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.]

Assume that the above curve $a(t) \in S^2$ does not self-intersect, and therefore divides the sphere into two regions. Prove that those two regions of S^2 have equal area.

Exercise 3. Geodesics.

Take a rectangular strip of paper and draw a line down the middle. Make a Möbius band in the usual way by joining two opposite edges along opposite directions. The drawn line is a closed curve. Show that however you move the strip in $\mathbb{R}^3$, that closed curve is never planar. [*Hint. If the curve is planar, consider the outward normal to the curve in the given plane.*]

Exercise 4. Critical points and Euler characteristic.

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \cos 2\pi x + \cos 2\pi y$. Find the critical points and classify them into minima, saddle points and maxima.

Explain why f defines a smooth function on the torus $\mathbb{R}^2/\mathbb{Z}^2$, and compute the Euler characteristic of the torus by appropriately counting the critical points.

Exercise 5. Riemann curvature, Ricci curvature, scalar curvature.

Recall in Exercise sheet 2 we defined the tangential derivative, which in the basis $X_1 = \partial_x F$, $X_2 = \partial_y F$ *defines the Christoffel symbols:*

$$\nabla_i X_j = \Gamma^k_{ij} X_k \tag{*}$$

where from now on we use the Einstein summation convention*: you sum over repeated indices (so above, we sum over k since it appears once as an upper index and once as a lower index).*

The Riemann curvature tensor R^m_{ijk} *measures how much the tangential derivatives fail to commute. It is defined by:*

$$\boxed{\begin{aligned} R(X_i, X_j)X_k &= \nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k \\ &= R^m_{ijk} X_m \end{aligned}} \tag{**}$$

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It's often useful to dot the above with another basis vector X_ℓ, which defines

$$\boxed{R_{ijk\ell} = R(X_i, X_j)X_k \cdot X_\ell = I(R(X_i, X_j)X_k, X_\ell)}$$

You can pass from one to the other by the lowering/raising of indices using the Riemannian metric $g_{ij} = I_{ij} = X_i \cdot X_j$. Explicitly $R_{ijk\ell} = R_{ijk}^m g_{m\ell}$, which you can undo by using the inverse matrix g^{ij} of g_{ij} which satifies $g^{ij}g_{jk} = \delta_k^i$ (summing over j).

By substituting (*) into (**), show that R is determined completely by the Christoffel symbols and the Riemannian metric $g_{ij} = I_{ij}$ (the first fundamental form):

$$\boxed{\begin{array}{rcl} R_{ijk}^m & = & \partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{jk}^p \Gamma_{ip}^m - \Gamma_{ik}^p \Gamma_{jp}^m \\ R_{ijk\ell} & = & (\partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{jk}^p \Gamma_{ip}^m - \Gamma_{ik}^p \Gamma_{jp}^m) g_{m\ell}. \end{array}}$$

Notice that by Exercise Sheet 2, the Γ_{ij}^k are determined by g_{ij}, so R only depends on the Riemannian metric g_{ij}. Therefore R doesn't change under isometry, even if you pick a different isometric embedding of the surface into $\mathbb{R}^3$.

Explain why $R_{ijk\ell}$ is anti-symmetric in the indices i, j.

Recall by Exercise Sheet 2 that

$$\partial_i I(v, w) = I(\nabla_i v, w) + I(v, \nabla_i w).$$

Since $I(v, w)$ is a smooth function, its partial derivatives commute: $\partial_j \partial_i I(v, w) = \partial_i \partial_j I(v, w)$. Deduce that $R_{ijk\ell}$ is anti-symmetric in k, ℓ.

Deduce that R is determined by just one value: R_{1212}.

Recall by Exercise Sheet 2 that

$$\partial_i X_j = \nabla_i X_j + I\!I_{ij}\, n = \Gamma_{ij}^k X_k + I\!I_{ij}\, n.$$

Since F is smooth, the partial derivatives commute: $\partial_2 \partial_1 \partial_1 F = \partial_1 \partial_1 \partial_2 F$, so $\partial_2 \partial_1 X_1 = \partial_1 \partial_1 X_2$. From this, and the above equation, deduce by brute force calculation that

$$0 = (\partial_2 \partial_1 X_1 - \partial_1 \partial_1 X_2) \cdot X_2 = R_{2112} - \det I\!I_F.$$

Deduce, using $K = \frac{\det I\!I_F}{\det I_F}$ (from lectures), that

$$\boxed{R_{1212} = -K \det I_F}$$

Finally, deduce Gauss' *Theorema Egregium*: the Gaussian curvature K only depends on the Riemannian metric, so it is the same for two isometric surfaces.

The Ricci curvature *is defined as the metric trace of $R_{ijk\ell}$ in the j, ℓ indices, explicitly:*

$$R_{ik} = R_{ijk\ell} g^{j\ell} = R_{ijk}^j$$

and the scalar curvature *is the metric trace of the Ricci curvature, explicitly:*

$$R = g^{ij} R_{ij}$$

as usual summing over repeated indices.

Show that for surfaces in $\mathbb{R}^3$,

$$\boxed{\mathrm{R}_{ij} = -K g_{ij} \quad \text{and} \quad R = -2K}$$

Cultural Remark. *The above ideas are very important, for example the Einstein field equations for general relativity are*

$$R_{ij} - \frac{1}{2} g_{ij} R + g_{ij} \Lambda = \frac{8\pi G}{c^4} T_{ij}$$

where the left-hand side encodes the geometry of the universe and the right-hand side encodes the physical properties of the universe. The symbols are: G = Newton's gravitational constant, c = speed of light, Λ = cosmological constant, T_{ij} = stress-energy tensor (which measures the matter/energy content of spacetime). More of this in **C7.5/C7.6 General Relativity**.

In a vacuum, these equations become $R_{ij} = 0$ when the cosmological constant is zero, and $R_{ij} = \Lambda g_{ij}$ (so a multiple of the metric) otherwise. Manifolds with a vanishing Ricci tensor are called Ricci-flat manifolds*, and manifolds with a Ricci tensor proportional to the metric are called* Einstein manifolds*. They are objects of great interest nowadays in geometry.*