B3.2 GEOMETRY OF SURFACES - EXERCISE SHEET 2

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Exercise 1. Tori and figure 8 loops.
A figure 8 loop consists of two circles touching at a point. Show that a torus can be obtained by attaching a disc onto a figure 8 loop.

Exercise 2. The Euler characteristic constrains graphs.
Given five points in the plane, show that it is impossible to connect each pair by paths which do not cross. Is it possible for five points in a torus?

Exercise 3. The Euler characteristic constrains Platonic solids.
Using the Euler characteristic, show that there are no more than five Platonic solids.[1]

Exercise 4. The classification of elliptic curves.
Recall from lectures that there are bijections

$$\frac{\left\{\begin{array}{c}\textit{Riemann surfaces}\\ \textit{homeomorphic to a torus}\end{array}\right\}}{\textit{biholomorphisms}} \longleftrightarrow \frac{\left\{\begin{array}{c}\textit{Quotients}\\ \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)\ \textit{with}\ \tau\in\mathbb{H}\end{array}\right\}}{\textit{biholomorphisms}} \longleftrightarrow \mathbb{H}/PSL(2,\mathbb{Z}),$$

where the second map is $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \leftrightarrow [\tau]$, *and where* $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\pm I$ *acts on the upper half-plane* $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ *by Möbius maps.*

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$, show that $\mathrm{Im}(Az) = \frac{1}{|cz+d|^2} \cdot \mathrm{Im}(z)$. Deduce that, given a constant K, only finitely many $c, d \in \mathbb{Z}$ satisfy $\mathrm{Im}(Az) > K$.

It turns out by some easy group theory that $SL(2,\mathbb{Z})$ *is generated by the matrices* $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, *although we will not need this fact. The corresponding Möbius maps* $S(z) = -1/z$ *and* $T(z) = z + 1$ *are rather useful in this exercise.*

Show that $\mathbb{H}/PSL(2,\mathbb{Z})$ is a topological space homeomorphic to $\mathbb{C}$, by first showing that each point of $\mathbb{H}/PSL(2,\mathbb{Z})$ has a representative inside the "strip"

$$\{\tau \in \mathbb{H} : |\mathrm{Re}(\tau)| \leq 1/2, |\tau| \geq 1\}$$

and then checking that the only remaining identifications are on the boundary of the strip.
Hint. Try to maximize the imaginary part for the orbit of z under the action.

Does $PSL(2,\mathbb{Z})$ act freely[2] on $\mathbb{H}$? Briefly comment on why the natural local coordinates from $\mathbb{H}$ do *not* make $\mathbb{H}/PSL(2,\mathbb{Z})$ into a topological surface (let alone a Riemann surface).

Cultural remark. *By Exercise sheet 1 equations like* $w^2 = 4z^3 - g_2 z - g_3$ *determine Riemann surfaces homeomorphic to a torus. It turns out that this is biholomorphic to* $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ *if we take coefficients* $g_2 = 60\sum(m+n\tau)^{-4}$ *and* $g_3 = 140\sum(m+n\tau)^{-6}$ *summing over all integers* $(m,n) \neq (0,0)$. *It also turns out that* $\mathbb{H}/PSL(2,\mathbb{Z})$ *can be made into a Riemann surface via the biholomorphism* $\mathbb{H}/PSL(2,\mathbb{Z}) \cong \mathbb{C}, \tau \mapsto j(\tau) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2}$, *called the* elliptic modular function *or* Klein's j-invariant.

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[1] A Platonic solid is a convex polyhedron with congruent faces consisting of regular polygons and the same number of faces meet at each vertex.

[2] A group G acts freely on X if stabilizers are trivial, explicitly: if $g \bullet x = x$ then $g = 1$.