

# LECTURE 5

## HOMOMORPHISMS

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Def Lie group homomorphism  $\varphi: G \rightarrow H$  means 1)  $\varphi$  hom of gp's  
2)  $\varphi$  smooth map

- When  $H = GL(n, \mathbb{R})$ ,  $\varphi$  is called a representation of G

- Lie group isomorphism if  $\varphi$  bijective and  $\varphi^{-1}$  Lie gp hom

(so:  
1)  $\varphi$  iso of gp's  
2)  $\varphi$  diffeo)

Warning Let  $\mathbb{R}_{\text{disc}} = \mathbb{R}$  with discrete topology (each point is an open set) is Lie gp using + identity:  $\mathbb{R}_{\text{disc}} \rightarrow \mathbb{R}$  is a bijective Lie gp hom, inverse not continuous!  
(Exp/naturality tells you nothing since  $T_0 \mathbb{R}_{\text{disc}} = \{0\}$ )

Rmk Local diffeo + bijective = diffeo

Lemma 1  $\varphi: G \rightarrow H$  bijective Lie gp hom and  $\varphi^{-1}$  continuous near 1  $\Rightarrow \varphi$  Lie gp iso

Pf By naturality of exp:  $(\text{nbhd of } 0 \in T_1 G) \xrightarrow{D_1 \varphi} (\text{nbhd of } 0 \in T_1 H)$

May not be diffeo  
May need to pick smaller  
(nbhd of  $1 \in G$ ) =  $V \subseteq \varphi^{-1}(U)$   
 $\Rightarrow$  replace  $U$  by  $\varphi(V)$   
 $\varphi(V) = (\varphi^{-1})^{-1}(V)$  open  
since  $\varphi^{-1}$  cts near 1.  
This fails in Warning above

$$\varphi^{-1}(U) = (\text{nbhd of } 1 \in G) \xrightarrow{\varphi} U = (\text{nbhd of } 1 \in H)$$

$\varphi$  bijective  $\Rightarrow D_1 \varphi$  bijective near 0 hence iso (since linear map)  
inverse function theorem  $\Rightarrow \varphi$  local diffeo near 1  $\in G$

$$\varphi \text{ hom} \Rightarrow (\phi_{\varphi(g)} \circ \varphi \circ \phi_{g^{-1}})(h) = \varphi(g) \varphi(g^{-1}h) = \varphi(h)$$

$\Rightarrow \varphi$  local diffeo near  $g$  (since  $\phi$  maps are diffeos and )

(Alternative proof:  $D_1 \phi_{\varphi(g)} \circ D_1 \varphi \circ D_g \phi_{g^{-1}}: T_g G \rightarrow T_1 G \rightarrow T_1 H \rightarrow T_{\varphi(g)} H$ )

The proof also shows:

$= D_g \varphi$  is iso, then apply inverse function thm  $\blacksquare$

(for  
 $\varphi: G \rightarrow H$   
Lie gp hom)

Lemma 2

$D_1 \varphi$  iso  $\Rightarrow D_g \varphi$  iso for any  $g$   $\blacksquare$

Lemma 3

$\varphi$  locally homeo near 1  $\Rightarrow \varphi$  local diffeo near any  $g$ .

Exercise

$\varphi$  bijective and  $\dim G = \dim H \Rightarrow \varphi$  Lie gp iso

(Hint: use injectivity of  $\varphi: V \rightarrow U$  above to get  $D_1 \varphi$  injective, then use  $\dim G = \dim H$ )

Harder exercise Manifolds are usually required to be second countable ( $\exists$  countable basis for the topology). If we require Lie gps to be 2nd countable, is it true that

Bijective Lie gp hom  $\Rightarrow$  Lie gp iso?

Idea: above if  $D_1 \varphi(T_1 G) \neq T_1 H$  then it is a strictly lower dimensional vector subspace, so  $\varphi(V) \subseteq U$  is a strictly lower dimensional submanifold in  $H$ , so you need uncountably many left translates  $\phi_h(\varphi(V))$  to get a disjoint cover of  $U$  (by non-open sets). Then  $\varphi^{-1}(\phi_h(\varphi(V)))$  give uncountably many disjoint open sets covering  $V$  contradicting  $G$  is 2nd cble  $\Rightarrow \varphi^{-1}(\phi_{h \circ g}(\varphi(V))) = \varphi^{-1}\varphi(\phi_g V) = \phi_g(V)$  where  $c(g) = h$ .

Def Lie algebra homomorphism  $\psi: (V, [\cdot, \cdot]_V) \rightarrow (W, [\cdot, \cdot]_W)$  means

1)  $\psi$  linear map (homomorphism of vector spaces)

2)  $[\psi x_1, \psi x_2]_W = \psi [x_1, x_2]_V \quad \text{all } x_1, x_2 \in V$

- When  $W = \text{Mat}_{n \times n}(\mathbb{R})$ ,  $[B, C]_W = BC - CB$ ,  $\psi$  is called a representation of V

- Lie algebra isomorphism if  $\psi$  also bijective (hence  $\psi$  iso of v-s.)

MORE ABSTRACTLY FOR REPRESENTATIONS CAN REPLACE :

For Lie gps:	$GL(n, \mathbb{R})$	$\text{Aut}(\mathbb{R})$
For Lie algs:	$\text{Mat}_{n \times n}(\mathbb{R})$	$\text{End}(\mathbb{R}) = \text{Hom}(\mathbb{R}, \mathbb{R})$

where  $\mathbb{R}$  is a vector space

$$\varphi(g)(r) + (g)(r)$$

Often call  $\mathbb{R}$  the representation and write  $\begin{matrix} g \cdot r \\ x \cdot r \end{matrix}$  instead of

## EXAMPLES

1)  $\gamma: \mathbb{R} \rightarrow G$ , 1-param. subgroups are Lie gp homs

example:  $\mathbb{R} \rightarrow S^1$ ,  $x \mapsto e^{2\pi i x}$  (or  $x \bmod \mathbb{Z}$  if view  $S^1 = \mathbb{R}/\mathbb{Z}$ )

2)  $SU(2) \rightarrow SO(3)$  on Q-sheet 2

3)  $A_g: G \rightarrow G$

$$A_g(h) = g h g^{-1}$$

Lie group isomorphism

(the inverse is  $A_g^{-1} = A_{g^{-1}}$ )

4)

$$\text{Ad}: G \rightarrow \text{Aut}(T_e G) \cong \text{Aut}(\text{Lie } G)$$

$$\text{Ad}(g) = D_{g^{-1}} A_g$$

ADJOINT  
REPRESENTATION  
OF  $G$

$\nwarrow D_{g^{-1}} A_g$  is an automorph since has inverse  $D_{g^{-1}} A_g^{-1}$  (chain rule)

5)

$$\text{ad} = D_{g^{-1}} \text{Ad}: \text{Lie } G \cong T_e G \rightarrow \text{End}(T_e G)$$

Why is  $D_{g^{-1}} \text{Ad}$  a Lie alg hom?

ADJOINT  
REPRESENTATION  
OF  $\text{LIE}(G)$   
(in Lecture 6 will prove that)  
 $(\text{ad } X)(Y) = [X, Y]$

Theorem For  $\varphi: G \rightarrow H$  Lie group hom

$$\varphi: \text{Lie } G \cong T_e G \xrightarrow{D_e \varphi} T_e H \cong \text{Lie } H \text{ a Lie algebra hom}$$

Rmk: not obvious since  $\varphi$  usually not a diffeo, so can't push-forward a v.f.

FROM NOW ON ABBREVIATE  $\mathfrak{g} = \text{LIE}(G)$

Consequences By naturality of  $\exp$ :

Cor

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{Ad}(g)} & \mathfrak{g} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{dg} & G \end{array}$$

$$\begin{array}{ccc} g & \xrightarrow{\text{ad}} & \text{End}(g) \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\text{Ad}} & \text{Aut}(\mathfrak{g}) \end{array}$$

$$g \cdot \exp(X) \cdot g^{-1} = \exp(\text{Ad}(g) \cdot X)$$

$$\text{Ad}(\exp X) = \exp(\text{ad } X)$$

Example For  $G = GL(n, \mathbb{R})$   $\mathfrak{g} = \text{Mat}_{n \times n}(\mathbb{R})$

$$A_A(x) = Ax A^{-1} \text{ and } \text{Ad}(A) \cdot B = ABA^{-1}$$

1st box says:  $A e^B A^{-1} = e^{ABA^{-1}}$  (holds because  $(ABA^{-1})^n = A B^n A^{-1}$ )

Exercise What does 2nd box say, using result from lecture 6 that  $\text{ad } X = [X, \cdot]$

this  $\exp$  we know  
 $\exp(B) = I + B + \frac{B^2}{2!} + \dots$

## PROOF OF THEOREM

$$Z \in \text{Lie } G \Rightarrow Z|_g = D_g \phi^G \cdot Z|_1$$

$\phi^G_g$  means  
left-translation in  $G$   
 $\phi^G_g(\tilde{g}) = g\tilde{g}$

$$\text{call } \tilde{Z} = \varphi(Z), \text{ so: } \tilde{Z}|_h = D_h \phi^H \cdot (D_g \varphi \cdot Z|_1) \stackrel{\substack{\text{chain} \\ \text{rule}}}{=} D_g \varphi \cdot D_{\tilde{g}} \phi^G \cdot Z|_1 = (D\varphi \cdot Z)|_g$$

$$\Rightarrow \tilde{Z}|_{\varphi(g)} = D_{\tilde{g}} \underbrace{(\phi^H_{\varphi(g)} \circ \varphi)}_{\substack{\text{because } \varphi \text{ hom:} \\ \varphi(g)\varphi(\bullet) = \varphi(g \cdot \bullet)}} \cdot Z|_1 \stackrel{\substack{\text{chain} \\ \text{rule}}}{=} D_g \varphi \cdot D_{\tilde{g}} \phi^G \cdot Z|_1 \stackrel{\substack{\text{since left-inv} \\ Z|_g}}{=} (D\varphi \cdot Z)|_g$$

$$\Rightarrow \boxed{\tilde{Z}|_{\varphi(\cdot)} = D\varphi \cdot Z}$$

In general such vector fields  $Z, \tilde{Z}$   
are called  $\varphi$ -related

Rmk If  $\varphi$  was a diffeo, this would say that  $\tilde{Z}$  is the pushforward  
of  $Z$ . If  $\varphi$  not diffeo, then  $D\varphi \cdot Z$  need not be a vector field

Proposition  
(proof later)

If  $X, \tilde{X}$  and  $Y, \tilde{Y}$  are  $\varphi$ -related then  $[X, Y], [\tilde{X}, \tilde{Y}]$  are  $\varphi$ -related

continue proof:

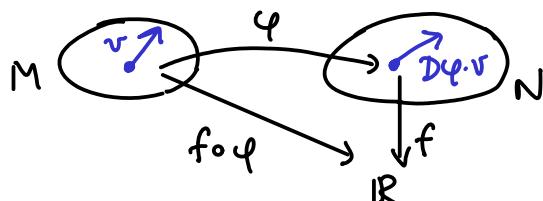
$$[\tilde{X}, \tilde{Y}]|_{\varphi(\cdot)} = D\varphi \cdot [X, Y] = \overbrace{[X, Y]}^{\substack{\text{above equation for } Z = [X, Y]}}|_{\varphi(\cdot)}$$

$$\begin{aligned} &\Rightarrow [\varphi(X), \varphi(Y)]|_{\varphi(g)} = (\varphi[X, Y])|_{\varphi(g)} \quad \forall g \in G \\ &\xrightarrow{g=1} \Rightarrow [f(X), f(Y)]|_1 = (\varphi[X, Y])|_1 \\ &\Rightarrow [f(X), f(Y)] = \varphi[X, Y] \quad \begin{array}{l} \text{(since left-invariant vector fields)} \\ \text{are determined uniquely by value at 1)} \end{array} \end{aligned}$$

## PROOF OF PROPOSITION

Given:  $\varphi: M \rightarrow N$  some v.f.  $X, Y$  on  $M$  want:  $[\tilde{X}, \tilde{Y}]|_{\varphi(\cdot)} = D\varphi \cdot [X, Y]$   
smooth map of manifolds  $\tilde{X}|_{\varphi(\cdot)} = D\varphi \cdot X$  "  $\tilde{Y}|_{\varphi(\cdot)} = D\varphi \cdot Y$  "  $\tilde{X}, \tilde{Y}$  "  $N$

Need TRICK:



PROOF OF TRICK Locally:

$$v = \sum a_j \frac{\partial}{\partial x_j} \text{ and } \varphi = (\varphi_1, \dots, \varphi_n): \mathbb{R}^m \dashrightarrow \mathbb{R}^n$$

$$v \cdot (f \circ \varphi) = \sum a_j \frac{\partial}{\partial x_j} (f \circ \varphi)$$

$$\stackrel{\text{chain rule}}{=} \sum a_j \frac{\partial f}{\partial x_i} \cdot \frac{\partial \varphi_i}{\partial y_j} = (D\varphi \cdot v) \cdot f$$

matrix for  $D\varphi$

For  $v \in T_m M$  and  $f: N \rightarrow \mathbb{R}$   
 $v \cdot (f \circ \varphi) = (D_m \varphi \cdot v) \cdot f \in \mathbb{R}$   
for vector field  $X$  on  $M$ :  
 $X \cdot (f \circ \varphi) = (D\varphi \cdot X) \cdot f$   
as functions  $M \rightarrow \mathbb{R}$

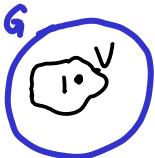
means:  

$$\begin{cases} ((D\varphi \cdot X) \cdot f)(m) = \\ = (D_m \varphi \cdot X|_m) \cdot f \end{cases}$$

$$\begin{aligned}
 (D\varphi \cdot [x, y]) \cdot f &= [x, y] \cdot (f \circ \varphi) \\
 \stackrel{\text{TRICK}}{=} X \cdot (Y \cdot (f \circ \varphi)) - Y \cdot (X \cdot (f \circ \varphi)) \\
 &= X \cdot ((D\varphi \cdot Y) \cdot f) - \text{switch } X, Y \\
 \stackrel{\text{TRICK}}{=} X \cdot ((\tilde{y} \cdot f) \circ \varphi) &- " \\
 &= (D\varphi \cdot X) \cdot (\tilde{y} \cdot f) - " \\
 \stackrel{\text{TRICK}}{=} (\tilde{X} \cdot (\tilde{y} \cdot f)) \Big|_{\varphi(1)} &- " \\
 &= [\tilde{X}, \tilde{y}] \Big|_{\varphi(1)} \quad \blacksquare
 \end{aligned}$$

Example question sheet 2:  $\varphi: SU(2) \rightarrow SO(3)$  double cover (in particular a diffeo near  $1 \in SU(2)$ )  
 $\Rightarrow \psi = D_1 \varphi: su(2) \rightarrow so(3)$  Lie algebra isomorphism!  
 $\{2 \times 2 \text{ complex}\}$   $= \{3 \times 3 \text{ real skew-symmetric}\}$   
 $\text{skew Hermitian}$  (connected component of  $1 \in G$ )

Lemma A neighbourhood  $V \subseteq G_0$  of 1 generates  $G_0$  as a group

 Pf Can assume  $V$  is open (interior( $V$ ) is smaller than  $V$ ,  $1 \in \text{Int}(V)$ )  
 $\langle V \rangle = \text{subgroup generated by } V \text{ in } G_0$   
 $\Rightarrow \langle V \rangle \subseteq G_0$  open subset (since  $v_1^{\pm 1} \dots v_k^{\pm 1}$  has nbhd  $v_1^{\pm 1} \dots v_{k-1}^{\pm 1} \cdot V \subseteq G_0$ )  
 $\Rightarrow$  cosets  $g \cdot \langle V \rangle$  are open (since  $\not\exists g$  diffeo)  
 $\Rightarrow \langle V \rangle$  closed subset (since complement of open set  $\bigcup g \cdot \langle V \rangle$  is disjoint union of cosets of  $\langle V \rangle$ .  $\bigcup g \cdot \langle V \rangle$  homeom. to  $\bigcup v_1^{\pm 1} \dots v_{k-1}^{\pm 1} \cdot (V^{\pm 1})$  open)  
 $\Rightarrow \langle V \rangle$  connected component (since open & closed)  
 $\Rightarrow \langle V \rangle = G_0$   $\blacksquare$

Theorem Let  $G$  be connected

A Lie gp hom  $\varphi: G \rightarrow H$  is uniquely determined by  $D_1 \varphi: T_1 G \rightarrow T_1 H$   
(meaning: if  $\varphi, \tilde{\varphi}: G \rightarrow H$  Lie gp homs with  $D_1 \varphi = D_1 \tilde{\varphi}$  then  $\varphi = \tilde{\varphi}$ )

Pf Naturality of exp: (small nbhd  $0 \in T_1 G$ )  $\xrightarrow{D_1 \varphi} T_1 H$   
 $\exp \downarrow \text{Diffeo}$   $\downarrow \exp$   
 $V = (\text{small nbhd of } 1 \in G) \xrightarrow{\varphi} H$

$\Rightarrow \varphi$  determined by  $D_1 \varphi$  on  $V$

$\varphi \text{ hom} \rightarrow \Rightarrow \varphi$  determined by  $D_1 \varphi$  on  $\langle V \rangle = G_0 = G$   $\blacksquare$

Warning ("not everything is determined at the identity")

$su(2) \cong so(3)$  but  $SU(2) \not\cong SO(3)$ : different topologically:

$$\overset{\text{''}}{S^3} \quad \overset{\text{''}}{\mathbb{R}P^3}$$

FACT (non-examinable)  $S^3$  is simply connected (all loops are contractible)  
 $\mathbb{R}P^3$  is not simply connected.