

ADJOINT REPRESENTATION

LAST TIME

$A_g: G \rightarrow G$ conjugation by g $A_g(h) = ghg^{-1}$

$Ad: G \rightarrow \text{Aut}(\mathfrak{g})$, $Ad(g) = D_1 A_g$

$ad = D_1 Ad: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$

$\mathfrak{g} = \text{Lie}(G)$

We will need 2 TRICKS from LECTURE 1: ① $D\varphi \cdot v = \frac{\partial}{\partial s} \Big|_{s=0} \varphi(\gamma(s))$
(where $\varphi: M \rightarrow N$, $f: M \rightarrow \mathbb{R}$, $v = [\text{curve } \gamma(s)]$) ② $v \cdot f = \frac{\partial}{\partial s} \Big|_{s=0} f(\gamma(s))$

Theorem $\underbrace{ad(X)}_{\in \text{End}(\mathfrak{g})} \cdot Y = [X, Y] \quad X, Y \in \mathfrak{g}$

pf. $ad(X) \cdot Y \stackrel{\text{def}}{=} \underbrace{(D_1(Ad) \cdot X)}_{\in \text{End}(\mathfrak{g})} \cdot Y$

$\stackrel{\text{①}}{=} \frac{\partial}{\partial s} \Big|_0 Ad(\gamma_X(s)) \cdot Y$ ← recall $\gamma_X'(s) = X$
for 1-param. subgp $\gamma_X(s)$

$\stackrel{\text{def}}{=} \frac{\partial}{\partial s} \Big|_0 D_1 A_{\gamma_X(s)} \cdot Y$

$\stackrel{\text{①}}{=} \frac{\partial}{\partial s} \Big|_0 \frac{\partial}{\partial t} \Big|_0 A_{\gamma_X(s)}(\gamma_Y(t))$

$\stackrel{\text{def}}{=} \frac{\partial^2}{\partial s \partial t} \Big|_{(0,0)} \gamma_X(s) \gamma_Y(t) \underbrace{\gamma_X(s)^{-1}}_{= \gamma_X(-s)} \leftarrow \text{Question sheet 2}$

$\stackrel{\text{①}}{=} \frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} \gamma_X(s) \gamma_Y(t) \gamma_X(0) - \frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} \gamma_X(0) \gamma_Y(t) \gamma_X(s)$

$= \frac{\partial}{\partial s} \Big|_0 \frac{\partial}{\partial t} \Big|_0 F^Y(t, \gamma_X(s)) - \frac{\partial}{\partial t} \Big|_0 \frac{\partial}{\partial s} \Big|_0 F^X(s, \gamma_Y(t))$

↑
partial derivs commute and recall $\gamma_X(0) = 1$ and $F^Y(t, g) = g \cdot \gamma_Y(t)$ is flow of Y (LECTURE 3)

$\stackrel{\text{def. of flow}}{=} \frac{\partial}{\partial s} \Big|_0 Y \Big|_{\gamma_X(s)} - \frac{\partial}{\partial t} \Big|_0 X \Big|_{\gamma_Y(t)}$

$= [X, Y] \Big|_1$, by next Lemma ■

partial derivatives commute and chain rule for composition

$\mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \rightarrow G$

$s \mapsto (s, -s)$

$(x_1, x_2) \mapsto \gamma_X(x_1) \gamma_Y(t) \gamma_X(x_2)$

$\Rightarrow \frac{\partial}{\partial s} = \sum \frac{\partial x_i}{\partial s} \frac{\partial}{\partial x_i}$
 $= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}$

$$\text{Lemma } X, Y \in \mathfrak{g} \Rightarrow [X, Y] \Big|_1 = \frac{\partial}{\partial s} \Big|_0 Y \Big|_{\gamma_X(s)} - \frac{\partial}{\partial s} \Big|_0 X \Big|_{\gamma_Y(s)}$$

Pf locally $Y = \sum b_i(x) \frac{\partial}{\partial x_i}$ so $Y \Big|_{\gamma_X(s)} = \sum b_i(\gamma_X(s)) \frac{\partial}{\partial x_i}$
 $\frac{\partial}{\partial s} \Big|_0 Y \Big|_{\gamma_X(s)} = \sum \frac{\partial}{\partial s} \Big|_0 (b_i \circ \gamma_X) \frac{\partial}{\partial x_i} \stackrel{②}{=} \sum (X \cdot b_i) \frac{\partial}{\partial x_i}$ ■

Proof of Thm also showed:

$$\text{Corollary } X, Y \in \mathfrak{g} \Rightarrow \frac{\partial^2}{\partial s \partial t} \Big|_0 \gamma_X(s) \gamma_Y(t) \gamma_X(-s)$$

Def A matrix group is a closed subgroup of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ (or linear group) (or $\text{Aut}(\mathbb{R})$ for v.s. \mathbb{R})

Rmk By LECTURE 7, this condition ensures they are Lie groups.

Examples $O(n), SO(n), U(n), SU(n), SL(n, \mathbb{R}), \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}, \dots$

EXAMPLE 1 $[X, Y] = XY - YX$ for matrix groups ($X, Y \in \text{Mat}_{n \times n}$)

$$\gamma_X(s) = 1 + sX + \sigma(s) \quad \left(f \text{ is "little-oh-es" if } \lim_{s \rightarrow 0} \frac{f(s)}{s} = 0 \right)$$

since $\gamma_X(0) = 1$ since $\gamma_X'(0) = X$ hence $\frac{\partial}{\partial s} \Big|_0 f(s) = 0$

$$\Rightarrow [X, Y] = \frac{\partial^2}{\partial s \partial t} \Big|_0 \gamma_X(s) \gamma_Y(t) \gamma_X(-s) = \frac{\partial^2}{\partial s \partial t} \Big|_0 (1 + sX + \sigma(s))(1 + tY + \sigma(t))(1 - sX + \sigma(-s))$$

(The s^2, t^2 terms vanish if do $\frac{\partial}{\partial t}, \frac{\partial}{\partial s}$) $\Rightarrow \frac{\partial^2}{\partial s \partial t} \Big|_0 (1 + tY + st(XY - YX) + \sigma(st)) = XY - YX$ ■

EXAMPLE 2 G abelian ($S^1 = U(1), T^n, \mathbb{R}^n, \dots$)

$$\Rightarrow \mathcal{A}_g(h) = hgh^{-1} = hh^{-1}g = g$$

$$\Rightarrow \mathcal{A}_g = \text{Id}$$

$$\Rightarrow \text{Ad}(g) = D, \mathcal{A}_g = \text{Id} \quad (\text{"D of linear map is the linear map"})$$

$$\Rightarrow \text{ad} = D, \text{Ad} = 0 \quad (\text{"D of constant map } g \mapsto \text{Id} \text{ is } 0\text{"})$$

$$\Rightarrow [\cdot, \cdot] \equiv 0$$

$$\Rightarrow \mathfrak{g} \text{ abelian Lie algebra} \quad (\text{so } \cong (\mathbb{R}^n, [\cdot, \cdot] = 0))$$

USING EXP TO DETERMINE \mathfrak{g} AND G

Lemma If $H \xrightarrow{\text{Lie group hom}} G$ embedding then the 1-param. subgps of H are precisely those $\gamma_X(s) \subseteq G$ which lie in H .

Proof Naturality $\mathfrak{h} \xrightarrow{\exp} G$ $\mathfrak{g} \xrightarrow{\exp} G$ $sY \mapsto sX$ since embedding can view $H \subseteq G$ and $\mathfrak{h} \subseteq \mathfrak{g}$ so $Y \equiv X, \gamma_Y^H(s) \equiv \gamma_X^G(s) \subseteq H$

converse: if $\gamma_X^G(s) \subseteq H$ then $\gamma_X^G: \mathbb{R} \rightarrow H$ is a Lie gp hom hence 1-param subgp. in H (smooth in H because smooth in G)

Consequences

- Can identify $\mathfrak{h} = \text{Lie}(H)$ with a vector subspace of \mathfrak{g} :
 $\mathfrak{h} \cong \{X \in \mathfrak{g} : \gamma_X(s) \subseteq H \text{ for small (hence all) } s \in \mathbb{R}\}$
 Lie alg. iso. (respect bracket by above Corollary)
- exp for H agrees with exp for G : $\exp(X) = \gamma_X(1) \in H$ if $X \in \mathfrak{h} \subseteq \mathfrak{g}$

EXAMPLE 3 $\mathfrak{o}(n) = \{X \in \text{Mat}_{n \times n}(\mathbb{R}) : X^T + X = 0\}$

Proof $\gamma_X(s) \subseteq O(n) \Rightarrow 1 = \gamma_X(s)^T \gamma_X(s)$
 \uparrow
 in $GL(n)$ $= (1+sX)^T(1+sX) + \sigma(s)$
 $= 1 + s(X^T + X) + \sigma(s)$ hence $X^T + X = 0$.

($\mathfrak{o}(n) = \text{Lie } O(n) \cong \mathfrak{so}(n)$
 since $O(n), SO(n)$ locally diffeo near 1)

converse:

$X^T + X = 0 \Rightarrow \gamma_X(s)^T \gamma_X(s) = \exp(sX)^T \exp(sX) \stackrel{\text{exp series for } \mathfrak{gl}(n)}{=} \exp(sX^T) \exp(sX) = \exp(-sX) \exp(sX) \stackrel{\text{inverses (Q. sheet 2)}}{=} 1$
 $\Rightarrow \gamma_X(s) \subseteq O(n)$

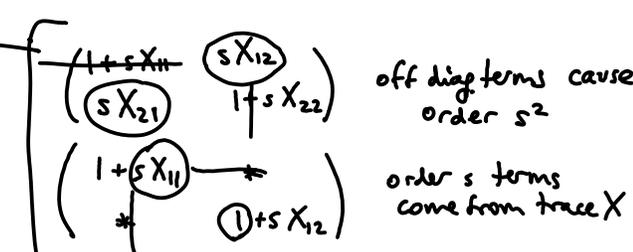
SAME PROOF SHOWS $\mathfrak{u}(n) = \{X \in \text{Mat}_{n \times n}(\mathbb{C}) : X^* + X = 0\}$

EXAMPLE 4 $\mathfrak{sl}(n) = \{X \in \text{Mat}_{n \times n} : \text{Trace}(X) = 0\}$ ($\mathfrak{sl}(n) = \text{Lie } SL(n)$, work over \mathbb{R} or \mathbb{C})

Pf $\det \gamma_X(s) = \det(1+sX) + \sigma(s) = 1 + s \cdot \text{Tr}(X) + \sigma(s)$

$\Rightarrow \frac{\partial}{\partial s} \Big|_0 \det \gamma_X(s) = \text{Tr}(X)$

$\Rightarrow \frac{\partial}{\partial t} \Big|_t \det \gamma_X(t) = \frac{\partial}{\partial s} \Big|_0 \det \gamma_X(t+s)$
 chain rule $\underbrace{\gamma_X(t) \gamma_X(s)}_{\gamma_X(t) \gamma_X(s)}$
 $= \det \gamma_X(t) \cdot \frac{\partial}{\partial s} \Big|_0 \det \gamma_X(s)$
 $= \det \gamma_X(t) \cdot \text{Tr}(X)$



Now: $\gamma_X(s) \in SL(n) \Rightarrow \det \gamma_X(s) = 1 \Rightarrow 1 + s \cdot \text{Tr}(X) + \sigma(s) = 0 \Rightarrow \text{Tr}(X) = 0$.

converse: $\text{Tr } X = 0 \Rightarrow \det \gamma_X(t)$ constant in $t \Rightarrow \det \gamma_X(t) = \det \gamma_X(0) = 1 \Rightarrow \gamma_X(t) \in SL(n)$

SAME PROOF SHOWS: $\mathfrak{su}(n) = \{X \in \mathfrak{u}(n) : \text{Tr}(X) = 0\}$

Theorem Let G be connected.

- $\exp: \mathfrak{g} \rightarrow G$ is a group hom $\Leftrightarrow G$ abelian
- G abelian $\Leftrightarrow G \cong \text{torus} \times \text{vector space}$

Cor 1 G abelian $\Rightarrow \exp: \mathfrak{g} \rightarrow G_0$ surjective hom. onto $G_0 = \text{conn. comp. of } 1 \in G$

Cor 2 G compact connected abelian $\Rightarrow G \cong T^n$

Cultural Rmk In non-connected case in 2) get torus \times vector space \times discrete abelian group

PROOFS Thm 2 \Rightarrow Cor 2: because a vector space $\cong \mathbb{R}^k$ is non-compact ($k \neq 0$)

Pf 1: \Rightarrow : $(\mathfrak{g}, +)$ is an abelian group $\Rightarrow \exp(\mathfrak{g})$ abelian

But $\exp(\mathfrak{g})$ generates $G_0 = G$ (LECTURE 5 : $\exp \mathfrak{g} \supseteq$ nbhd V of $1 \in G$)

Thm 1 \Rightarrow Cor 1 $\langle \exp(\mathfrak{g}) \rangle = G_0$ and, since image of a hom is a subgp, get $\exp(\mathfrak{g}) = \langle \exp(\mathfrak{g}) \rangle$

Pf 1: \Leftarrow : G abelian \Rightarrow multiplication $\mu: G \times G \rightarrow G$ is a Lie grp hom

\Rightarrow naturality of \exp :

$$\begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} & \xrightarrow{D, \mu} & \mathfrak{g} \\ \exp \downarrow & & \downarrow \exp \\ G \times G & \xrightarrow{\mu} & G \end{array} \quad \begin{array}{ccc} (X, Y) & \xrightarrow{D, \mu} & D, \mu \cdot (X, Y) \\ \downarrow & & \downarrow \\ (\exp X, \exp Y) & \xrightarrow{} & \exp(X) \exp(Y) \end{array}$$

$$\begin{aligned} \mu((g_1, h_1) \cdot (g_2, h_2)) &= \mu(g, g_2, h_1, h_2) = \\ &= g, g_2, h, h_2 \stackrel{G \text{ abelian}}{=} g, h, g_2, h_2 = \mu(g, h_1) \cdot \mu(g_2, h_2) \end{aligned}$$

(Rmk general fact $\text{Lie}(G_1 \times G_2) = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \xrightarrow{\exp} G_1 \times G_2$ is just \exp in each entry. Indeed $\gamma_{X_1+X_2}(t) = (\gamma_{X_1}(t), \gamma_{X_2}(t)) \in G_1 \times G_2$ since solve flow equation in each entry)

$$D, \mu \cdot (X, Y) \stackrel{\textcircled{1}}{=} \frac{\partial}{\partial t} \Big|_0 \mu(\gamma_X(t), \gamma_Y(t)) \stackrel{\text{chain rule}}{=} \frac{\partial}{\partial x_1} \mu(\gamma_X(x_1), 1) + \frac{\partial}{\partial x_2} \mu(1, \gamma_Y(x_2)) = X + Y$$

$\Rightarrow \exp(X+Y) = \exp(X) \exp(Y)$ so \exp is hom \blacksquare

Pf Thm 2: \Rightarrow Idea is to use the 1st isomorphism theorem for groups

We already know $\text{Im}(\exp: \mathfrak{g} \rightarrow G) = G_0 = G$ (by "Thm 1 \Rightarrow Cor 1"). Need find $\text{Ker}(\exp)$

Claim $K := \text{Ker}(\mathfrak{g} \xrightarrow{\exp} G)$ is a discrete subgroup of the vector space \mathfrak{g} .
 \uparrow (any point is an open set)

proof $\exp: \mathfrak{U} \rightarrow \mathfrak{V}$ diffeo, and for $X \in K$: $\exp(\underbrace{X+U}_{\text{nbhd of } X}) \stackrel{\text{exp is hom by Thm 1.}}{=} \underbrace{\exp(X)}_1 \cdot \underbrace{\exp(U)}_{\text{only get } \exp(u)=1 \text{ if } u=1 \in U}$

$\Rightarrow (X+U) \cap K = \{X\}$ (Note: $X+U$ is an open set around X in K for the subspace topology for $K \subseteq \mathfrak{g}$) \blacksquare

Def Discrete subgps of a vector space are called lattices.

FACT: discrete subgroups of a vector space are generated (as group, so over \mathbb{Z} not \mathbb{R})

\uparrow by a finite collection of linearly independent vectors.

(This is proved by induction on dim of the vector space. We take it as a fact for this course)

$$\Rightarrow K = \text{span}_{\mathbb{Z}}(X_1, \dots, X_k) \cong \mathbb{Z}^k \subseteq \mathbb{R}^k = \text{span}_{\mathbb{R}}(X_1, \dots, X_k)$$

$$\begin{array}{ccc} \mathfrak{g} & \cong & \mathbb{R}^k \times \mathbb{R}^{n-k} \\ \uparrow \text{complete to a basis} & & \uparrow \\ K & \longrightarrow & \mathbb{Z}^k \times 0 \end{array}$$

$$\xrightarrow{\text{1st isomorphism theorem}} \frac{\mathfrak{g}}{K} \cong \mathbb{T}^k \times \mathbb{R}^{n-k} \xrightarrow{\cong} \text{Image}(\exp) = G \text{ by 1st iso. theorem} \blacksquare$$

Above proof showed in general: Lemma $D_{(\mu)} \mu \cdot (X, Y) = X + Y$ for $\mu: G \times G \xrightarrow{\text{multiply}} G$