

LECTURE 9

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THE SUBGROUP–SUBALGEBRA CORRESPONDENCE

G Lie group, $\mathfrak{g} = \text{Lie } G$.

Chevalley's Theorem There is a 1-to-1 correspondence

$$\begin{array}{c} \{\text{Lie subalgebras } \mathfrak{h} \subseteq \mathfrak{g}\} \longleftrightarrow \{\text{connected Lie subgroups } H \subseteq G\} \\ \mathfrak{h} = \text{Lie}(H) \longleftrightarrow H \\ \mathfrak{h} \longmapsto H = \langle \exp \mathfrak{h} \rangle = \text{subgp generated by } \exp \mathfrak{h} \end{array}$$

EXAMPLE

dim	$\mathfrak{h} \subseteq \mathfrak{g}$	$H \subseteq G$
0	$\{0\}$	$\{1\}$
1	$\mathbb{R} \cdot X$ (Lie subalgebra) (since $[X, X] = 0$)	$\gamma_X(\mathbb{R})$ (image of the (1-parameter subgp))
$n = \dim G$	\mathfrak{g}	$G_0 \subseteq G$

EXAMPLE $G = T^n = \mathbb{R}^n / \mathbb{Z}^n \Rightarrow [\cdot, \cdot] = 0$

\Rightarrow any vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra

Recall $\exp : \mathbb{R}^n \rightarrow T^n$ is the homomorphism $\pi(v) = v \bmod \mathbb{Z}^n$

$\Rightarrow \langle \exp \mathfrak{h} \rangle = \exp(\mathfrak{h}) = \mathfrak{h} \bmod \mathbb{Z}^n (\cong \mathfrak{h} / \mathfrak{h} \cap \mathbb{Z}^n)$

\Rightarrow correspondence is: $\begin{pmatrix} \text{vector subspaces} \\ \mathfrak{h} \subseteq \mathbb{R}^n = \text{Lie}(T^n) \end{pmatrix} \xleftrightarrow{1:1} \begin{pmatrix} \text{abelian subgroup} \\ \mathfrak{h} \bmod \mathbb{Z}^n \subseteq T^n \end{pmatrix}$

MORE EXAMPLES Q. sheet 4: $SO(3)$, $SL(2, \mathbb{Z})$.

have the form
subtorus \times vectorspace
e.g. $\mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2$ is $\cong \mathbb{R}$
 $\mathbb{R} \cdot (1, 3) \subseteq T^2$ is $\cong S^1$

Remarks • It's because we want the above correspondence that we do not require Lie subgrps to be submfds (Lecture 7, Q.sheet 3)

- The correspondence is difficult to prove because $H = \langle \exp \mathfrak{h} \rangle$ need not be a submfd. In general we need to define a new topology on H , which may not be the subspace topology, in order to prove that H is a Lie group!

(Example from Lecture 7, $\mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2$: the subspace topology has less open sets than the usual topology on \mathbb{R})



Connected is necessary: $O(3), SO(3) \subseteq GL(3, \mathbb{R})$ have Lie algs $\mathfrak{o}(3) = \mathfrak{so}(3) \subseteq \mathfrak{gl}(3, \mathbb{R})$.

Proof of uniqueness: H connected $\Rightarrow H$ generated by nbhd of $1 \in H$

$\Rightarrow \exp(\mathfrak{h})$ generates H (since contains nbhd of 1)
 $\Rightarrow H = \langle \exp(\mathfrak{h}) \rangle$ is the only possible choice if you want $\text{Lie } H = \mathfrak{h}$.

Proof of existence:

① Consider $D = \text{span}(\mathfrak{h}) \subseteq TG$. Pick basis X_1, \dots, X_d of \mathfrak{h} .

Notice: at each $g \in G$, $D_g = \text{span}(X_1|_g, \dots, X_d|_g) \subseteq T_g G$ is a d -dim'l v.s. and locally near $g \in G$ there are vector fields Y_1, \dots, Y_d with $D = \text{span}(Y_1, \dots, Y_d)$ e.g. take $Y_j = X_j$. Such D are called a d -dim'l distribution on the manifold G .

② Say that a vector field X on G is in D , written $X \in D$, if $X|_g \in D_g$ all $g \in G$.

Claim 1 D is integrable (or involutive), meaning: $[X, Y] \in D \quad \forall X, Y \in D$

Proof all $X \in D$ are pointwise in span of X_1, \dots, X_d hence

$$X = \sum a_j(x) X_j \Rightarrow [X, Y] = \sum_{i,j} a_i b_j \underbrace{[X_i, X_j]}_{\in \mathfrak{h} \text{ since Lie subalg}} + a_i (X_i \cdot b_j) X_j - b_j (X_j \cdot a_i) X_i \in D$$

③

Local Frobenius Theorem

d -dim'l integrable distributions are locally of the form:

$$\exists x_1, \dots, x_m \text{ local coords for the manifold with } D_x = \text{span}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$$

Proof idea (NON-EXAMINABLE)

Locally $D = \text{span}(Y_1, \dots, Y_d)$. By integrating Y_1 get local coordinate $x_1 = \text{time coord. of flow of } Y_1$. Using the flow of Y_1 , one can build local coordinates y_1, y_2, \dots, y_m with $y_1 = \frac{\partial}{\partial y_1}$ and $y_i = x_i$. Then modify other v.f. $Z_j = Y_j - (y_j \cdot x_1) Y_1$ ($j \geq 2$) - notice $Z_j \cdot x_1 = 0$.

The slice $\{x_1=0\}$ is locally a submfld S and $D' = \text{span}(Z_2, \dots, Z_d)$ is a $(d-1)$ -dim'l integrable distribution on S (since $Z_j \cdot x_1 = 0$ have $Z_j \in TS$)

Then use an induction on dim of distribution to get local coords x_2, \dots, x_m on S .

Extend x_2, \dots, x_m to local coords near S by projecting to S (in y -coord system)

By construction $Y_1 = \frac{\partial}{\partial x_1}$ but need to check $Z_j \cdot x = 0$ for coords $x = x_{d+1}, \dots, x_m$

We know this on S (by induction) so we need to show it holds also near S .

$$\text{Observe: } \frac{\partial}{\partial x_1} Z_j \cdot x = Y_1 \cdot (Z_j \cdot x) = \underbrace{[Y_1, Z_j]}_{\in D} \cdot x \text{ since } Y_1 \cdot x = 0 \text{ (for } x = x_{d+1}, \dots, x_m\text{)}$$

$$\text{use } Y_1 \cdot x = 0 \Rightarrow \frac{\partial}{\partial x_1} Z_j \cdot x = \sum f_{jk} \cdot Z_k \cdot x \text{ some functions } f_{jk}$$

Now fix values of x_2, \dots, x_m , then $Z_j \cdot x = y_j(x_1)$ and get system of ODE's.

$$y'_j(x_1) = \sum f_{jk} y_k \text{ hence unique solution given initial condition}$$

Initial condition is $y_j(x_1) = Z_j \cdot x = 0$ on $S = \{x_1=0\}$ (by induction) so $y_j \equiv 0$ unique solution

$\Rightarrow D = \text{span}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$ (since $D \cdot x = 0$ all $x = x_{d+1}, \dots, x_m$) (Note: we are not claiming that $Z_j = \frac{\partial}{\partial x_j}$)

④ locally can integrate D meaning \exists submfld $S \subseteq G$ with $T_S S = D_S$

$$\text{namely: } S = \left\{ x = (x_1, \dots, x_d, \underbrace{x_{d+1}, \dots, x_m}_{\text{constants}}) \right\} \leftarrow \text{called slice}$$

By ③ these S are the only connected integral manifolds of D (meaning $T_S S = D_S \forall s \in S$)
 $\Rightarrow H$ near $l \in G$ is the unique slice of D through l .

⑤ piece together slices of D starting with this one) to build the manifold H .

Rmk 1: slices are embedded, so we simply define the topology and manifold structure as the subspace topology and submanifold structure of $S \subseteq G$.

Definition A leaf L is a connected integral manifold meaning:

a manifold L together with an injective immersion $L \xrightarrow{\varphi} G$ such that $D\varphi \cdot TL = D$.

Examples: • a slice is an embedded leaf. $\curvearrowleft (\text{D}\varphi \text{ injective})$

- $\mathbb{R} \cdot (1, \sqrt{2})$ is a leaf in T^2

Recall (Lecture 7) φ immersion $\Leftrightarrow \varphi$ local embedding

Claim 2 A leaf is a union of slices, and each slice is an open subset of the leaf.

Pf Using φ cts, for small connected $U \subseteq L$ have $\varphi(U) \subseteq$ local model ④ so $\varphi(U) =$ some slice.

Since φ immersion, $\varphi: U \rightarrow \varphi(U) \subseteq G$ local embedding (for small U) so $U \cong \varphi(U)$ diffeo
 so the topology & manifold structure on U is the same as for slice (Rmk 1) ■

⑥ Rmk 1 + Claim 2 \Rightarrow topology and manifold structure on leaves is determined by G

\Rightarrow finer topology on G called leaf topology given by taking leaves as basis of open sets.

Example • $\mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2$  \curvearrowleft all these segments are now open sets!
 (leaves)

\Rightarrow Let $H =$ the connected component of l in the leaf topology on G
 = maximal connected leaf in G through l

$\Rightarrow H$ is a connected manifold (with the leaf topology) and $H \rightarrow G$ is an immersion.

Non-examinable FACT A map $S \xrightarrow{\alpha} H$ is smooth \Leftrightarrow composition $S \xrightarrow{\alpha} H \rightarrow G$ smooth
 $(\Leftarrow \text{requires some work to show } \alpha \text{ cts})$

⑦ Claim H is a subgp

connected components
 for the leaf topology

Pf $(\phi_g^G)_* X = X$ for all $X \in g$ $\Rightarrow (\phi_g^G)_* D = D \Rightarrow \phi_g^G$ permutes the maximal leaves
 (Q.Sheet 1) \curvearrowleft (Indeed: $L \xrightarrow{\varphi} G$ leaf $\Rightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi_g^G} G$ leaf since injective, $D\phi_g^G \circ D\varphi$ injective,
 and $D\phi_g^G \circ D\varphi \cdot TL = D\phi_g^G \cdot D = D$)

For $h \in H$: $\phi_{h^{-1}}^G \cdot H$ leaf containing $l \Rightarrow \phi_{h^{-1}}^G \cdot H \subseteq H \Rightarrow h^{-1} \in H$
 $\Rightarrow \phi_{(h^{-1})^{-1}}^G \cdot H \subseteq H$ so $h \cdot h' \in H$ all $h' \in H$ ■

⑧ Claim group operations in H are smooth

Pf Want inversion $H \xrightarrow{i} H$ smooth. By above FACT need show composite $S = H \xrightarrow{i} H \xrightarrow{\varphi} G$ is smooth. This composite equals the composition $H \xrightarrow[\text{smooth}]{\varphi} G \xrightarrow[\text{smooth}]{i} G$ hence smooth ✓

Want multiplication $H \times H \xrightarrow{m} H$ smooth. By FACT need show $S = H \times H \xrightarrow{m} H \xrightarrow{\varphi} G$ smooth.

This equals the composition $H \times H \xrightarrow[\text{smooth}]{\varphi \times \varphi} G \times G \xrightarrow[\text{smooth}]{m} G$ hence smooth ✓ ■