

COMPLETE REDUCIBILITY

Def A rep  $V$  is completely reducible if  $V = \bigoplus n_i V_i$  is a sum of irreps.

Question Is every rep a sum of irreps?

Answer No:  $G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C} \right\}$  (abelian!)

$V = \mathbb{C}^2$  is reducible, and subreps are:

$$0, W = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{C} \right\}, V$$

but  $V \neq W \oplus$  irrep

More details:  
If  $\begin{pmatrix} x \\ y \end{pmatrix} \in$  subrep  $W$   
then:  
 $\begin{pmatrix} 1 & -\frac{x}{y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} \in W$   
so  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} \in W$   
but  $W$  is v-subspace, so  
 $\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ y \end{pmatrix} \in W$   
span  $\mathbb{C}^2$  so  $W = \mathbb{C}^2$

Lemma For  $\mathbb{F} = \mathbb{C}$ ,  $G$  abelian  $\Rightarrow$  Irreps are 1-dimensional

Pf For  $V$  irrep,  $G$  abelian, the multiplication  $\phi_g: V \rightarrow V, \phi_g(v) = gv$  is  $G$ -linear:

$$\phi_g(g'v) = gg'v = g'gv = g' \phi_g(v)$$

$$\xrightarrow{\text{Schur}} \phi_g = \lambda_g \cdot \text{Id} \quad \text{some } \lambda_g \in \mathbb{C}.$$

$$\Rightarrow \phi_g(\mathbb{C}v) = \lambda_g \cdot \mathbb{C}v = \mathbb{C}v \quad \text{for all } g \in G$$

$$\Rightarrow \mathbb{C}v \text{ subrep of } V, \text{ so } V = \mathbb{C}v \text{ (since irrep)} \blacksquare$$

AIM: prove that for compact  $G$ , reps are completely reducible.

The proof for finite groups  $G$  is as follows:

① Given an inner product  $(\cdot, \cdot): V \times V \rightarrow \mathbb{F}$  (e.g. pick basis, so  $V \cong \mathbb{F}^d$ , then use standard inner product on  $\mathbb{F}^d$ )

can produce a  $G$ -invariant inner product  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$

by averaging  $\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} (g \cdot v, g \cdot w).$

② If  $V$  irrep, done ✓

If  $V$  reducible, let  $W \subseteq V$  be a subrep  $\neq 0, V$ . Then  $W^\perp$  is a subrep

proof:  $\langle gv, w \rangle = \langle gv, gg^{-1}w \rangle = \underbrace{\langle v, g^{-1}w \rangle}_{\substack{\text{G-inverse} \\ \in W}} = 0 \quad \text{for all } w \in W$   
 $\Rightarrow gv \in W^\perp \blacksquare$

③ Induction on  $\dim V \Rightarrow$  can completely reduce  $W, W^\perp \Rightarrow$  can completely reduce  $V = W \oplus W^\perp$ . ■

Theorem 1 If  $V$  admits a  $G$ -invt inner product, then  $V$  is completely reducible

Proof By steps ② & ③ above. ■

↙ PROOF NON-EXAMINABLE - SEE NON-EXAMINABLE HAND-OUT (integrals  
are limits of  
Riemann sums)

Theorem  $G$  compact Lie group

$\Rightarrow \exists$  unique normalized left-invariant integral over  $G$ , meaning:

To any continuous function  $f: G \rightarrow \mathbb{R}$  it associates a value  $\int_G f = \int_{g \in G} f(g) \in \mathbb{R}$   
such that •  $\int_G 1 = 1$

- If  $f > 0$  then  $\int_G f > 0$  (positivity)
- $\int_G (f_1 + \lambda f_2) = \int_G f_1 + \lambda \int_G f_2$  for  $\lambda \in \mathbb{R}$  (linearity)
- $\int_G f \circ \phi_h = \int_{g \in G} f(hg) = \int_G f$  (left-invariance)

Rmk 1) linearity + positivity  $\Rightarrow$  monotonicity:

- if  $f > g$  then  $\int_G f > \int_G g$  (proof: consider  $f-g > 0$ )

(using normalization, this also holds with " $\geq$ ". Proof: if  $f > 0$  then  $f > -\varepsilon$  so  $\int_G f > -\varepsilon \int_G 1 = -\varepsilon$ )

2) In fact the integral is bi-invariant, i.e. left and right invariant. Indeed it satisfies

- $\int_{g \in G} f(gh) = \int_G f$  (right-invariance)
- $\int_{g \in G} f(g^{-1}) = \int_G f$  (inversion-invariance)
- $\int_{G_1} f \circ \varphi = \int_{G_2} f$  for cts  $f: G_2 \rightarrow \mathbb{R}$  and Lie gp iso  $\varphi: G_1 \rightarrow G_2$  (isomorphism-invariance)

3) Can integrate continuous maps  $f: G \rightarrow \mathbb{F}^d$  by integrating in each entry so  $\int f \in \mathbb{F}^d$   
 $\Rightarrow \dots$  " " " "  $f: G \rightarrow V$  (pick basis, so  $V \cong \mathbb{F}^d$ ) so  $\int_G f \in V$

Exercise:  $\frac{\text{LINEARITY}}{(V \text{ any v.s.)}} \quad \frac{\varphi: V \rightarrow V \text{ linear}}{f: G \rightarrow V} \Rightarrow \int \varphi \circ f = \varphi \left( \int f \right)$

Corollary  $G$  compact  $\Rightarrow$  for any rep  $V$  there is a  $G$ -invt inner product  
 $\Rightarrow$  all reps are completely reducible

Pf Pick any inner product  $(\cdot, \cdot)$  on  $V$  (e.g. standard i.p. on  $\mathbb{F}^d \cong V$ )

Define

$$\langle v, w \rangle = \int_{g \in G} (gv, gw)$$

Linear in each entry since  $G$ -action linear,  $(\cdot, \cdot)$  bilinear, and  $\int_G$  is linear.

Positive definite since  $\langle v, v \rangle = \underbrace{\int (gv, gv)}_{>0 \text{ (for } v \neq 0\text{)}} > 0$  using positivity of  $\int_G$ .

$G$ -invariant: for  $h \in G$ ,

$$\langle hv, hw \rangle = \int_{g \in G} (ghv, ghw) = \int_{g \in G} f(gh) = \int_G f = \int_{g \in G} (gv, gw) = \langle v, w \rangle$$

Define  $f: G \rightarrow \mathbb{R}$

$$f(g) = (gv, gw) \quad (\text{fixed } v, w)$$

right-invariance

# CHARACTERS

Def The character  $\chi_v = \chi_{\rho} : G \rightarrow \mathbb{F}$  of a rep  $\rho : G \rightarrow \text{Aut } V$  is

$$\boxed{\chi_v(g) = \text{Tr}(\rho(g))}$$

$\text{Tr} = \text{Trace.}$

Q.sheet 5: •  $\chi_v$  smooth

•  $\chi_v(1) = \dim V$

•  $\chi_v(h^{-1}gh) = \chi_v(g)$

•  $V \cong W \Rightarrow \chi_v = \chi_w$

•  $\chi_{V \oplus W}(g) = \chi_v(g) + \chi_w(g)$

•  $\chi_{V \otimes W}(g) = \chi_v(g) \cdot \chi_w(g)$

•  $\chi_{V^*}(g) = \chi_v(g^{-1})$

•  $\chi_{\overline{V}}(g) = \overline{\chi_v(g)}$

Lemma 1 ( $\mathbb{F} = \mathbb{C}$ )  $G$  compact  $\Rightarrow \chi_v(g^{-1}) = \overline{\chi_v(g)}$

Pf  $\chi_v(g^{-1}) = \chi_{V^*}(g) = \overline{\chi_V(g)} = \overline{\chi_v(g)}$ .

$V^* \cong \overline{V}$  Q.sheet 5, using  $G$  compact. ■

## FIXED POINTS

Def  $v \in V$  is a fixed point of the  $G$ -action if  $g \cdot v = v$  for all  $g \in G$ .

$$\Rightarrow V^G = \{\text{fixed points}\} \subseteq V \quad \text{subrep}$$

For finite groups  $G$  you build fixed points by averaging:

$$V^G = \left\{ \frac{1}{|G|} \sum_{g \in G} g \cdot w : w \in V \right\}$$

Proof:  $h \in G \Rightarrow h \cdot \left( \frac{1}{|G|} \sum g \cdot w \right) = \frac{1}{|G|} \sum hg \cdot w = \frac{1}{|G|} \sum gw$  since  $\phi_h : G \xrightarrow{\sim} G$  bijection.

Conversely:  $v$  fixed  $\Rightarrow \frac{1}{|G|} \sum_{g \in G} g \cdot v \underset{=v}{\approx} v = \underbrace{\frac{1}{|G|} \sum_{g \in G} v}_{=1} = v$  ■

Thm For compact Lie group  $G$ ,

$$\boxed{V^G = \left\{ \int_{g \in G} g \cdot w : w \in V \right\} \subseteq V}$$

Pf:

$\supseteq$ : For  $h \in G$ ,  $\begin{matrix} V & \longrightarrow & V \\ v & \mapsto & h \cdot v \end{matrix}$  is linear  $\Rightarrow h \int_{g \in G} g \cdot w = \int_{g \in G} h \cdot g \cdot w = \int_{g \in G} g \cdot v$ .

$\subseteq$ :  $v$  fixed  $\Rightarrow \int_{g \in G} g \cdot v = \int_g v = \left( \int_G 1 \right) \cdot v = v$  ■

LINEARITY  
(RMK 3)

Lemma 2  $\dim V^G = \sum_{g \in G} \chi_v(g)$

$$\underline{\text{Pf}} \quad \int \chi_v(g) = \int \text{Tr}(p(g)) = \underbrace{\text{Tr} \int p(g)}_{\substack{\text{LINEARITY} \\ (\text{Rmk 3})}} \underbrace{\int}_{\substack{\text{integrate each} \\ \text{matrix entry}}} p(g)$$

TRICK: averaging operator  $\varphi: V \rightarrow V$ ,  $\varphi(v) = \int g \cdot v$  is a projection onto  $V^G$  meaning  $\varphi^2 = \varphi$  (indeed if  $v \in V^G = \text{Image}(\varphi)$  then  $\varphi(v) = \int \underbrace{gv}_{=v} = \int 1 = v$ )

For projection maps,  $\text{Tr}(\varphi) = \dim_{\mathbb{F}} \text{Im}(\varphi)$ , since  $V = \text{Im}(\varphi) \oplus \text{Ker}(\varphi)$   
 $v \mapsto \varphi(v) + (v - \varphi(v))$

$$\Rightarrow \text{Tr} \int p(g) = \text{Tr}(\varphi) = \dim_{\mathbb{F}} \text{Im } \varphi = \dim_{\mathbb{F}} V^G \blacksquare$$

$$\begin{array}{c} \uparrow \\ \varphi = \text{Id} \end{array} \quad \begin{array}{c} \uparrow \\ \varphi = 0 \end{array}$$

## ORTHOGONALITY RELATIONS

Theorem For compact Lie group  $G$ ,

$$\langle \chi_v, \chi_w \rangle \underset{\text{define}}{=} \int_{g \in G} \overline{\chi_v(g)} \chi_w(g) = \dim \text{Hom}_G(V, W)$$

Cor •  $\langle \chi_v, \chi_w \rangle$  defines an inner product on  $\text{span}_{\mathbb{F}} \{ \chi_v : V \text{ rep} \}$

•  $V_i$  non-equivalent irreps  $\xrightarrow{\text{Schur}}$   $\chi_{V_i}$  are orthogonal, so linearly independent

• Lecture 11 :  $V \cong W$  irreps  $\Rightarrow \int \overline{\chi_v} \cdot \chi_w \begin{cases} = 1 & \text{if } \mathbb{F} = \mathbb{C} \\ \geq 1 & \text{if } \mathbb{F} = \mathbb{R} \end{cases}$   
 (Cor of Schur) (equivalent)

Pf Thm Work over  $\mathbb{F} = \mathbb{C}$  (if  $\mathbb{F} = \mathbb{R}$  just think of  $G \rightarrow \text{Aut}(\mathbb{R}^n) \subseteq \text{Aut}(\mathbb{C}^n)$ )

$$\text{TRICK} \quad \text{Hom}_G(V, W) = \text{Hom}_{\mathbb{C}}(V, W)^G$$

recall  $(g\varphi)(v) = (g \circ \varphi \circ g^{-1})(v)$  so  $g\varphi = \varphi \Leftrightarrow \varphi(gv) = g\varphi(v)$  all  $v \in V$  all  $g \in G$

$$\Rightarrow \dim \text{Hom}_G(V, W) = \int \chi_{\text{Hom}_{\mathbb{C}}(V, W)}(g) \quad (\text{Lemma 2})$$

$$= \int \chi_{V^*}(g) \chi_W(g) \quad \begin{array}{l} (\text{Lecture 11 : } \text{Hom}_{\mathbb{F}}(V, W) \cong V^* \otimes W) \\ (\text{and use properties of characters}) \end{array}$$

$$= \int \overline{\chi_V(g)} \chi_W(g) \quad (\text{Lemma 1}) \blacksquare$$

Thm  $G$  compact  $\Rightarrow$  any rep is determined uniquely (up to equivalence) by character

$$\underline{\text{Pf}} \quad V \cong \bigoplus n_i \underbrace{V_i}_{\substack{\text{irreps} \\ \text{complete reducibility}}} \Rightarrow \chi_V = \sum n_i \chi_{V_i} \Rightarrow n_i = \frac{\langle \chi_V, \chi_{V_i} \rangle}{\langle \chi_{V_i}, \chi_{V_i} \rangle} \begin{cases} = 1 & \mathbb{F} = \mathbb{C} \\ \geq 1 & \mathbb{F} = \mathbb{R} \end{cases} \blacksquare$$