

LECTURE 14

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ASSUME G COMPACT
 G CONNECTED

MAXIMAL TORI

Def A Lie subgp $T \subseteq G$ is called a torus if $T \cong \mathbb{R}^n / \mathbb{Z}^n$ (Lie gp iso)
T is a maximal torus if $T \subseteq T' \subseteq G \Rightarrow T = T'$ for any torus T' .

EXAMPLE $T = \{\text{diagonal matrices } \in U(n)\} \subseteq G = U(n)$ max torus

Lemma

- ① Max T exist (Pf $\{1\} \subsetneq T_1 \subsetneq T_2 \subsetneq \dots$ must stop since the $\dim \text{Lie}(T_j) < \dim g$ increase)
- ② Every torus lies in a max torus (Pf By Pf of ①.)
- ③ $T \text{ max} \Rightarrow$ conjugates gTg^{-1} are max tori (Pf $gTg^{-1} \subseteq T' \Leftrightarrow T \subseteq g^{-1}T'g$).
- ④ Tori are always embedded (Pf T compact $\Rightarrow \text{Image}(T \xrightarrow{\text{smooth}} G)$ compact so closed).

Lemma $\{\text{maximal tori}\} = \{\text{maximal connected abelian subgps } A \subseteq G\}$

Pf $T \subseteq A \subseteq \overline{A} \subseteq G \Rightarrow \overline{A}$ compact connected abelian \Rightarrow torus $\Rightarrow T = A = \overline{A}$ ■

Fact T max torus $\Rightarrow T$ is a maximal abelian subgrp \leftarrow (uses G connected)

Converse is false $SO(3) \supseteq A = \left\{ \begin{pmatrix} \pm 1 & \pm 1 \\ & \pm 1 \end{pmatrix} \right\}$ max abelian (disconnected).

Cor T max \Rightarrow Centralizer $Z(T) = \{z \in G : zx = xz \text{ all } x \in T\} = T$

Pf $T \subseteq Z(T)$. If $z \in Z(T) \setminus T$ then $\langle T \cup \{z\} \rangle$ is larger abelian subgp than T ■

EXAMPLE If $A \in U(n)$ commutes with diafonal matrices then A is diagonal!

T torus \Rightarrow rep: $\rho = \text{Ad}|_T : T \rightarrow \text{Aut}(g)$ $\rho(x) = \text{Ad}(x) = D, \xrightarrow{\text{Ad}} (\text{Ad}(h) = ghg^{-1})$

T compact \Rightarrow completely reduce $g \cong \bigoplus n_i V_i$ real irreps $V_i, n_i \in \mathbb{N}$

Irreps of tori: $V_0 = \mathbb{R}$ trivial rep T acts by Id
 $V_a = \mathbb{R}^2, \rho(x) = \text{rotation by } 2\pi\theta_a(x) = \begin{pmatrix} \cos 2\pi\theta_a(x) & -\sin 2\pi\theta_a(x) \\ \sin 2\pi\theta_a(x) & \cos 2\pi\theta_a(x) \end{pmatrix}$
where $\theta_a(x) = \langle a, x \rangle = a_1 x_1 + \dots + a_n x_n$ and $a \neq 0 \in \mathbb{Z}^n$

Since $V_a \cong V_b \Leftrightarrow a = -b$, don't distinguish $a, -a$ so don't distinguish $\theta_a, -\theta_a = \theta_{-a}$

$\Rightarrow g = g_0 \oplus \bigoplus_{a \neq 0} g_a$ $\text{Ad}(x)$ acts by Id on $g_0 = n_0 V_0$
 $\text{Ad}(x)$ " " rot'n by $2\pi\theta_a(x)$ on each V_a in $g_a = n_a V_a$.

Def For $n_a \neq 0$, $\theta_a : T \rightarrow S^1$ called roots of G

Rmks ① Lie alg rep $\text{ad}|_t : t = \text{Lie } T = \mathbb{R}^n \rightarrow \text{End}(g), \text{ad}(x) \cdot X = [x, X]$

$\tilde{\theta}_a : t = \mathbb{R}^n \rightarrow \mathbb{R} = \text{Lie}(S^1), \tilde{\theta}_a(x) = a_1 x_1 + \dots + a_n x_n$ roots of g ($n_a \neq 0$)

② Many books work with $g \otimes_{\mathbb{R}} \mathbb{C}$ instead of g (so $T \rightarrow \text{Aut}(g) \subseteq \text{Aut}(g \otimes_{\mathbb{R}} \mathbb{C})$)

$\Rightarrow \dim_{\mathbb{C}} (\text{irreps } V_a^{\mathbb{C}} \text{ of } T) = 1 \Rightarrow V_a \otimes_{\mathbb{R}} \mathbb{C} \cong V_a^{\mathbb{C}} \oplus V_{-a}^{\mathbb{C}}$ (diagonalize rot'n by $\begin{pmatrix} e^{2\pi i \theta_a} & 0 \\ 0 & e^{-2\pi i \theta_a} \end{pmatrix}$)

$\Rightarrow g \otimes_{\mathbb{R}} \mathbb{C} = n_0 V_0^{\mathbb{C}} \oplus \bigoplus_{a \in \mathbb{Z}^n \setminus 0} n_a V_a^{\mathbb{C}}$ with $n_a = n_{-a}$ since g real rep of T .

$V_0^{\mathbb{C}}, V_a^{\mathbb{C}}$ (for $n_a \neq 0$) called weight eigenspace with weight $0, 2\pi i \tilde{\theta}_a \in g^*$ since
 $\rho(x)v = 0 \cdot v, e^{2\pi i \tilde{\theta}_a(x)} \cdot v$ respectively for $v \in V_0, V_a$, similarly at Lie algebra level:
 $\text{ad}(x) \cdot v_a = [x, v_a] = (\text{differentiate}) = 0, 2\pi i \tilde{\theta}_a(x) v_a$ respectively.

③ Lie subalgebra $t = \text{Lie } T \subseteq g$ (after $\otimes_{\mathbb{R}} \mathbb{C}$) is called Cartan subalgebra since t abelian and ad_t acts diagonally (i.e. $g \otimes \mathbb{C} = \bigoplus \text{weight spaces}$) and maximal ($g_0 = t$)

Lemma T maximal $\iff t = g_0 \subseteq g$ $(t = \text{Lie}(T))$

Pf " \Rightarrow ": T abelian $\Rightarrow \text{Ad}_g = \text{Id}$ on $T \Rightarrow \text{Ad}(x) = D, \text{Ad}_g = \text{Id}$ on $t \Rightarrow t \subseteq g_0$

If $y \in g_0 \Rightarrow \text{Ad}(x)y = y \Rightarrow \exp(y) = \exp(\text{Ad}(x) \cdot y) = x \exp(y)x^{-1}$ all $x \in T$.

Same holds for $sy, s \in \mathbb{R}$ so T commutes with subgrp $H = \{\exp(sy) : s \in \mathbb{R}\}$

so $\langle TuH \rangle$ is a connected abelian subgrp larger than T unless $y \in T$ ■

" \Leftarrow ": $T \subseteq T' \subseteq G$ larger torus $\Rightarrow t \subseteq \text{Lie } T' \subseteq g'_0 \subseteq g_0$

Assumption $t = g_0 \Rightarrow t = \text{Lie } T' \Rightarrow T = T'$ \blacksquare $\begin{matrix} \uparrow \\ \text{1st part} \\ \text{of proof} \end{matrix} \quad \begin{matrix} \uparrow \\ \text{Ad}(x')y = y \quad \forall x' \in T' \\ \text{so also for } x' = x \in T \end{matrix}$
 $\begin{matrix} \uparrow \\ \text{decompose } g \text{ using } T' \end{matrix}$

EXAMPLE

Recall (Q. Sheet 3) for matrix groups G , $\text{Ad}(A) \cdot B = ABA^{-1}$.

$$G = U(2)$$

$$T = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} : \lambda_k = e^{2\pi i x_k}, (x_1, x_2) \in T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \right\}$$

$$U(2) = \left\{ B \in \text{Mat}_{2 \times 2}(\mathbb{C}) : \overline{B}^T = -B \right\} = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & id \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$$= \underbrace{\mathbb{R} \cdot \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}}_{g_0} \oplus \underbrace{\mathbb{R} \cdot \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}}_{g_a} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} : z \in \mathbb{C} \right\}}_{g_z}$$

proof $\text{Ad} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_1 \lambda_1^{-1} i & 0 \\ 0 & \lambda_2 \lambda_2^{-1} i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$

$$\text{Ad} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \text{ similarly}$$

$$\text{Note: } g_0 = \left\{ \begin{pmatrix} ia & 0 \\ 0 & id \end{pmatrix} \right\} = t \text{ hence } T \text{ maximal.}$$

Now find $a \in \mathbb{Z}^2$:

$$\text{Ad} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \lambda_1 \lambda_2^{-1} z \\ -\bar{z} & 0 \end{pmatrix}$$

$$\Rightarrow g_a \cong \mathbb{C} \cong \mathbb{R}^2 \text{ is the rep } z \mapsto \lambda_1 \lambda_2^{-1} z = e^{2\pi i(x_1 - x_2)} z \\ \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \mapsto z \mapsto \begin{pmatrix} \text{Re } z \\ \text{Im } z \end{pmatrix} \quad = \text{rotation by } 2\pi i(x_1 - x_2)$$

$$\Rightarrow \theta_a(x) = a_1 x_1 + a_2 x_2 = x_1 - x_2 \text{ root}$$

$$\Rightarrow a = (1, -1) \in \mathbb{Z}^2$$

Typically write $g_{x_1 - x_2}$ instead of $g_{(1, -1)}$.

$\Rightarrow U(2) \cong g_0 \oplus g_{x_1 - x_2}$ has one root: $x_1 - x_2$. ■

Rmk Lie algebra approach is: abbreviate $y_1 = 2\pi i x_1, y_2 = 2\pi i x_2$ then:

$$\text{ad} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} = \left[\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right] = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} - \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} = \begin{pmatrix} 0 & (y_1 - y_2)z \\ -\bar{z} & 0 \end{pmatrix}$$

CONJUGATES OF MAX T COVER G

Theorem Any $h \in G$ lies in a conjugate of T ($h = g x g^{-1}$ some $x \in T$)

EXAMPLE Any $A \in U(n)$ is diagonalizable, so conjugate to some $(\lambda_1 \dots \lambda_n) \in T$.

Rmk Uses G connected since $gTg^{-1} \subseteq G_0$ cannot reach $h \in G \setminus G_0$.

Proof (Non-examinable)

Trick $f = \phi_h : G/T \rightarrow G/T$, $f(gT) = hgT$ (just a smooth map, G/T manifold)

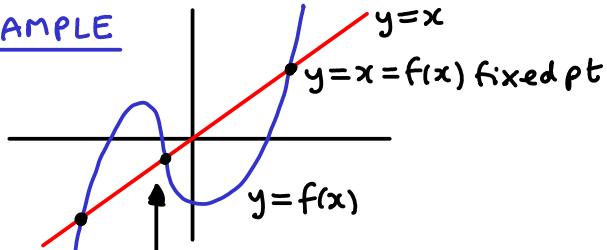
gT fixed point of $f \Leftrightarrow f(gT) = gT \Leftrightarrow hgT = gT \Leftrightarrow g^{-1}hg \in T$ ✓

Remains to show \exists fixed point.

Trick 2 For a smooth map $f: M \rightarrow M$ of a manifold, the number of fixed points (counted with multiplicity) does not change if we continuously deform f .

$f(x) = x$ count x as $+1$ if $\det(Id - D_x f) > 0$ (more complicated)
 -1 " " " < 0 if $\det = 0$: have multiplicity
Lefschetz fixed pt thm

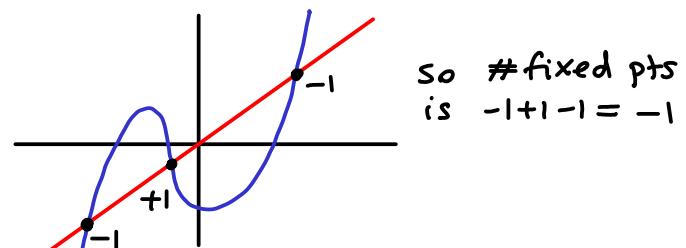
EXAMPLE



here $x - f(x)$ increases, so

$$\frac{d}{dx}(x - f(x)) = 1 - f'(x) > 0 \text{ so count as } +1$$

Now continuously deform f to



fixed pts is still -1 . ✓

Trick 3 Q.2 Sheet 3 : almost any $T \in T$ generates $\overline{\langle T \rangle} = T$. Pick such a T .
 \hookrightarrow (closure of subgp gen by T)

Since G connected mfd, it is path-connected, so deform f by moving h to T .

Remains to show : $f: G/T \rightarrow G/T$, $f(gT) = \tau gT$ has $\#(\text{fixed pts}) \neq 0$.

Claim $\{\text{fixed points}\} = \{nT : n \in N(T)\}$ ← Normalizer $N(T) = \{g \in G : gTg^{-1} = T\}$

Pf nT fixed $\Leftrightarrow n^{-1}\tau n \in T \Leftrightarrow \overline{\langle n^{-1}\tau n \rangle} = n^{-1}\overline{\langle \tau \rangle}n = n^{-1}Tn \subseteq T$ ■

Rmk $T \subseteq N(T) \subseteq G$ closed subgroup so compact Lie subgp. (equality since $n^{-1}Tn$ max torus)

Claim $N(T)_o = T$

Pf $n \in N(T)_o \Rightarrow T \rightarrow T$, $g \mapsto ngn^{-1}$ is a Lie gp hom depending on a continuous parameter n . But homs $T \rightarrow T$ are parametrized by a discrete parameter in $\mathbb{Z}^n \times \dots \times \mathbb{Z}^n$ (Q.2 Sheet 3). So ngn^{-1} independent of n up to deforming. So move n to 1 $\Rightarrow ngn^{-1} = g \Rightarrow ng = gn$. If $T \not\subseteq N(T)_o \Rightarrow$ could create connected abelian subgp larger than T by taking $\langle T \cup \{\exp(sy) : s \in \mathbb{R}\} \rangle$ for $y \in \text{Lie } N(T)_o \setminus T$ ■

Consequence fixed points are cosets nT of $T = N(T)_0$ in $N(T)$

\Rightarrow finitely many since $N(T)/N(T)_0$ is discrete + compact.

(or directly: $N(T)$ compact
 $N(T)_0 \subseteq N(T)$ open
and cosets cover $N(T)$)

Trick 4 $\det(I - D_{nT} f)$ is independent of n .

Pf Consider $\psi_n^{-1} \circ f \circ \psi_n$ where $\psi_n =$ right-multiplication by n on G/T (NOTE $gTn = gnT$ since $n \in N(T)$)

$$\psi_n^{-1} \circ f \circ \psi_n(gT) = \psi_n^{-1} f(gTn) = \psi_n^{-1}(\tau gTn) = \tau gT \cancel{\psi_n^{-1}} = f(gT)$$

$$\Rightarrow \psi_n^{-1} \circ f \circ \psi_n = f$$

$$\begin{aligned} \Rightarrow \det(I - D_T f) &= \det(I - D_{nT} \psi_n^{-1} \circ D_{nT} f \circ D_T \psi_n) = \det(D\psi_n^{-1} \circ (I - D_{nT} f) D\psi_n) \\ &\quad \text{chain rule} \qquad \qquad \qquad \nwarrow \psi_n(T) = Tn = nT \\ &= \det(I - D_{nT} f) \blacksquare \end{aligned}$$

Remains to calculate sign ($\det(I - D_T f)$).

Trick 5 $A_\tau : G \rightarrow G$, $A_\tau(g) = \tau g \tau^{-1}$

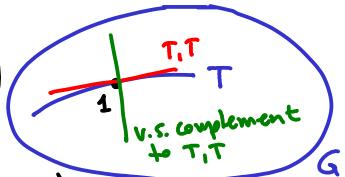
$$\Rightarrow A_\tau : G/T \rightarrow G/T, A_\tau(gT) = \tau gT \tau^{-1} = \tau gT = f(gT)$$

Since tangent space $T_1 G \cong T_1 T \oplus T_{T_1}(G/T)$,

$D_T f = D_1(A_\tau)$ restricted to a vector space complement of $T_1 T \subseteq T_1 G$

$$= D_1 A_\tau \Big|_{\bigoplus \sigma_a} = \text{Ad}(\tau) \Big|_{\bigoplus \sigma_a} = \bigoplus \left(\text{rotation by } 2\pi \theta_a(\tau) \text{ on } \sigma_a \right)$$

(omit $\sigma_0 = t \cong T_1 T$)



$$\begin{aligned} \det(I - D_T f) &= \prod_a \det \begin{pmatrix} 1 - \cos 2\pi \theta_a(\tau) & \sin 2\pi \theta_a(\tau) \\ -\sin 2\pi \theta_a(\tau) & 1 - \cos 2\pi \theta_a(\tau) \end{pmatrix} \\ &= \prod_a 2 \underbrace{(1 - \cos 2\pi \theta_a(\tau))}_{>0 \text{ since } \theta_a(\tau) \neq 0 \pmod{\mathbb{Z}}} \end{aligned}$$

otherwise $\theta_a(\tau) = \theta_a(T) = 0 \pmod{\mathbb{Z}}$
but $\theta_a \neq 0$ for $a \neq 0$.

\Rightarrow multiplicity of all nT is $+1$.

\Rightarrow # fixed points > 0 (if there is no σ_a , then $\sigma = t$, so $G = T$) ■

Corollaries

① All maximal tori are conjugate: T, T' max $\Rightarrow T' = gTg^{-1}$ some g

② Every element $h \in G$ lies in some max tori

③ Decomposition $\sigma \cong \bigoplus \sigma_0 \oplus \bigoplus_a \sigma_a$ is independent of choice of max T .

So roots θ_a do not depend on choice and $\dim(\text{max } T) = \dim(\sigma_0)$ called rank(G).

Pf ① $T' = \langle \tau' \rangle$ some τ' . By Thm, $\tau' = gxg^{-1}$ some $x \in T$. So $T' \subseteq \langle gxg^{-1} \rangle \subseteq gTg^{-1}$ ■

② $h \in gTg^{-1}$ some g , by Thm.

③ $T' = gTg^{-1} \Rightarrow$ Claim $\sigma'_a = \text{Ad}(g) \cdot \sigma_a \cdot \text{Ad}(g)^{-1}$ so $\sigma'_a \cong \sigma_a$ equivalent reps.

Pf $\text{Ad}(gxg^{-1}) \cdot \text{Ad}(g) \cdot y \cdot \text{Ad}(g)^{-1} = \text{Ad}(g) \text{Ad}(x) y \text{Ad}(g^{-1})$ and $\text{Ad}(x)y \in \sigma_a$ iff $y \in \sigma_a$ ■

Similarly for σ_0 . Since $\sigma'_a \cong \sigma_a$ also $n'_a = n_a$ so characters $= 2\cos(2\pi \theta_a(x))$ same

so $\theta'_a = \pm \theta_a$ (and recall we don't distinguish $\theta_a, -\theta_a = \theta_{-a}$). ■