

ASSUME: T MAX TORUS
 G COMPACT CONNECTED

WEYL GROUP

$$W(T) = \left\{ \mathcal{A}_n|_T : T \rightarrow T \quad \text{where } n \in N(T) \right\} \subseteq \text{Aut}(T)$$

$$x \mapsto nxn^{-1}$$

Recall $N(T) = \text{normalizer} = \{n \in G : nTn^{-1} = T\}$

Rmk Independent (up to isomorphism) of choice of max torus T :
 $N(gTg^{-1}) = gN(T)g^{-1}$ and $W(gTg^{-1}) = \mathcal{A}_g \circ W(T) \circ \mathcal{A}_g^{-1}$

Lemma $W(T) \cong N(T)/T$ via $\mathcal{A}_n|_T \mapsto nT$

Pf $N(T) \rightarrow W(T)$ surjective hom so done by 1st iso thm
 $n \mapsto \mathcal{A}_n|_T$ if can show kernel = T .

kernel: $n \in N(T)$ with $\mathcal{A}_n|_T = \text{id}$ so $nxn^{-1} = x$ all $x \in T$

$\Rightarrow n \in T$ otherwise $\langle T \cup \{n\} \rangle$ larger abelian subgp than T .

\uparrow (reproving $Z(T) = T$ see Lecture 14) ■

Rmk By Lecture 14 $T = N(T)_0$ so $N(T)/T = N(T)/N(T)_0$

is discrete and compact, hence finite. So $W(T)$ is a finite group

Cor Characters χ of G are determined by their restrictions $\chi|_T$
and the $\chi|_T$ are invariant under the Weyl group

Pf $\chi(h) \stackrel{\uparrow}{=} \chi(gxg^{-1}) \stackrel{\leftarrow}{=} \chi(x) = \chi|_T(x)$ ✓
(recall trace is conjugation invariant)

(recall thm: any $h \in G$ lies in a conjugate of T , say $h = gxg^{-1}$, $x \in T$)

$$(\mathcal{A}_n \cdot \chi|_T)(x) = (\chi|_T \circ \mathcal{A}_n^{-1})(x) = \chi|_T(n^{-1}xn) = \chi|_T(x) \quad \checkmark \quad \blacksquare$$

Theorem $R(G) \rightarrow R(T)^W = \{\text{Weyl invariant virtual characters}\}$
 $\chi \mapsto \chi|_T$ is an isomorphism!

Cor above shows injective hom. Surjectivity is harder (won't prove it). In practice, you don't use surjectivity: first you find reps of G giving all possible characters in $R(T)^W$ then by injectivity you know you have found all reps.

Example Representation theory of $U(n)$

$T = \{\text{diagonal matrices}\} \subseteq U(n)$

Claim 1 $W(T) = S_n = \text{symmetric group acting on } T \text{ by permuting diagonal entries.}$

Pf $A_n(x) = n \times n^{-1}$ does not change the eigenvalues of $x \in T$ (diagonal entries).

Recall (Q. sheet 3) $T = \overline{\langle x \rangle}$ if $x = \begin{pmatrix} e^{2\pi i x_1} & & \\ & \dots & \\ & & e^{2\pi i x_n} \end{pmatrix}$ with $1, x_1, \dots, x_n$ lin. indep./ \mathbb{Q}

A_n permutes the distinct entries $e^{2\pi i x_j}$

But $A_n(x)$ determines A_n on $T = \overline{\langle x \rangle}$ by continuity

$\Rightarrow A_n = \text{a permutation of the diagonal entries}$

Conversely: all permutations arise because all transpositions arise:

2x2 case: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ so $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N(T)$

$n \times n$ case: use matrix with $\begin{cases} 1 \text{ on diagonal except in positions } (i,i), (j,j) \\ \text{transposes } \lambda_i, \lambda_j \\ \text{diagonal entries of } T \end{cases} \begin{cases} 1 \text{ in entries } (i,j), (j,i) \\ 0 \text{ else} \end{cases}$ ■

claim 2 $R(T)^{W(T)} = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{S_n} = \mathbb{Z}[P_1, \dots, P_n, P_n^{-1}]$ where

the P_j are the elementary symmetric polynomials in n variables:

$$P_1 = \sum t_j, \quad P_2 = \sum_{i < j} t_i t_j, \quad \dots, \quad P_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} X_{i_1} X_{i_2} \dots X_{i_k}, \quad \dots, \quad P_n = X_1 X_2 \dots X_n$$

Pf t_j corresponds to rep: $(\lambda_1, \dots, \lambda_n) \in T \longmapsto \lambda_j = e^{2\pi i x_j}$

$\Rightarrow S_n$ acts by permuting the t_1, \dots, t_n .

Fact from algebra: $\mathbb{Z}[t_1, \dots, t_n]^{S_n} = \mathbb{Z}[P_1, \dots, P_n]$ poly ring in elem. sym. polys

If $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{S_n}$ then (large power of P_n) $\cdot f$ will not have negative powers of t_j , so $f \in P_n^{-N} \cdot \mathbb{Z}[P_1, \dots, P_n] \subseteq \mathbb{Z}[P_1, \dots, P_n, P_n^{-1}]$.

Conversely $\mathbb{Z}[P_1, \dots, P_n, P_n^{-1}] \subseteq \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{S_n}$, hence equality. ■

Claim 3 There exists a rep V_k of $U(n)$ with character $\chi_{V_k} = P_k$

Pf $U(n)$ acts on $V = \mathbb{C}^n$ (standard rep: $\rho(g) = g \in U(n) \subseteq GL(n, \mathbb{C}) \Rightarrow U(n)$ acts on $V^{\otimes 2}$. On generators: $g \cdot (v_1 \otimes v_2) = gv_1 \otimes gv_2$. But not irrep: has a subrep:

$$\Lambda^2 V = \left\{ \text{tensors } \sum v_i \otimes w_j \text{ such that symmetric group } S_2 \text{ acting} \right. \\ \left. \text{by permuting factors } v_i, w_j \text{ acts by sign(permutation) } \cdot \text{Id} \right\}$$

$$\Rightarrow \Lambda^2 V = \text{span}_{\mathbb{C}} \{ v \otimes w - w \otimes v : v, w \in V \}$$

(transposition (12) acts by $w \otimes v - v \otimes w$ so by $-\text{Id} = \text{sign} \cdot \text{Id}$)

Convenient to abbreviate $v \wedge w = v \otimes w - w \otimes v$

$$\Rightarrow \Lambda^2 V = \text{vector space over } \mathbb{C} \text{ with basis } (e_i \wedge e_j)_{1 \leq i < j \leq n}$$

with $U(n)$ -action $g \cdot (e_i \wedge e_j) = ge_i \wedge ge_j$

Rmk we stipulate that the symbol \wedge is linear in each entry and antisymmetric: $e_j \wedge e_i = -e_i \wedge e_j$, so $ge_i \wedge ge_j$ makes sense.

alternating product $\wedge^k V = \{z \in V^{\otimes k} : \sigma \cdot z = \text{sign}(\sigma) \cdot z \text{ for all } \sigma \in S_k\}$
 ↖ act by permuting tensor factors

↗ \cong vector space over \mathbb{C} with basis the symbols $(e_{i_1} \wedge \dots \wedge e_{i_k})_{1 \leq i_1 < \dots < i_k \leq n}$
 $U(n)$ -action $g \cdot (e_{i_1} \wedge \dots \wedge e_{i_k}) = g e_{i_1} \wedge \dots \wedge g e_{i_k}$
 $(v_1 \wedge \dots \wedge v_k \equiv \sum_{\sigma \in S_k} \text{sign}(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)})$

In particular, $\wedge^n V = \mathbb{C} \cdot e_1 \wedge e_2 \wedge \dots \wedge e_n$ is 1-dimensional.

Character restricted to T:

$g = (\lambda_1, \dots, \lambda_n) \in T \Rightarrow g \cdot e_i = \lambda_i e_i \Rightarrow \chi_{V_1} = \sum \lambda_i$ ← trace = \sum evaluates

Also $g \cdot (e_i \wedge e_j) = g e_i \wedge g e_j = \lambda_i e_i \wedge \lambda_j e_j = \lambda_i \lambda_j (e_i \wedge e_j) \Rightarrow \chi_{V_2} = \sum_{i < j} \lambda_i \lambda_j$
 ↖ \wedge is multi-linear

$g \cdot (e_{i_1} \wedge \dots \wedge e_{i_k}) = (\lambda_{i_1} \dots \lambda_{i_k}) e_{i_1} \wedge \dots \wedge e_{i_k} \Rightarrow \chi_{\wedge^k V} = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k} = P_k(\lambda_1, \dots, \lambda_n)$

In particular: $\chi_{\wedge^n V}(\lambda_1, \dots, \lambda_n) = \lambda_1 \lambda_2 \dots \lambda_n = \det(g)$ (1-dimensional rep) ■

Claim 4

$$R(U(n)) = \mathbb{C} [\chi_V, \chi_{\wedge^2 V}, \dots, \underbrace{\chi_{\wedge^n V}}_{\det}, \underbrace{\chi_{\wedge^n V}^{-1}}_{(\det)^{-1} = \overline{\det}}]$$

Notice: we only use that $R(U(n)) \rightarrow R(T)^{W(T)}$ is injective (which we proved in general) and we deduced surjectivity in this example by Claim 3.

Claim 5 The $V, \wedge^2 V, \dots, \wedge^n V, \overline{\wedge^n V}$ are irreps ($\wedge^n V, \overline{\wedge^n V}$ obviously since 1-dimensional)

Pf Suppose not: $\wedge^k V = U_1 \oplus U_2$ sum of subreps.

If we knew that $e_1 \wedge e_2 \wedge \dots \wedge e_k \in U_1$, then in fact $U_1 = \wedge^k V, U_2 = \{0\}$ because the matrix $g \in U(n)$ with columns $(e_{i_1} | e_{i_2} | \dots | e_{i_k} | \text{other basis vectors})$ acts by $g e_m = e_{im}$ so $g \cdot (e_1 \wedge \dots \wedge e_k) = e_{i_1} \wedge \dots \wedge e_{i_k}$ which is a basis for $\wedge^k V$.

Trick Consider action of max tons $T \subseteq U(n) \rightarrow \text{Aut}(\wedge^k V)$. Then $\wedge^k V = \bigoplus V_a$ where $V_a = \{v \in \wedge^k V : x \cdot v = \chi_a(x) v\}$ are weight spaces ($a \in \mathbb{Z}^n$):

$V_a =$ sum of 1-dim irreps/ \mathbb{C} each with character $\chi_a(x) = e^{2\pi i \langle x, a \rangle}$. Similarly

$U_1 = \bigoplus V_a', U_2 = \bigoplus V_a''$ with $V_a', V_a'' \subseteq V_a$ (unique decomposition into irreps of T)

But we know $V_a = \mathbb{C} e_{i_1} \wedge \dots \wedge e_{i_k}$ for $a = (1, 1, \dots, 1, 0, \dots, 0)$ (1 in first k entries) (indeed $V_a = \mathbb{C} e_{i_1} \wedge \dots \wedge e_{i_k}$ where a has 1 in entries i_1, \dots, i_k and 0 elsewhere)

$\Rightarrow V_a' = V_a$ or $V_a'' = V_a$ so $e_1 \wedge \dots \wedge e_k \in U_1$ or U_2 (as opposed to being $U_1 + U_2$) ■

Rmk There is a more systematic approach to finding irreps for G by looking for "highest weight vectors" in terms of a certain ordering of weights $a \in \mathbb{Z}$. For $U(n)$, the lexicographic order works so $(1, 1, \dots, 1, 0, \dots, 0)$ is the highest weight arising in $\wedge^k V$ and $e_1 \wedge \dots \wedge e_k$ the highest weight vector.

More examples

← since $\det = 1$, lose $\det, \overline{\det}$

$$R(SU(n)) = \mathbb{Z} [X_V, X_{\wedge^2 V}, \dots, X_{\wedge^{n-1} V}] \text{ where } V = \mathbb{C}^n \text{ standard}$$

$$R(SO(2n+1)) = \mathbb{Z} [X_V, X_{\wedge^2 V}, \dots, X_{\wedge^n V}], V = W \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{2n+1}, W = \mathbb{R}^{2n+1} \text{ standard}$$

Some more facts

$$1) \quad \mathcal{Q}(G) \xleftarrow{1:1} \mathcal{Q}(T)^W = C(T)^W \quad (\text{since } T \text{ is abelian, } \mathcal{Q}(T) = C(T))$$

$$(f: G \rightarrow \mathbb{C}) \longmapsto (f|_T: T \rightarrow \mathbb{C})$$

$$\left(\begin{array}{l} f(h) = f(x) \\ \text{if } h = gxg^{-1} \end{array} \right) \longleftarrow (f: T \rightarrow \mathbb{C})$$

Rmk need check $f(h)$ well-defined and continuous. The key step to show well-definedness is fact:

$$x_1, x_2 \in T \text{ are conjugate in } G \iff w \cdot x_1 = x_2 \text{ some } w \in W(T) \\ (\text{i.e. they are conjugate in } N(T))$$

Therefore $f(x_1) = f(w \cdot x_1) = f(x_2)$ since f is Weyl-invariant.

Example For $SU(2)$, $T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\} \cong S^1$. Then $W(T) = S_2$ (just like for $U(2)$)

$$\mathcal{Q}(SU(2)) = \mathbb{C}[e^{2\pi i x}, e^{-2\pi i x}]^{S_2} = \mathbb{C}[2 \cos(2\pi x)] = \text{span}_{\mathbb{C}} \{ \cos(2\pi a x) : a \in \mathbb{Z} \} \\ (\text{Compare Q. Sheet 6})$$

2) Weyl group action on T permutes the roots

meaning: $(w \cdot \theta_a)(x) = \theta_a(w^{-1} \cdot x) = \theta_{a'}(x)$ all $x \in T$, some root $\theta_{a'}$.

Pf $\text{Ad}(w^{-1} \cdot x) = \text{Ad}(n^{-1} x n) = \text{Ad}(n)^{-1} \text{Ad}(x) \text{Ad}(n)$ hence the rep $\text{Ad}_T \circ w^{-1} \cong \text{Ad}_T$ and $\text{Ad}(n)^{-1} \cdot V_a \cdot \text{Ad}(n)$ is an irred summand so $\cong V_{a'}$. ■

Example $U(n)$ roots are $x_k - x_\ell$ where $\lambda_j = e^{2\pi i x_j}$ are diag entries of T . $W(T) = S_n$ permutes the diagonal entries, so permutes roots.

3) The permutation action on the roots uniquely determines elements of $W(T)$

(So $W(T) \cong$ subgroup of S_m where $m = \#$ roots of G)

Pf G connected Lie gr \Rightarrow Lie gr hom $w = \mathcal{A}_n|_T: T \rightarrow T$ is determined by $D_w = D_{\mathcal{A}_n|_T} = \text{Ad}(n): t \rightarrow t$ ($t = \text{Lie } T$). Since $\mathcal{A}_n|_T = \text{id}$ on $\text{Centre}(G) \subseteq T$ have $D_w = \text{id}$ on $\text{Centre}(\mathfrak{g}) \subseteq t$. In Lecture 16 will show:

$$\text{dual } \mathfrak{g}^* = \text{span}_{\mathbb{R}}(\text{roots } \theta_a) \oplus \text{Centre}(\mathfrak{g})^*$$

hence $(D_w)^*$ determined by action on roots, so result follows. ■

Weyl integration formula (Non-examinable)

Would like to recover $\langle X_1, X_2 \rangle = \int_{g \in G} \overline{X_1(g)} X_2(g)$ from an inner product on T

$$f \in \mathcal{Q}(G) \Rightarrow \int_{g \in G} f(g) = \int_{x \in T} f(x) \delta(x) \text{ where } \delta(x) = \frac{1}{|W(T)|} \prod_{\text{roots}} |e^{\pi i \theta_a(x)} - e^{-\pi i \theta_a(x)}|$$

Example $G = U(n)$: $\int_0^1 \dots \int_0^1 f(x_1, \dots, x_n) \frac{1}{n!} \prod_{k < \ell} |e^{2\pi i x_k} - e^{2\pi i x_\ell}|^2 dx_1 \dots dx_n$

$G = SU(2)$: $\int_0^1 f(x) \frac{1}{2} |e^{\pi i (2x)} - e^{-\pi i (2x)}|^2 dx = 2 \int_0^1 f(x) \sin^2(2\pi x) dx$