

LECTURE 16

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C3.5 LIE GROUPS, HT2015, Oxford.

ASSUME: T MAX TORUS
 G COMPACT CONNECTED

Weyl-invariant inner product on $t = \text{Lie}(T)$

Recall $G \rightarrow \text{Aut}(V) \Rightarrow \exists G\text{-invariant inner product on } V$
 $\text{Ad}: G \rightarrow \text{Aut}(g) \Rightarrow \quad " \quad " \quad " \quad (\cdot, \cdot) \text{ on } g$

Cor $\exists W(T)\text{-invariant inner product on } t = \text{Lie}(T)$

hence $W(T) \cong$ finite subgroup of $O(n)$ action on $(\mathbb{R}^n, (\cdot, \cdot)_{\text{standard}}) \cong t$

Pf $w = A_n|_T \in W(T)$ acts by $D_w = D, A_n = \text{Ad}(n)$ on $t \subseteq g$ ($n \in N(T)$) ■

Rmk Recall Lie group hom $w = A_n|_T$ is determined by Lie algebra hom $D_w = \text{Ad}(n)$.

Example $G = U(n)$ $(X, Y) = \text{Trace}(\bar{X}^T Y) = -\text{Trace}(XY)$ on $g = \{X: \bar{X}^T = -X\}$
 $t = \left\{ \begin{pmatrix} ix_1 & & \\ & \ddots & \\ & & ix_n \end{pmatrix} : x_i \in \mathbb{R} \right\} \Rightarrow (X, Y) = \sum x_j y_j$ standard inner product on $t \cong \mathbb{R}^n$

Finding the Weyl group combinatorially

$$\begin{aligned} T_\alpha &= \text{Ker}(\text{root } \theta_\alpha: T \rightarrow S' = \mathbb{R}/\mathbb{Z}) \\ t_\alpha &= \text{Lie } T_\alpha = \text{Ker}(\theta_\alpha: t \cong \mathbb{R}^n \rightarrow \mathbb{R}) \\ &= \text{the hyperplane orthogonal to } \alpha \in \mathbb{R}^n \cong t \quad (\text{equation: } \langle x, \alpha \rangle = 0) \end{aligned} \quad \left. \begin{array}{l} \theta_\alpha(x) = \langle x, \alpha \rangle \\ = x_1 \alpha_1 + \dots + x_n \alpha_n \end{array} \right\}$$

Example $U(n)$ for $\theta_\alpha = \theta_{x_k - x_\ell}: T_\alpha = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_k = \lambda_\ell \right\} \quad t_\alpha = \{x \in \mathbb{R}^n : x_k = x_\ell\}$

Rmk (exercise) $\text{Centre}(G) = \bigcap_{\text{roots}} T_\alpha$, $\text{Centre}(g) = \bigcap_{\text{roots}} t_\alpha = \bigcap \text{Ker } \theta_\alpha$

Example $\text{Centre}(U(n)) = \left\{ \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \right\} = S^1 \cdot \text{Id}$, $\text{Centre}(u(n)) = \left\{ \begin{pmatrix} ix_1 & & \\ & \ddots & \\ & & ix_n \end{pmatrix} \right\} = i\mathbb{R} \cdot \text{Id}$

Fact (tricky) For any root θ_α , there exists $w \neq \text{Id}$ in W fixing each point of T_α

Cor $W(T)$ acting on t contains all reflections in the hyperplanes t_α

Pf $W(T)$ acts by orthogonal matrices $\Rightarrow w$ from Fact must be the reflection in t_α ■

Fact $W(T) \subseteq \text{Aut}(t)$ is generated by the reflections in the hyperplanes t_α

Example $U(n)$ reflection in $t_\alpha = \{x \in \mathbb{R}^n : x_k = x_\ell\}$ swaps x_k, x_ℓ coordinates
 $\Rightarrow W(T) \subseteq \text{Aut}(\mathbb{R}^n)$ is group S_n = permutation matrices which permute coords

2x2 Example $w = \text{conjugation by } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $w \cdot \begin{pmatrix} ix_1 & 0 \\ 0 & ix_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} ix_1 & 0 \\ 0 & ix_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} ix_2 & 0 \\ 0 & ix_1 \end{pmatrix}$

Rmk Generating reflections have formula: $S_\alpha: t \rightarrow t$, $S_\alpha(x) = x - \frac{2\theta_\alpha(x)}{\langle \theta_\alpha^*, \theta_\alpha^* \rangle} \theta_\alpha^*$

where $\theta_\alpha^* \in t^*$ are duals of $\theta_\alpha \in t^*$ via inner product (\cdot, \cdot) so: $(\theta_\alpha^*, Y) = \theta_\alpha(Y)$.

Rmk $t = \text{centre}(g) \oplus \text{span}_{\mathbb{R}} \theta_\alpha^*$, and the dual $t^* = \text{centre}(g)^* \oplus \text{span}_{\mathbb{R}} (\text{roots } \theta_\alpha)$

Pf Orthog. complement to $\text{span}_{\mathbb{R}} \theta_\alpha^*$ is $\{x \in t : (x, \theta_\alpha^*) = 0\} = \{x \in t : \theta_\alpha(x) = 0\} = \bigcap \text{Ker } \theta_\alpha$ ■

KILLING FORM

Recall (Q. sheet 2) :

$$\langle x, y \rangle = \text{Trace}(\text{ad}(x) \text{ad}(y))$$

recall $\text{ad}(x) \in \text{End}(V)$
 $\text{ad}(x)(y) = [x, y]$

is bilinear map $V \times V \rightarrow \mathbb{F}$
any Lie algebra V .

Thm For G compact,

$$\begin{aligned} \langle x, x \rangle &\leq 0 \quad \text{all } x \in g \\ &= 0 \quad \text{if and only if } x \in \text{Centre}(g) \end{aligned} \quad \xrightarrow{\text{Q. sheet 4}} \quad \text{Centre}(g) = \text{Ker}(\text{ad}) = \text{Lie}(\text{Centre } G)$$

Pf Using $\text{Ad}(G)$ -invariant metric (\cdot, \cdot) from above, $g \cong \mathbb{R}^m$ and

$$\begin{aligned} \text{Ad}(g) &\in O(m) \quad \text{so } \text{ad}(g) \in \sigma(m) \text{ by naturality:} \\ &\Rightarrow A = \text{ad}(x) = \text{skew-symmetric matrix} \\ &\Rightarrow \langle x, x \rangle = \text{Tr}(AA) = \text{Tr}(-ATA) = \sum - (A^T)_{ij} A_{ji} = \sum - A_{ji}^2 \leq 0 \\ &= 0 \quad \text{iff all } A_{ij} = 0, \text{ that is } A = \text{ad}(x) = 0. \quad \blacksquare \end{aligned}$$

Cor For G compact, $g = \text{Centre}(g) \oplus g'$ where $g' \subseteq g$ is an ideal on which the Killing form is negative definite. $\begin{array}{l} \text{ideal } W \subseteq V: \bullet \text{vector subspace } W \subseteq V \\ (\text{Q. sheet 4}) \quad \bullet [v, w] \in W \text{ for all } v \in V, w \in V \end{array}$

Pf $g' = \text{Centre}(g)^\perp \leftarrow$ using Ad-invariant metric (\cdot, \cdot)

$\text{Ad}(g)$ acts by orthogonal matrices so sends $g' \rightarrow g'$, so $\text{ad} = D, \text{Ad} : g \rightarrow \text{End}(g')$

$\Rightarrow \text{ad}(x)(y) = [x, y] \in g'$ for $x \in g, y \in g'$ so g' is an ideal.

By above Thm, $\langle y, y \rangle < 0$ for $y \neq 0 \in g'$ since $y \notin \text{Centre}(g)$ \blacksquare

CLASSIFICATION OF COMPACT LIE GROUPS

NON-EXAMINABLE

Recall (Q. Sheet 4)

Lie algebra V called simple if V not abelian and only ideals are $0, V$
 V semi-simple if $V = \bigoplus$ simple Lie algebras.

Q. Sheet 4 : for connected Lie group G ,

g simple $\Leftrightarrow G$ simple (meaning: not abelian and $\{1\}$ is the only non-trivial connected normal Lie subgp)

Fact Killing form on V is non-degenerate $\Rightarrow V$ semi-simple

Cor G compact $\Rightarrow g = (\text{abelian summand}) \oplus (\text{semi-simple summand})$

Rmk • So don't have nasty summands ("solvable summands") in Lie algebra.
• Lie algebra theory can classify all simple Lie algebras!

Consequence: can classify compact Lie groups!

G compact connected

FACT 1 If G simply-connected and of simple, then G can be:

$SU(n)$

$Spin(n) = \text{universal cover of } SO(n)$

$Sp(n) = \text{symplectic group}$ (Lecture 2)

$G_2, F_4, E_6, E_7, E_8 \leftarrow \text{exceptional Lie groups.}$

(Example
 $SU(2) = Spin(3)$
= univ. cover of $SO(3)$)

Rmk If omit assumption of simple, then G is a product of above gps

FACT 2 G has a finite cover $G' \xrightarrow{\pi} G$ (meaning $\ker \pi$ finite)

with $G' \cong \text{torus} \times \text{finite product of groups from above list.}$

$\cong \text{finite product of copies of } S^1 \text{ and groups from above list.}$

Rmk By Q1 of Q.sheet 5, $G \cong G'/\Gamma$ where $\Gamma \subseteq \text{Centre}(G')$ is a finite group (since discrete and compact).

The wikipedia page "List of simple Lie groups" has lots of info and examples about this classification.

———— END OF COURSE ———