ritter@maths.ox.ac.uk

Homework 1.

You are encouraged to collaborate on these exercises.

Question 1. Show that the *tangent bundle* $TG = \bigsqcup_{g \in G} T_g G$ of a Lie group G is canonically identifiable with $G \times T_I G$.
Hint. consider the left translation map $\phi_g : G \to G$, $\phi_g(h) = gh$.

Deduce that any Lie group of dimension n has n non-vanishing vector fields which are linearly independent at each point of G.

Deduce that the 2-dimensional sphere S^2 *cannot* be a Lie group.
Hint. you may quote the "hairy ball theorem" – google it!

Show that the 3-dimensional sphere S^3 is a Lie group by considering

$$SU(2) = \{2 \times 2 \text{ complex matrices with } A^\dagger A = I, \ \det A = 1\}$$

where $A^\dagger$ denotes the conjugate transpose of A.
Hint. Verify that $SU(2)$ *is the set of matrices* $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ *with* $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$.

Cultural remark. The only spheres which are also Lie groups are S^0, S^1, S^3.

Question 2. Suppose G_1, G_2 are Lie groups. Show that $G_1 \times G_2$ is a Lie group in a natural way.

Deduce that the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ is a Lie group.

Find a map $\pi : \mathbb{R}^n \to T^n$ that allows you to identify $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ (the quotient group).

Not all vector fields on $\mathbb{R}^n$ give rise to vector fields on T^n if you apply $D\pi$, but which ones do? Are these all the vector fields on T^n?

Find out which vector fields on T^n are *left-invariant*, meaning

$$D_h \phi_g \cdot X|_h = X|_{gh}$$

for all $h, g \in G$, where ϕ_g is defined in Question 1.

Question 3. Use the implicit function theorem (at the end of Lecture 1) applied to

$$\varphi : GL(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R}) = \{n \times n \text{ symmetric matrices }\}, \ \varphi(A) = A^T A,$$

to prove that the orthogonal group $O(n)$ is a Lie group, to find the dimension of $O(n)$ and to find the tangent space $T_I O(n)$.

Show that $O(n)$ is compact. *Hint. You may quote the Heine-Borel theorem.*

Question 4. Let $\varphi : M \to N$ be a *diffeomorphism* of manifolds (a smooth map with smooth inverse). For a vector field X on M define the *push-forward* vector field $Z = \varphi_* X$ on N by

$$Z|_y = D_x \varphi \cdot X|_x$$

where $x = \varphi^{-1}(y)$. Show that for any function $f : N \to \mathbb{R}$,

$$(\varphi_* X) \cdot f = (X \cdot (f \circ \varphi)) \circ \varphi^{-1}.$$

Deduce that $[\varphi_* X, \varphi_* Y] \cdot f = \varphi_* [X, Y] \cdot f$, and deduce that

$$[\varphi_* X, \varphi_* Y] = \varphi_* [X, Y].$$

Check that this last identity holds in the simple case: $M = N = \mathbb{R}$, $X = \frac{\partial}{\partial x}$, $Y = x \frac{\partial}{\partial x}$, $\varphi(x) = 2x$.

Let G be a Lie group. Prove the following characterization of left-invariant vector fields:

$$X \in \mathrm{Lie}\, G \ \Leftrightarrow \ (\phi_g)_* X = X \quad \text{ for all } g \in G,$$

and deduce that, if $X, Y \in \mathrm{Lie}\, G$, then also $[X, Y] \in \mathrm{Lie}\, G$.

Remark. It's tricky to show $[\varphi_* X, \varphi_* Y] = \varphi_* [X, Y]$ *directly using coordinates, try it if you are brave.*