ritter@maths.ox.ac.uk

Homework 2.

You are encouraged to collaborate on these exercises.

Question 1. Viewing quaternions as matrices, show that quaternions satisfy the rules

$$|h_1 h_2| = |h_1| \cdot |h_2| \qquad |h^{-1}| = |h|^{-1}.$$

Viewing $\mathbb{H}$ as a real 4-dimensional vector space, check that $|h|$ is the usual norm on $\mathbb{R}^4$.
Show that *(using Lecture 2 and Question sheet 1)*

$$\mathrm{Sp}(1) = SU(2).$$

For $h \in \mathbb{H} \setminus \{0\}$ define

$$\mathcal{A}_h \colon \mathbb{H} \to \mathbb{H}, p \mapsto hph^{-1}.$$

Show that $\mathcal{A}_h$ is an orthogonal map (viewing $\mathbb{H}$ as $\mathbb{R}^4$). *(Hint. recall Example 11 from Lecture 2.)*
By considering the orthogonal complement of $\mathbb{R} = \mathbb{R} \cdot 1 \subset \mathbb{H}$, deduce that $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \subset \mathbb{H} \setminus \{0\}$ acts on $\mathbb{R}^3$ by rotations.
Writing quaternions as $r + v$, where $r \in \mathbb{R}$ and $v \in \mathbb{R}^3 = \mathrm{span}_{\mathbb{R}}(i, j, k)$, show that

$$v_1 v_2 = -v_1 \bullet v_2 + v_1 \times v_2$$

for $v_1, v_2 \in \mathbb{R}^3$, where $\bullet$ is dot product in $\mathbb{R}^3$, and $\times$ is cross product in $\mathbb{R}^3$.
Show that any $h \in \mathrm{Sp}(1)$ can be written as

$$h = \cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2}) v$$

for a unit vector $v \in \mathbb{R}^3$ and for some $\theta \in \mathbb{R}$. Show that in this case $vv = -1$ and $\mathcal{A}_h(v) = v$.
Describe the rotation determined by h. *(Hint. consider an orthonormal basis $w_1, w_2, v \in \mathbb{R}^3$.)*
Deduce that there is a smooth surjective homomorphism

$$\mathrm{SU}(2) \to \mathrm{SO}(3)$$

and explain briefly in what sense SU(2) "covers" SO(3) twice.
Show that SO(3) as a manifold is a solid ball $B^3 \subset \mathbb{R}^3$ of radius π having identified the antipodal points on the boundary of the ball (this boundary is a sphere of radius π in $\mathbb{R}^3$). This space is called *real projective space*, $\mathbb{R}P^3$.
Taking inspiration from the construction of polar coordinates, show that $\mathbb{R}P^3$ can be identified with the space of straight lines in $\mathbb{R}^4$ through the origin. Finally, show that the map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ corresponds to the map

$$S^3 \to \mathbb{R}P^3, \quad (x \in S^3 \subset \mathbb{R}^4) \mapsto (\text{the straight line in } \mathbb{R}^4 \text{ through the two points } 0 \text{ and } x).$$

Question 2. Check these properties of $\exp : \mathrm{Lie}(G) \to G$.

(1) $\mathrm{Image}(\exp) \subset G_0 =$ connected component of $1 \in G$;
(2) $\exp((t+s)v) = \exp(tv)\exp(sv)$ for all $t, s \in \mathbb{R}$;
(3) $(\exp v)^{-1} = \exp(-v)$;
(4) If $g = \exp(v)$ then it has an n-th root: $\exp(\frac{1}{n}v)$;
(5) Show that the following map is not surjective

$$\exp : \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$$

by considering the eigenvalues of the square root (if it existed) of $g = \begin{pmatrix} -4 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$.

Cultural remark. *For any compact connected Lie group G,* exp *is surjective.*

Question 3. *Remark. Abbreviate* $\mathfrak{g} = \mathrm{Lie}(G)$. *By Lecture 5 you know that* $\mathbf{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is a Lie algebra homomorphism because it is the derivative* $D_1\mathrm{Ad}$ *of a Lie group homomorphism.*

Prove directly that $\mathbf{ad}$ is a Lie algebra homomorphism by using the fact that $\mathbf{ad}(X) \cdot Z = [X, Z]$.

Show that

$$v_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

is a basis for $\mathfrak{so}(3) \subset \mathrm{Mat}_{3\times 3}(\mathbb{R})$.

By computing all brackets $[v_i, v_j]$, show that

$$\mathfrak{so}(3) \cong (\mathbb{R}^3, \text{cross product}), \; v_i \mapsto \text{standard basis vector } e_i$$

is a Lie algebra isomorphism.

Via this isomorphism we identify $\mathrm{End}(\mathfrak{so}(3))$ with 3×3 matrices. Compute the matrices $\mathbf{ad}(v_i)$.

By computing $\langle v_i, v_j \rangle$ show that the **Killing form**

$$\langle v, w \rangle = \mathrm{Trace}(\mathbf{ad}(v)\mathbf{ad}(w)) \in \mathbb{R}$$

is a negative definite scalar product on $\mathfrak{so}(3)$.

Remark. Observe

$$\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{o}(3),$$

since $SU(2)$, $SO(3)$, $O(3)$ are locally diffeomorphic near 1.

Cultural Remark. *for a compact Lie group, the Killing form is negative definite on* $\mathfrak{g}/\ker \mathbf{ad}$ *(here we quotiented by the centre* $Z(\mathfrak{g}) = \mathrm{Lie}(Z(G)) = \ker \mathbf{ad}$ *because the Killing form is zero if* $\mathbf{ad}(v) = 0$*).*