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Homework 3.

You are encouraged to collaborate on these exercises.

Question 1. Show that the subgroups of $S^1 = \mathbb{R}/\mathbb{Z}$ are: S^1 or one of two types:

(1) a finite subgroup generated by a rational number;
(2) an infinite subgroup which is dense in S^1.

Describe geometrically the 1-parameter sugroups of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. In particular, give an example of a subgroup $\mathbb{R} \subset T^2$ which is not a submanifold.[1]

Question 2. Let $\varphi : T^n \to S^1$ be a Lie group homomorphism. Show that $D_1\varphi$ has integer entries. *(Hint. use naturality of* exp*, and try the case $n = 1$ first if you get stuck.)*

Determine all Lie group homomorphisms

$$\varphi : T^n \to S^1$$

and all Lie group homomorphisms

$$T^n \to T^n.$$

(Hint. given $D_1\varphi \in \mathbb{Z}^n$, can you construct a homomorphism φ? is it unique?)

Let $v \in \mathbb{R}^n$. If the subgroup $\langle v \rangle$ generated by v is not dense in $T^n = \mathbb{R}^n/\mathbb{Z}^n$, show that $v \in \ker(\varphi : T^n \to S^1)$ for some non-trivial φ.
(Hint. what Lie group can $T^n/\overline{\langle v \rangle}$ be, using the final results of Lecture 6?)

Show that the following statements are equivalent for $v = (v_1, \dots, v_n) \in \mathbb{R}^n$:

(1) $1, v_1, \dots, v_n$ are linearly dependent over $\mathbb{Q}$;
(2) $\sum a_i v_i \in \mathbb{Z}$ for some $a_i \in \mathbb{Z}$, where not all a_i are zero;
(3) $\langle v \rangle$ is not dense in T^n.

Deduce that almost any $v \in T^n$ will generate a dense subset of T^n!

Question 3. Using the formulas from Lecture 5, obtain the formula

$$\boxed{\exp(X)\exp(Y)\exp(-X) = \exp(Y + [X,Y] + \tfrac{1}{2!}[X,[X,Y]] + \tfrac{1}{3!}[X,[X,[X,Y]]] + \cdots)}$$

Show that for a matrix group,

$$\mathrm{Ad}(g) \cdot X = gXg^{-1}$$

where $g \in G, X \in \mathfrak{g}$.

Consider the subgroup $T \subset U(n)$ of diagonal unitary matrices. Show that T is a torus and that T lies in the image of $\exp : \mathfrak{u}(n) \to U(n)$. Deduce that

$$\exp : \mathfrak{u}(n) \to U(n)$$

is surjective.
(Hint. Recall from linear algebra that a unitary matrix has a basis of unitary eigenvectors.)

Question 4. Suppose

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$$

as a vector space. Let

$$\mathfrak{g} \to G, \quad \psi(v_1, \dots, v_k) = \exp(v_1) \cdots \exp(v_k).$$

Show that[2]

$$D_0\psi \cdot (X_1, \dots, X_k) = X_1 + \cdots + X_k,$$

and deduce that ψ is a local diffeomorphism near 0.

[1] $N \subset M$ is a submanifold if the inclusion is an embedding, i.e. a homeomorphism onto the image (in the subspace topology) and the derivative of the inclusion is injective.

[2] where we naturally identify $T_0\mathfrak{g} = \mathfrak{g}$, [curve $0 + tX$] $\leftrightarrow X$.

Therefore, for small $X, Y \in \mathfrak{g}$, we can uniquely define $f(X,Y) \in \mathfrak{g}$ by the equation

$$\exp X \cdot \exp Y = \exp(f(X,Y)).$$

Intuitively $f(X,Y)$ is telling you what group multiplication in G looks like in $\mathfrak{g}$ via $\log = (\exp)^{-1}$.

By Taylor[3] expanding f near $(0,0)$, show that there is a bilinear map $B : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ such that

$$f(X,Y) = X + Y + \tfrac{1}{2}B(X,Y) + \text{higher order terms.}$$

Using $\exp(Z)^{-1} = \exp(-Z)$, show that B is antisymmetric. Using the formula of Q.3, show

$$B(X,Y) = [X,Y].$$

Cultural Remark.

$$f(X,Y) = \exp^{-1}(\exp(X)\exp(Y)) = X + Y + \tfrac{1}{2}[X,Y] + \text{higher}$$

is called the ***Baker-Campbell-Hausdorff formula****. A hard theorem states that the higher order terms can all be expressed in terms of Lie brackets involving X and Y (see Wikipedia). This proves the remarkable fact that the local group structure of G (multiplication for elements near 1) is determined by the Lie algebra $\mathfrak{g}$.*

[3]Recall Taylor says: $f(X,Y) = f(0,0) + D_0(f) \cdot \binom{X}{Y} + \binom{X}{Y}^T \cdot \text{Hessian}_0(f) \cdot \binom{X}{Y} + \cdots$. To ensure that the Hessian term does not have x_i^2, y_i^2 terms, consider $f(X,0)$ and $f(0,Y)$.