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Homework 4.

You are encouraged to collaborate on these exercises.

Question 1. Let $\varphi : G_1 \to G_2$ be a Lie group homomorphism. Show that

$$\ker \varphi \subset G_1$$

is a closed (hence embedded) Lie subgroup with Lie algebra

$$\ker(D_1\varphi) \subset \mathfrak{g}_1.$$

*A vector subspace $J \subset (V, [\cdot,\cdot])$ of a Lie algebra is called an **ideal** if*

$$[v, j] \in J \text{ for all } v \in V, j \in J.$$

Show that ideals are Lie subalgebras. Show that for a Lie subgroup $H \subset G$, with H, G connected,

$$\boxed{H \subset G \text{ is a normal subgroup} \Leftrightarrow \mathfrak{h} \subset \mathfrak{g} \text{ is an ideal}}$$

Hints. for $\Leftarrow$ use the formula from Question 1. For $\Rightarrow$ use that formula but put tX, sY instead of X, Y and show that the curve $e^{t\,\mathrm{ad}\,X} \cdot Y$ lies in $\mathfrak{h}$.

*The **centre** of a Lie algebra $(V, [\cdot,\cdot])$ is*

$$Z(V) = \{v \in V : [v, w] = 0 \text{ for all } w \in V\}.$$

For G connected, prove that the centre of the group G is[1]

$$\boxed{Z(G) = \ker(\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}))}$$

Deduce that the centre of G is a closed (hence embedded) Lie subgroup of G which is abelian, normal and has Lie algebra

$$\mathrm{Lie}(Z(G)) = Z(\mathfrak{g}).$$

Finally deduce that, for G connected,

$$\boxed{G \text{ is abelian} \Leftrightarrow \mathfrak{g} \text{ is abelian}}$$

Question 2. Show that

$$[X, Y] = 0 \Rightarrow \exp(X + Y) = \exp(X)\exp(Y).$$

(Hint. By Lecture 8, Lie subalgs of $\mathfrak{g}$ correspond to connected Lie subgps of G. Consider span(X, Y).)

Prove that if G is a Lie group **with $\mathbf{Z}(\mathbf{G}) = \{\mathbf{1}\}$** then G can be identified with a Lie subgroup of $GL(m, \mathbb{R})$, some m, so $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(m, \mathbb{R})$.

If $(V, [\cdot,\cdot])$ is a Lie algebra **with $\mathbf{Z}(\mathbf{V}) = \{\mathbf{0}\}$**, show that V is the Lie algebra of some Lie group.
(Hint. consider $\mathbf{ad} : V \to \mathrm{End}(V), \mathbf{ad}(X) \cdot Y = [X, Y]$, and use the theorem in the previous hint.)

Cultural remark 1. *A big theorem (**Ado's theorem**) states that any Lie algebra V has a faithful representation into some $\mathfrak{gl}(m, \mathbb{R})$ (that is, an injective Lie algebra homomorphism $V \to \mathfrak{gl}(m, \mathbb{R})$). The same arguments you used above imply that there is a Lie subgroup of $GL(m, \mathbb{R})$ with Lie algebra V. So one could reduce the study of Lie algebras to studying matrices with the bracket $[B, C] = BC - CB$.*

Cultural remark 2. *Another big theorem (**Lie's third theorem**) states: if you impose the topological condition that the Lie group should be simply-connected[2] then you also get uniqueness:*

$$\{ \text{ Lie algebras } V \}/\text{Lie alg isos} \overset{1:1}{\longleftrightarrow} \{ \text{ connected simply-connected Lie groups } G \}/\text{Lie gp isos}$$

That condition is necessary, since the double cover $SU(2) \to SO(3)$ illustrates two different Lie groups with isomorphic Lie algebras (but only $SU(2)$ is simply connected).

[1] Recall the centre of a group is $Z(G) = \{g \in G : hg = gh \text{ for all } h \in G\} = \{g \in G : hgh^{-1} = g \text{ for all } h \in G\}$.

[2] meaning continuous loops can always be continuously deformed to a point.

All connected Lie groups having a given Lie algebra are obtained from the corresponding simply-connected Lie group by quotienting by a central discrete sugroup. In the example, $SO(3) = SU(2)/\{\pm I\}$.

Cultural Remark 3. *Not all Lie groups are matrix groups. The Heisenberg group*

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

is a simply-connected matrix group (as a manifold, it's just $\mathbb{R}^3$), but it turns out that the quotient

$$H / \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

does not admit a faithful representation into any $\mathfrak{gl}(m, \mathbb{R})$.

Question 3. Find all the connected Lie subgroups of $SO(3)$.
Hint. Use the results from Q.3 of Question sheet 2.

Question 4. Given any real or complex matrix X, show that

$$\det e^X = e^{\mathrm{Tr}}(X).$$

(Hint. Recall from linear algebra, that over $\mathbb{C}$ any matrix is conjugate to an upper triangular matrix.)

Deduce that

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in \mathrm{Mat}_{n \times n}(\mathbb{R}) : \mathrm{Tr}(X) = 0\}.$$

Deduce that

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a basis of $\mathfrak{sl}(2, \mathbb{R})$ and check that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Why is the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ not isomorphic to $\mathfrak{so}(3)$?

Which connected Lie subgroup of $SL(2, \mathbb{R})$ corresponds to the Lie subalgebra $\mathbb{R} \cdot (f - e)$?

Which connected Lie subgroup of $SL(2, \mathbb{R})$ corresponds to the Lie subalgebra $\mathrm{span}(h, e)$?

*A Lie group is called **simple** if it is connected, non-abelian, and has no non-trivial connected normal subgroups. A Lie algebra V is called **simple** if it is non-abelian, and its only ideals are 0 and V.*

Prove in general the correspondence:

$$\{ \text{ connected normal subgroups of G } \} \overset{1:1}{\longleftrightarrow} \{ \text{ideals of } \mathfrak{g} \}$$

Deduce that a connected Lie group is simple if and only if its Lie algebra is simple.

By considering $\mathfrak{sl}(2, \mathbb{R})$, show that $SL(2, \mathbb{R})$ is a simple Lie group.

Question 5. Let V_n be the vector space of homogeneous[3] polynomials of degree n in two variables z_1, z_2. Show that $SU(2)$ acts on V_n by

$$(A \cdot p)(z) = p(zA),$$

where $p \in V_n$, $A \in SU(2)$, and zA is matrix multiplication of the row-vector $z = (z_1, z_2)$ with A. Deduce that the V_n are representations[4] of the Lie group $SU(2)$ of dimension $n + 1$.

Cultural Remark. *In fact, these are all the irreducible[5] representations of $SU(2)$. Here V_0 is the trivial representation, V_1 is the standard representation, and V_n is called the n-th symmetric power of V_1.*

By considering the double cover $SU(2) \to SO(3)$, and using the cultural remark, show that the irreducible representations of $SO(3)$ are precisely the spaces V_{2n} of odd dimension $2n + 1$.

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[3]meaning: the total degree of each term is the same, for example $3z_1^2 + 4z_1 z_2 - 5z_2^2$ is homogeneous of degree 2.

[4]Recall a representation R of a group G is a vector space R together with a Lie group homomorphism $\varphi : G \to \mathrm{Aut}(R)$.

[5]Irreducible means that the only vector subspaces $R' \subset R$ satisfying $g \cdot R' \subset R'$, for all $g \in G$, are $R' = 0$ and $R' = R$ (recall we abbreviate $g \cdot r' = \varphi(g)(r')$).