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Homework 6. - Do collaborate...

All Lie groups are assumed compact, and we work over $\mathbb{F} = \mathbb{C}$.
We'll prove some harder theorems on this sheet. Use the footnotes to guide you through the argument.

Question 1. Irreducibility criterion: prove that V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.
If V_1, V_2 are irreps of G_1, G_2, show $V_1 \otimes V_2$ is an irrep of $G_1 \times G_2$.
Claim:[1] Conversely, all irreps of $G_1 \times G_2$ have the form $V_1 \otimes V_2$, for irreps V_1, V_2 of G_1, G_2 respectively.

Question 2. Representation theory for SU(2) *Recall (Q.5 Sheet 4) $SU(2)$ acts by $(A \cdot p)(z) = p(zA)$ on $p \in V_n = \{$homogeneous degree n polys in z_1, z_2 over $\mathbb{C}\}$. We'll use the basis $P_j = z_1^j z_2^{n-j}$, $0 \leq j \leq n$.*
Claim 1.[2] The V_n are irreducible.
Claim 2.[3] The characters χ_n of the V_n are uniformly dense in $\mathrm{Cl}(SU(2))$.
Claim 3.[4] The V_n are the only irreps of $SU(2)$ (up to equivalence).

Question 3. Claim.[5] **Every compact Lie group admits a faithful rep into some $\mathbf{U(n)}$.**
Remark. $U(n) \to SO(2n)$ embeds via $A \mapsto \left(\begin{smallmatrix} \mathrm{Re}\, A & -\mathrm{Im}\, A \\ \mathrm{Im}\, A & \mathrm{Re}\, A \end{smallmatrix}\right)$, so we can replace $U(n)$ by $O(n)$ above.

Question 4. Claim 1.[6] The span over $\mathbb{C}$ of the image of $\chi : R(G) \to \mathrm{Cl}(G)$ is dense, that is: class functions f can be uniformly approximated by $\sum z_i \chi_{V_i}$ for $z_i \in \mathbb{C}$.
Claim 2.[7] The matrix entries of a faithful representation $\rho : G \to U(n)$, together with the conjugates of the entries, and with 1, generate the $\mathbb{C}$-algebra $\mathcal{F}(G)$ of all representative functions.
Claim 3.[8] Every irrep of G is a subrep of $V^{\otimes a} \otimes \overline{V}^{\otimes b}$, some $a, b \in \mathbb{N}$, where $V = \mathbb{C}^n$ is the faithful rep $\rho : G \to U(n)$. *Remark. This implies that $L^2(G)$ has countable dimension (see Lecture 13).*
Claim 4.[9] For a closed (so compact Lie) subgp $H \subset G$, any irrep of H is contained inside an irrep of G.

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[1] Let V be a rep of $G_1 \times G_2$. Then V is a rep of $G_2 = 1 \times G_2 \subset G_1 \times G_2$. Apply the canonical decomposition (Q.5 Sheet 5) to V, G_2. Define a G_1-action on $\mathrm{Hom}_{G_2}(V_2, V)$ for an irrep V_2 of G_2 so that the decomposition becomes $G_1 \times G_2$-linear. Apply complete reducibility to the G_1-mod $\mathrm{Hom}_{G_2}(V_2, V)$.

[2] By the irreducibility criterion of Q.1, V_n is irrep iff $\mathrm{Hom}_{SU(2)}(V_n, V_n) = 1$. So given $\varphi : V_n \to V_n$ $SU(2)$-linear map, need show $\varphi = c \cdot \mathrm{Id}$. Consider the diagonal matrices $D_\lambda \in SU(2)$ with entries λ, λ^{-1}. Compute the action of D_λ on P_j. Deduce that for $\lambda = e^{2\pi i/4n}$ the λ^{2j-n}-eigenspace of D_λ is spanned by P_j. Deduce that $\varphi(P_j) = c_j P_j$, some $c_j \in \mathbb{C}$. Consider the rotation $R_\theta \in SU(2)$ by θ. Expand $\varphi(R_\theta P_n) = R_\theta \varphi(P_n)$ to deduce that the c_j are all equal.

[3] Recall unitary matrices are diagonalizable. Deduce that any element in $SU(2)$ is conjugate to D_λ with $\lambda = e^{i\theta}$, uniquely up to changing θ to $-\theta$. Deduce that class functions $f : SU(2) \to \mathbb{C}$ are in $1:1$ correspondence with cts 2π-periodic even functions $\mathbb{R} \to \mathbb{C}$ via $\theta \mapsto f \circ D_{e^{i\theta}}$. So can abbreviate $\chi_n(D_{e^{i\theta}}) = \chi_n(\theta)$. Check $\chi_n(\theta) = \sum e^{i(2j-n)\theta}$. Compute that geometric sum, you should get $\chi_n(\theta) = \frac{\sin((n+1)\theta)}{\sin \theta}$, call this $c_n(\theta)$. Using trig identities, deduce $c_n = \cos(n\theta) + c_{n-1}(\theta)\cos\theta$. Deduce that the $\chi_n(\theta)$ generate (as an algebra) $1, \cos(\theta), \cos(2\theta), \ldots, \cos(n\theta)$. By basic Fourier analysis, <u>even</u> 2π-periodic continuous functions are uniformly approximated by $\cos(n\theta)$, $n \in \mathbb{N}$.

[4] Hint. Orthogonality relations and Claim 2.

[5] For a chain of strict inclusions $K_1 \supset K_2 \supset K_3 \supset \cdots$ of closed sub-manifolds, check that dim drops or the # of connected components drops each time. If K_j are closed subgps of G show the chain must stop. For a manifold M, and distinct points $p, q \in M$, explain why there is a cts function $f : M \to \mathbb{C}$ with $f(p) \neq f(q)$. Repeatedly apply this idea to $M = G$, using the Peter-Weyl theorem to approximate such f by representative functions $L \circ \rho$. The first step is to take $p = 1, q = g \neq 1$, to get $\rho_1 : G \to \mathrm{Aut}(V_1)$, with $K_1 = \ker \rho_1 \subset G$ a strict inclusion. You aim to end up with a faithful rep $V_1 \oplus \cdots \oplus V_m$.

[6] Use the canonical decomposition $\mathbf{ev} : \oplus H_i \otimes V_i \simeq V, \mathbf{ev}(\psi_i, v_i) = \psi_i(v_i)$ (Q.5 Sheet 5) where $H_i = \mathrm{Hom}_G(V_i, V)$, V_i the irreps. By Peter-Weyl $f \approx \mathrm{Tr}(\varphi \circ \rho)$, some $\varphi \in \mathrm{Hom}_G(V, V)$, $\rho : G \to \mathrm{Aut}(V)$. Check that $\varphi \circ \rho$ on V corresponds via the canonical decomposition to $\oplus(\varphi \otimes \rho_i)$ on $\oplus H_i \otimes V_i$. So the traces of those two maps agree. Final hint: $z_i = \mathrm{Tr}(\varphi \circ : H_i \to H_i)$.

[7] Let $\mathcal{M}(G)$ be the algebra generated. By Stone-Weierstrass show $\mathcal{M}(G) \subset C(G)$ is dense. Deduce $\mathcal{M}(G)$ is dense in $\mathcal{F}(G)$. Aim: $\mathcal{M}(G) \subset \mathcal{F}(G)$ closed in sup-norm. Now $\|f\|^2 = \langle f, f \rangle = \int_G \overline{f}(g) f(g) \leq (\sup_G |f|)^2$, so sup-closure$(\mathcal{M}(G))$ is a subset of $\|\cdot\|$-closure$(\mathcal{M}(G))$. Deduce: if $\mathcal{M}(G)$ is $\|\cdot\|$-closed then both closures equal $\mathcal{M}(G)$. Recall $\mathcal{F}(G) = \oplus \mathcal{F}_{V_i}(G)$ is an orthogonal direct sum over irreps V_i of G. Orthogonal projection $\pi_i : \mathcal{F}(G) \to \mathcal{F}_{V_i}(G)$ satisfies $\|\pi_i(f - m)\| \leq \|f - m\|$ for all $m \in \mathcal{M}(G)$. Deduce that for $f \in \overline{\mathcal{M}(G)}$ (the $\|\cdot\|$-closure), $\pi_i(f) \in \overline{\pi_i(\mathcal{M}(G)} \subset \mathcal{F}_{V_i}(G)$. Deduce, since $\dim \mathcal{F}_{V_i}(G) < \infty$, that $f \in \mathcal{M}(G)$.

[8] The matrix entries of $V^{\otimes a} \otimes \overline{V}^{\otimes b}$ are monomials of degree a in matrix entries of V and of degree b in matrix entries of $\overline{V}$. By Claim 2, they generate $\mathcal{F}(G)$, as a, b vary. If W were an irrep contradicting Claim 3, then by orthogonality $\int_G \overline{f_W} f_V = 0$ for all $f_W \in \mathcal{F}_W(G)$, $f_V \in \mathcal{F}_{V^{\otimes a} \otimes \overline{V}^{\otimes b}}(G)$. By Peter-Weyl and Claim 2 this is impossible.

[9] Apply Claim 3 to the faithful rep $H \to G \to \mathrm{Aut}(V)$.

OPTIONAL QUESTIONS (hand in if you like)

Optional Question 1. A Lie group that is not a matrix group. Consider

$$H = \{\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) : a,b,c \in \mathbb{R}\} \qquad N = \{\left(\begin{smallmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}.$$

Check that $N \subset H$ is a closed normal subgroup, so H/N is a Lie group.

This is of interest in quantum mechanics, because H/N is faithfully represented by the group of operators $M_c S_b T_a$ (in that order!) acting on the Hilbert space $L^2(\mathbb{R})$, generated by translation $T_a f(x) = f(x+a)$, rescaling $S_b f(x) = e^{2\pi i b} f(x)$, and multiplication $M_c f(x) = e^{2\pi i c x} f(x)$. Check H/N is iso to this group.

Check that if you replace $n \in \mathbb{Z}$ by $n \in \mathbb{R}$ you obtain a circle $T \subset \mathrm{Centre}(H/N)$, and check[10] that each element in T is a commutator $ghg^{-1}h^{-1}$.

Prove in general that given a circle $T \subset \mathrm{Centre}(G)$, any rep of G is a sum $V = \oplus V_a$, such that T acts on V by $e^{2\pi i a x}$ where $a \in \mathbb{Z}$.

Prove in general that if elements of T can be written as commutators, then[11] in fact only V_0 is non-zero and therefore T is in the kernel of the representation.

Deduce that the Heisenberg group H/N **is not a matrix group**.

Optional Question 2. Real representations vs complex representations.

We saw in Q.4 sheet 5 that a real rep is also a complex rep via $G \to GL(n,\mathbb{R}) \subset GL(n,\mathbb{C})$ so $\rho(g) = \overline{\rho(g)}$. More abstractly, this is the process of **complexifying** a real rep W: we get $W_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} W$ with $\mathbb{C}$-action $\lambda(z \otimes w) = (\lambda z) \otimes w$ and G-action $g(z \otimes w) = z \otimes gw$.

Deduce that real reps arise as complex reps which are **self-conjugate**:[12] $\overline{V} \simeq V$.

Check that V is self-conjugate iff χ_V is real-valued.

In the reverse direction, a complex rep V gives a real rep $V_{\mathbb{R}}$: just consider V as a vector space over $\mathbb{R}$. Less abstractly: $\rho : G \to GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$ where[13] $GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$, $A \mapsto \left(\begin{smallmatrix} \mathrm{Re}\,A & -\mathrm{Im}\,A \\ \mathrm{Im}\,A & \mathrm{Re}\,A \end{smallmatrix}\right)$.

Prove that

$$(W_{\mathbb{C}})_{\mathbb{R}} \simeq W \oplus W \qquad\qquad (V_{\mathbb{R}})_{\mathbb{C}} \simeq V \oplus \overline{V}.$$

Let $R_{\mathbb{R}}(G) = \{\sum n_i W_i : n_i \in \mathbb{Z}, W_i \text{ real irreps of } G\}$ denote the real representation ring. Deduce that

$$R_{\mathbb{R}}(G) \to R(G), \quad W \mapsto \mathbb{C} \otimes_{\mathbb{R}} W$$

is an injective homomorphism. Hence, if W, W' are real reps with $\mathbb{C} \otimes_{\mathbb{R}} W \simeq \mathbb{C} \otimes_{\mathbb{R}} W'$ as cx reps, then $W \simeq W'$ as real reps. Therefore, once you've found out which self-conjugate cx reps are real reps, you just need to use this criterion to determine which are equivalent.

A good example to play with, which is an alternative approach to Q.4 Sheet 5, is to find the real irreps of S^1 given that we easily know the complex irreps (and similarly for real irreps of T^n).

[10] *Hint.* $T_n M_1 T_n^{-1} M_1^{-1}$.

[11] *Hint.* consider determinants.

[12] Self-conjugate reps may fail to be real – they may be quaternionic, or a tensor of a real and a quaternionic rep.

[13] Even more explicitly, a complex basis $e_j^{\mathbb{C}}$ gives a real basis $e_1^{\mathbb{C}}, \ldots, e_n^{\mathbb{C}}, ie_1^{\mathbb{C}}, \ldots, ie_n^{\mathbb{C}}$. The first column of the inclusion is because $Ae_j^{\mathbb{C}} = (\mathrm{Re}\,A + i\mathrm{Im}\,A)e_j^{\mathbb{C}} = \mathrm{Re}\,(A)\,e_j^{\mathbb{C}} + \mathrm{Im}\,(A)\,ie_j^{\mathbb{C}}$. A simple example is:

$$e^{i\theta} \in GL(1,\mathbb{C}) \mapsto \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right) \in GL(2,\mathbb{R}).$$