

## LECTURE 1

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Let  $G$  be a group with operations

$$\begin{aligned} \mu: G \times G &\rightarrow G & \mu(g, h) &= gh \\ i: G &\rightarrow G & i(g) &= g^{-1} \end{aligned}$$

Def  $G$  is a Lie group if it is also a (smooth) manifold such that  $\mu, i$  are smooth maps.

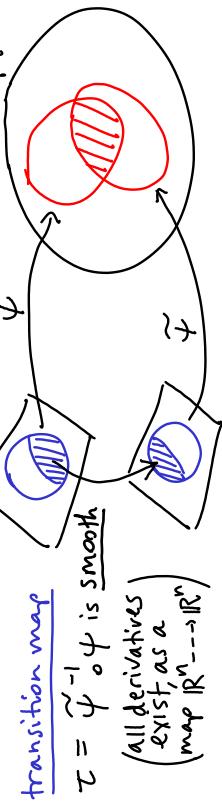
## CRASH COURSE ON MANIFOLDS

A manifold  $M = M^n$  of dimension  $n$  is a topological space which is locally parametrized by  $\mathbb{R}^n$ ,

A parametrization is a



abbreviate this by  
 $\psi: \mathbb{R}^n \dashrightarrow M$   
[not standard notation - only this course]  
such that on overlaps the parametrizations differ by smooth maps.



transition map  
 $\tau = \psi^{-1} \circ \psi$  is smooth  
(all derivatives exist as a map  $\mathbb{R}^n \dashrightarrow \mathbb{R}^n$ )

locally:  $\varphi(x) = \varphi(\underbrace{x_1, \dots, x_n}_{\text{(so really mean)}}) = (y_1(x), \dots, y_n(x))$

$(\psi_2^{-1} \circ \psi \circ \varphi_i)$   $x_i = \text{local coords near } p$   $y_i = \text{local coords near } \varphi(p)$

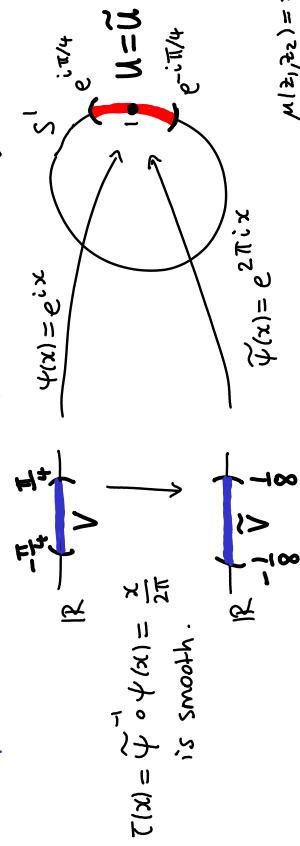
So:  $\varphi$  is smooth  $\Leftrightarrow$  the  $y_i(x)$  are smooth functions of  $x$ .

- $\varphi^{-1}: U \rightarrow V$  is called a chart

Def A parametrization  $\varphi: \mathbb{R}^n \dashrightarrow M$  defines local coordinates  $x_1, x_2, \dots, x_n$  on  $M$

namely:  $p \in U$  has coords  $\varphi^{-1}(p) = (x_1, \dots, x_n) \in \mathbb{R}^n$ .

Example  $M = S^1 = \text{circle} = \{z \in \mathbb{C} : |z| = 1\}$

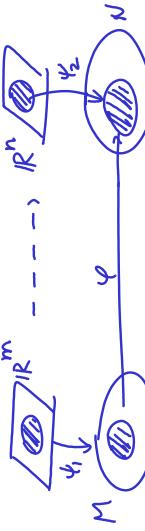


$\tau(x) = \tilde{\varphi} \circ \varphi(x) = \frac{x}{2\pi}$   
is smooth.

$\tilde{\varphi}(x) = e^{2\pi i x}$   
 $i(x) = e^{2\pi i x}$   
 $\mu(z_1, z_2) = z_1 \cdot z_2$   
 $i(z) = 1/z$

## SMOOTH MAPS

$S^1 = U(1) = \{1 \times \text{unitary matrices}\}$  is a Lie group: in the parametrizations of form  $x \mapsto e^{ix}$ :  $\mu$  becomes  $(x_1, x_2) \mapsto x_1 + x_2$ . ■



Def A continuous map  $\varphi: M \dashrightarrow N$  of manifolds is smooth if locally in some (and hence all) parametrizations the map  $R^n \dashrightarrow \varphi \dashrightarrow R^n$  is smooth.

## VECTORS

Def A tangent vector at  $p \in M$  is an equivalence class  $[\gamma]$  of smooth curves  $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ ,  $\gamma(0)=p$  (some  $\varepsilon > 0$ )

equivalent  $\gamma \sim \tilde{\gamma} \Leftrightarrow$  in some (hence all) parametrizations around  $p$ ,

$$\gamma'(0) = \tilde{\gamma}'(0) \in \mathbb{R}^n.$$

[locally]  $\gamma(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n \Rightarrow$  derivative  $\gamma'(0) = (x'_1(0), \dots, x'_n(0)) \in \mathbb{R}^n$

Example



$\gamma: \mathbb{R}^n \dashrightarrow M$ ,  $\gamma(0) = p$ , determines  $n$  obvious vectors at  $p$

$$[\gamma_1(t) = \psi(t, 0, \dots, 0)] \text{ called } \frac{\partial}{\partial x_1}$$

$$[\gamma_2(t) = \psi(0, t, 0, \dots, 0)] \text{ " } \frac{\partial}{\partial x_2}$$

$$\dots$$

$$[\gamma_n(t) = \psi(0, \dots, 0, t)] \text{ " } \frac{\partial}{\partial x_n}$$

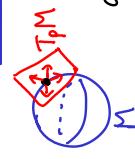
Rmk: Locally in  $\mathbb{R}^n$  those curves correspond to the standard basis vectors of  $\mathbb{R}^n$ :  $\gamma_1'(0) = \frac{\partial}{\partial t}|_0 (t, 0, \dots, 0) = (1, 0, \dots, 0) \in \mathbb{R}^n$

$$\gamma_j'(0) = \frac{\partial}{\partial t}|_0 (0, \dots, 0, t, 0, \dots, 0) = (0, \dots, 0, 1, 0, \dots, 0)$$

- don't really need  $\gamma(0) = p$ : if  $\psi(x) = p$  just translate:  $[\gamma_i(t) = \psi(x + (t, 0, \dots, 0))]$
- can add/scale vectors: if  $\psi(0) = p$ , then just add/scale the curves in  $\mathbb{R}^n$

$$\text{EXAMPLE: } 2 \frac{\partial}{\partial x_1} + 4 \frac{\partial}{\partial x_3} = 2 [\gamma_1] + 4 [\gamma_3] = [\gamma(1)] = \psi(2t, 0, 4t, 0, \dots, 0)$$

The tangent space  $T_p M$  at  $p \in M$  is the vector space of vectors at  $p$



$$T_p M \stackrel{\cong}{\longrightarrow} \mathbb{R}^n$$

$$a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n} = [\psi(x + (a_1 t, \dots, a_n t))] \mapsto \gamma'(0) = (a_1, \dots, a_n)$$

$$\gamma(t) = \text{curve in } \mathbb{R}^n, \gamma(0) = x$$

Rmk: The isomorphism depends on the choice of  $\psi$  with  $\psi(x) = p$ . We will see later that changing  $\psi$  to another parametrization  $\tilde{\psi}$  corresponds to multiplying  $\tilde{\gamma}'(0) \in \mathbb{R}^n$  by the derivative  $D\psi$  of the transition map  $\tau = \tilde{\psi}^{-1} \circ \psi$ .

## VECTORS

### Vectors act on functions

$$\boxed{\text{def } f = \left[ \frac{\partial}{\partial t}|_0 f(\gamma(t)) \right] \in \mathbb{R}}$$

For  $\gamma = [\text{curve } \gamma(t)] \in T_p M$ ,  $f: M \rightarrow \mathbb{R}$  defined near  $p$  locally it is just the obvious differentiation:

$$\gamma = a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n}$$

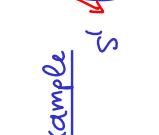
where  $\gamma(t) = (a_1 t, \dots, a_n t)$  is the local expression in parametrization  $\psi$  with  $\psi(p) = 0$

## VECTOR FIELDS

$$\boxed{\text{Def A vector field is a map } X: M \xrightarrow{p} T_p M \quad X|_p \in T_p M}$$

such that locally  $X|_x = a_1(x) \frac{\partial}{\partial x_1} + \dots + a_n(x) \frac{\partial}{\partial x_n}$  involves smooth functions  $a_i(x) \in \mathbb{R}$ .

$$\boxed{\text{here } \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}|_x = [\psi(x + (0, \dots, 0, t, \dots, 0))] \text{ is a vector at } x \text{ (position i) so varies with } x.]}$$

Example S' 

vector field  $X = \frac{\partial}{\partial \theta} \quad (\theta = \text{angle } \in [0, 2\pi])$  in a parametrization of type  $\psi(x) = e^{ix}$  have  $X|_x = \frac{\partial}{\partial x} = [\text{curve } t \mapsto e^{i(x+t)}]$

Vector fields act on functions:  $(X \cdot f)|_p = X|_p \cdot f$  so  $X \cdot f$  is a function

Locally it's  $X \cdot f = a_1(x) \frac{\partial f}{\partial x_1} + \dots + a_n(x) \frac{\partial f}{\partial x_n}$  gives a new function of  $x$

Rmk:  $X$  is determined locally by differentiating the coordinate functions  $x_i$ :

$$X \cdot x_i = a_i(x) \Rightarrow$$

A vector field is uniquely determined by how it acts on functions!

Rmk: When defining a concept locally, you always need to check that the choice of parametrization does not matter up to transition maps. So for  $f: M \rightarrow \mathbb{R}$  we want  $X \cdot f: M \rightarrow \mathbb{R}$  to be independent of the choices of  $\psi$  that locally define  $X \cdot f$ . (see Appendix if you care)

## DERIVATIVE MAP

Def The derivative (or differential) of  $\varphi: M \rightarrow N$  is  $D\varphi: TM \rightarrow TN$  (in particular  $D_p\varphi: T_p M \rightarrow T_{\varphi(p)} N$ )

$$D_p\varphi \cdot [\gamma] = [\varphi \circ \gamma]$$

CLAIM the derivative of  $\varphi$  is a linear map which is locally the matrix of partial derivatives of  $\varphi$ .

Proof Locally  $\varphi(x_1, \dots, x_m) = (y_1(x), \dots, y_n(x))$  as a map  $\mathbb{R}^m \dashrightarrow \mathbb{R}^n$  (we abusively write  $\varphi$  although it really is  $\varphi^{-1} \circ \varphi \circ \psi$ , for parametrizations  $\psi_1$  on  $M$  near  $p$  and  $\psi_2$  on  $N$  near  $\varphi(p)$ ). By definition,  $D\varphi \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = D\varphi \cdot [\dot{\gamma}_1(t) = 1, 0, \dots, 0] = [\varphi \circ \gamma_1(t)]$

Locally (that is using the above isomorphism  $T_p M \cong \mathbb{R}^m$ ,  $[\varphi \circ \gamma] \mapsto \gamma'(0)$ ) we just need to differentiate the curve in  $t$  at time  $t=0$ :

$$\frac{d}{dt}|_0 (\varphi(t, 0, \dots, 0)) = \left( \frac{\partial y_1}{\partial x_1}, \dots, \frac{\partial y_n}{\partial x_1} \right) \in \mathbb{R}^n$$

(which is now written in the basis  $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$  of  $T_{\varphi(p)} N \cong \mathbb{R}^n$ )

so explicitly:  $D_p\varphi \cdot \frac{\partial}{\partial x_i} = \left( \frac{\partial y_1}{\partial x_i}, \dots, \frac{\partial y_n}{\partial x_i} \right)$  and in general:

$$D_p\varphi \cdot \frac{\partial}{\partial x_j} = \left( \frac{\partial y_1}{\partial x_j}, \dots, \frac{\partial y_n}{\partial x_j} \right) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$$

(recall the columns of a matrix are the images of the standard basis) ■

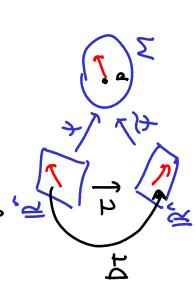
Claim  $M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\gamma} M_3 \Rightarrow D(\varphi \circ \gamma) = D\varphi \circ D\gamma$  (locally it is just the usual chain rule)

Proof  $D(\varphi \circ \gamma) \cdot [\gamma] = [\varphi \circ \gamma \circ \gamma] = D\varphi \cdot [\varphi \circ \gamma] = D\varphi \circ D\gamma \cdot [\gamma]$  ■

EXAMPLE The local expression of  $\varphi = \text{identity}: M \rightarrow M$  if we use param.  $\psi$  near  $p$  on domain, and param.  $\tilde{\psi}$  near  $\varphi(p) = p$  on the image, is the transition map  $\tau: \mathbb{R}^n \dashrightarrow \mathbb{R}^n$ ,  $\tilde{\tau} = \tilde{\psi}^{-1} \circ \psi$ .

Therefore  $D\varphi = \text{identity}$  is locally  $D\tau$ .

Hence if  $X^\psi, X^\tilde{\psi} \in \mathbb{R}^n$  are local expressions of the same vector  $X$  at  $p \in M$  in parametrizations  $\psi, \tilde{\psi}$  (see Appendix for more on this if you care) then  $\tilde{X}^\psi = D\tau \cdot X^\psi$

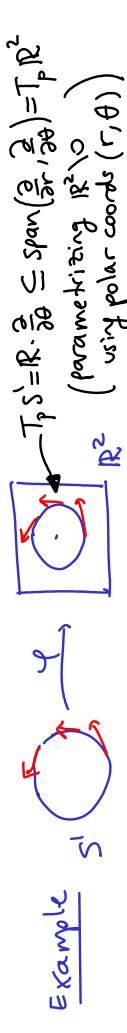


case  $\psi(x) = e^{ix}$

EXAMPLE  $M = S^1$  vector field " $\frac{\partial}{\partial \theta}$ " is  $X^\psi = \frac{\partial}{\partial x}$  in param.  $\psi(x) = e^{ix}$ . However mathematicians often want to view  $S^1$  as the quotient  $\mathbb{R}/\mathbb{Z}$  so they want to use  $\tilde{\psi}(x) = e^{2\pi ix}$  (so that  $\tilde{\psi}(\mathbb{Z}) = 1$ ). The local expression of " $\frac{\partial}{\partial \theta}$ " becomes:  $X^{\tilde{\psi}} = D\tau \cdot X^\psi = \frac{1}{2\pi} \cdot \frac{\partial}{\partial x}$  (since  $\tau(x) = \frac{x}{2\pi}$ ,  $D\tau = \frac{1}{2\pi}$ )

Def  $\varphi: M \rightarrow N$  is called an embedding if  $\varphi: M \rightarrow T_{\varphi(p)} N$  is a homeomorphism and  $D\varphi: T_p M \rightarrow T_{\varphi(p)} N$  is injective & open.

Rmks • Think of  $\varphi(M) \subseteq N$  as an identical copy of  $M$  inside  $N$  • Due to injective  $\Rightarrow D_p\varphi|_{T_p M}$  is a copy of  $T_p M$  in  $T_{\varphi(p)} N$  as a vector subspace



Example General linear group

$GL(n, \mathbb{R}) = \text{Lie group of invertible } n \times n \text{ real matrices} = \{A : n \times n \text{ matrix with } \det A \neq 0\}$  obvious parametrization near  $I$ :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad \leftarrow (a_{11}, a_{12}, \dots, a_{21}, \dots, a_{nn}) \in \mathbb{R}^n$$

For any matrix  $B = (b_{ij})$  have curve  $\gamma(t) = A + tB = (a_{ij} + tb_{ij})$  (still invertible for small  $|t|$ ) (so  $A + tB \in GL(n, \mathbb{R})$ )

$\Rightarrow$  Vector  $\frac{d}{dt}|_0 \gamma(t) = B \in \text{Mat}_{n \times n}(\mathbb{R}) = \{n \times n \text{ real matrices}\}$   $\Rightarrow$   $T_I GL(n, \mathbb{R}) = \text{Mat}_{n \times n}(\mathbb{R})$

## EXAMPLE orthogonal group

$O(n, \mathbb{R}) = \text{Lie group of orthogonal matrices} = \{A \in \text{Mat}_{n \times n} : A^T A = I\}$   
Not easy to write down a parametrization near  $I$

Note: the  $\Psi$  for  $GL(n, \mathbb{R})$  cannot work: ① The dimension is wrong  
and ② if you "wiggle" the  $a_{ij}$  then  $A$  may no longer be orthogonal!

(①)  $O(2, \mathbb{R}) = \{\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\} \sqcup \{\text{reflections } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}\}$   
has dimension 1 obviously, since we give parametrizations in  $GL(2, \mathbb{R})$ .  
Whereas  $GL(2, \mathbb{R})$  has dimension  $2^2 = 4 = \# \text{ entries}$ .

Trick Consider the embedding  $\varphi: O(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ ,  $A \mapsto A$   
and find  $T_I O(n, \mathbb{R})$  as vector subspace of  $T_I GL = \text{Mat}_{n \times n}$

For a curve  $Alt(t) \in O(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$  through  $A(0) = I$ ,

$$0 = \frac{d}{dt}|_0 Alt(t) = \underbrace{A'(0)^T}_{B^T} \underbrace{A(0)}_{\mathbb{I}} + \underbrace{A(0)^T}_{\mathbb{I}} \underbrace{A'(0)}_{B} = B^T + B$$

$\Rightarrow T_I O(n, \mathbb{R}) \subseteq$  vector subspace of skew symmetric matrices  
 $\{B \in \text{Mat}_{n \times n}(\mathbb{R}) : B^T + B = 0\}$

in fact, equality. One way to prove it is to check that  
 $\dim O(n, \mathbb{R}) = \dim(\{\text{skew } B\})$  since you are comparing vector spaces.

Question sheet: why is  $O(n, \mathbb{R})$  a manifold? Need:

### Implicit function theorem

Assume:  $\varphi: M \rightarrow N$  smooth,  $D_p \varphi: T_p M \rightarrow T_q N$  surjective  
for all  $p \in \varphi^{-1}(q) = \{p \in M : \varphi(p) = q\}$ .

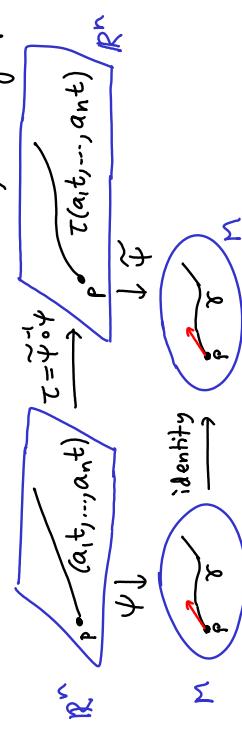
Then  $\bullet \varphi^{-1}(q) \subseteq M$  is a submanifold (inclusion into  $M$  is an embedding)  
 $\bullet \dim \varphi^{-1}(q) = m - n \leftarrow$  idea:  $q = g$  imposes  $n = \dim N$  conditions (independent equations)

$\bullet T_p(\varphi^{-1}(q)) = \text{Ker } D_p \varphi \leftarrow$  idea: if  $q = g$  is constant then  $\varphi(\text{curve}) = \text{constant} \Rightarrow \varphi(\text{curve}) = 0$  vector.

EXAMPLE:  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $\varphi(x, y) = x^2 + y^2$ . Take  $q = 1$  then  $\varphi^{-1}(1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$   
= circle  $S^1$ .  $D\varphi = \text{matrix } (2x \ 2y) : \mathbb{R}^2 \rightarrow \mathbb{R}$  is non-zero if  $x^2 + y^2 = 1$  hence  $D\varphi$  surjective.  
Implicit fn them  $\Rightarrow S^1$  is not a manifold!  $T_{(x,y)}S^1 = \text{Ker } ((2x \ 2y) \cdot \mathbb{R}^2 \rightarrow \mathbb{R})$   
as a vector subspace of  $\mathbb{R}^2 \equiv T_{(x,y)}\mathbb{R}^2$ . For example for  $(x_1, y_1) = (1, 0)$  get  $T_{(1,0)}S^1 = \text{span}((0, 1))$ .

## Appendix (NON-EXAMINABLE, also NOT IMPORTANT)

Question: how do vectors transform locally if change parametrization?



$$T_p M \cong \mathbb{R}^n \longrightarrow \mathbb{R}^n \cong T_p M$$

$$\sum a_i \frac{\partial}{\partial x_i} \cong (a_i)_{i=1, \dots, n} = \frac{\partial}{\partial t} |_{t=0} (a_i \circ t) \longrightarrow \frac{\partial}{\partial t} |_{t=0} (a_i \circ t) = \left( \sum_j \frac{\partial t_i}{\partial x_j} \cdot a_j \right) \cong \sum_{j,i} \frac{\partial t_i}{\partial x_j} \cdot a_j$$

$t \in \mathbb{R}^n$  has coordinates  $t_i(x) \in \mathbb{R}$

$\Rightarrow$  vectors transform by left-multiplication by the derivative DT of the transition map  $\tau: \mathbb{R}^n \dashrightarrow \mathbb{R}^n$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \begin{pmatrix} \frac{\partial \tau_1}{\partial x_1} & \frac{\partial \tau_1}{\partial x_2} & \cdots & \frac{\partial \tau_1}{\partial x_n} \\ \frac{\partial \tau_2}{\partial x_1} & \frac{\partial \tau_2}{\partial x_2} & \cdots & \frac{\partial \tau_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tau_n}{\partial x_1} & \frac{\partial \tau_n}{\partial x_2} & \cdots & \frac{\partial \tau_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Question: is  $X \cdot f = \sum a_i(x) \frac{\partial f}{\partial x_i} \in \mathbb{R}$  well-defined? (independent of choice of  $\psi$ )

Need more precise notation:  
For  $\psi: X^\psi = \sum a_i(x) \frac{\partial \psi}{\partial x_i}$ ,  $f^\psi(x) = f(\psi(x))$ ,  $X^\psi \cdot f^\psi = \sum a_i(x) \frac{\partial f^\psi}{\partial x_i}$

For  $\tilde{\tau}: X^{\tilde{\tau}} = DT \cdot X^\tau$ ,  $f^{\tilde{\tau}}(y) = f(\tilde{\tau}(y)) = f^\psi(\tau^{-1}(y))$

$X^{\tilde{\tau}} \cdot f^{\tilde{\tau}} = (DT \cdot X^\tau) \cdot (f^\psi \circ \tau^{-1}) = \underset{\substack{\text{CHAIN} \\ \text{RULE}}}{DT \cdot X^\tau} \cdot f^\psi \cdot D\tau^{-1} = X^{\tilde{\tau}} \cdot f^\psi$

$\Rightarrow X^{\tilde{\tau}} \cdot f^{\tilde{\tau}} = X^{\tilde{\tau}} \cdot f^\psi$  agree at points of  $M$ . ■  
check by writing it out with indices.

## LECTURE 2

### Examples of Lie groups

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- 0) Finite groups (and discrete groups) : not so interesting since 0-dimensional manifolds (just a set of points)
- 1)  $S^1 = \{1\}$
- 2)  $GL(n, \mathbb{R})$
- 3)  $O(n, \mathbb{R}) = \{\text{isometries of } \mathbb{R}^n \text{ which fix origin}\}$  include translations  $x \mapsto x + c$ .
- 4)  $\text{Isom}(\mathbb{R}^n) = \{\text{all Euclidean isometries of } \mathbb{R}^n\}$  Lie group
- 5)  $\mathbb{R}^n \xrightarrow{\text{Lie group}} \mu(x, y) = x + y$  and  $i(x) = -x$ .
- 6)  $G_1, G_2$  Lie groups  $\Rightarrow G_1 \times G_2$  Lie group
- 7)  $T^n = S^1 \times \dots \times S^1$  n-dimensional torus

### The skew-field of quaternions

(non-commutative multiplication)

Def Quaternions  $\mathbb{H}$  : 3 equivalent definitions :

matrices	$\mathbb{C}$ - v.s. dim=2	$\mathbb{R}$ - v.s. dim=4
$\{h = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}\}$	$\{h = a + bi : a, b \in \mathbb{C}\}$	$\{h = x_1 + x_2 i + x_3 j + x_4 k : x_i \in \mathbb{R}\}$
$\mathbb{C} = \{a + 0j\} \subseteq \mathbb{H}$	$\mathbb{C} = \{a + 0j\} \subseteq \mathbb{H}$	$\mathbb{C} = \{x_1 + x_2 i\} \subseteq \mathbb{H}$
how j multiplies	quaternion relations	$i^2 = j^2 = k^2 = -1$
$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$j^2 = -1$	$i \cdot j = -1$
$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$j \cdot z = \bar{z} \cdot j \text{ for } z \in \mathbb{C}$	$j \cdot k = -k \cdot j$
$j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		(not commutative : $ij = -ji = k$ )
$k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$		
$h^* := \begin{pmatrix} \bar{a} & -b \\ b & a \end{pmatrix}$ conjugate transpose	$h^* = \bar{a} - b j$	$h^* = x_1 - x_2 i - x_3 j - x_4 k$
$ h ^2 = \det =  a ^2 +  b ^2$	$ h ^2 = h^* h =  a ^2 +  b ^2$	$ h ^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$

scalar product  $\langle h_1, h_2 \rangle = h_1^* h_2 \in \mathbb{H}$  induces the norm  $|h| = \sqrt{\langle h, h \rangle}$   
Rmk similar to  $\mathbb{C}$ : e.g.  $h \neq 0 \Rightarrow h^{-1} = h^*/|h|^2$   
but careful about non-commutativity:  $(h_1 h_2)^* = h_2^* h_1^*$  what you expect matrices.

### More Examples of Lie groups

- 8)  $GL(1, \mathbb{H}) = \mathbb{H} \setminus \{0\}$  a "circle" in the world of quaternions
- 9)  $Sp(1) = \{h \in \mathbb{H} : |h|=1\} \subseteq \mathbb{H} \setminus \{0\}$  quaternion group
- 10) Generalize 5, 2, 3:

$\mathbb{R}^n$	$GL(n, \mathbb{R})$	$\text{Aut}(V) = \{F \text{-linear } V \rightarrow V\}$ vector space over $F = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H}$
$O(n, \mathbb{R})$	$G = \{A \in \text{Aut } V : \langle Av, Aw \rangle = \langle v, w \rangle \forall v, w \in V\}$	depending on a choice of scalar product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$
		(# = conjugate transpose.)
		over $\mathbb{R}$ just transpose.)
	11) In $\oplus$ : $V = \mathbb{F}^n$ with $\langle v, w \rangle = v^* w$	$v^* A^* Aw = v^* w$ all $v, w \in \mathbb{F}^n$
		Lie group $G \approx \{A \in GL(n, \mathbb{F}) : v^* A^* Aw = v^* w\}$
$\mathbb{F}$	$R$ orthogonal group $O(n) = O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A^T A = I\}$	$A^T A = I$
	$C$ unitary group $U(n) = \{A \in GL(n, \mathbb{C}) : A^* A = I\}$	$A^* A = I$
	$H$ symplectic group $Sp(n) = \{A \in GL(n, \mathbb{H}) : A^* A = I\}$	$A^* A = I$
		NON-EXAMINABLE fact: $Sp(n) \cong \{A \in U(2n) : A^T JA = J\}$
		• implicit function theorem $\Rightarrow$ these are Lie groups,
		and $T_I G = \{B \in \text{Mat}_{n \times n}(\mathbb{F}) : B^* + B = 0\}$
		• fact: $ Av  =  v  \quad \forall v \Leftrightarrow \langle A v, A w \rangle = \langle v, w \rangle \forall v, w \Leftrightarrow A^* A = I$
		For $O(n)$ geometrically this says linear maps which preserve lengths must preserve angles since $\langle v, w \rangle =  v  \cdot  w  \cdot \cos(\text{angle between } v, w)$
	12) special linear group $SL(n, \mathbb{F}) = \{A \in GL(n, \mathbb{F}) : \det A = 1\}$ volume preserving automorphisms	similarly: $SO(n) \subseteq O(n)$ impose $\det = 1$
		(can't do it for $Sp(n)$ since $\det$ doesn't make sense since $\mathbb{H}$ not commutative)
	Q. sheet 2: $\det = 1$ imposes $\text{trace}(B) = 0$ for tangent vectors $B \in T_G \subseteq \text{Mat}_{nn}$	Example $O(2) = \begin{matrix} SO(2) \\ \text{rotations} \end{matrix} \cup \begin{matrix} \text{reflections} \\ (\text{connected component of } I) \end{matrix}$
		$O(2) \cong SO(2) \times \{\pm 1\}$ via $A \mapsto ((\det A), \det A)$

13) Def  $G_0 = \text{connected component of } 1 \text{ in } G$  lie group

- Lemma
- ①  $G_0$  is a Lie group
  - ② the cosets of  $G_0$  in  $G$  are the connected components of  $G$  and they give an open cover  $G = \bigsqcup_{gG_0} gG_0$
  - ③  $G_0$  is a normal subgroup of  $G$
- 
- ④ The quotient group  $G/G_0$  is a Lie group (discrete group with 0-manifold)

EXAMPLE  $G = O(n) \Rightarrow G_0 = SO(n)$  and  $G/G_0 \cong \{\pm 1\}$  (group with 2 elements)

Rmks • For subgroups  $H \leq G$  the space of cosets  $G/H$  may not be group.

$H \leq G$  called normal subgroup if  $hGH^{-1} \subseteq H \quad \forall h \in G$

This condition ensures that  $G/H$  is a group using  $\bar{g}_1H \cdot \bar{g}_2H = \bar{g}_1g_2H$

In general  $G \cong G_0 \times \frac{G}{G_0}$  is false! (see Appendix if you care)

Proof of Lemma

$\phi_g : G \rightarrow G, h \mapsto gh$  is a homeomorphism (inverse is  $\phi_{g^{-1}}$ )

(continuous – indeed smooth – since group multiplication is smooth)

$\Rightarrow \phi_g$  sends connected components to connected components  
 $\Rightarrow gG_0$  are connected components

Recall: any topological space = disjoint union of its connected components

Hence:  $g \in G_0 \Rightarrow g \in G_0 \cap gG_0 \Rightarrow G_0 = gG_0$  (since  $G_0 \cap gG_0 \neq \emptyset$ )  
 $\Rightarrow g^{-1}G_0 = G_0$

Therefore can restrict multiplication and inversion to  $G_0$ , proving ①

If  $C$  is a connected component and  $g \in C$ , then  $g \in C \cap gG_0 \neq \emptyset$ , so  $C = gG_0$   
 $\Rightarrow$  ② follows (using general fact: connected components of a manifold are always open sets)

For ③ use homeomorphism  $G \rightarrow G, h \mapsto ghg^{-1}$  (inverse  $h \mapsto g^{-1}hg$ )  
 $G_0$  connected component  $\Rightarrow gG_0g^{-1}$  connected component  
but  $1 \in G_0 \cap gG_0g^{-1} \neq \emptyset$  so  $G_0 = gG_0g^{-1}$ .

④ follows (not much content in ④ since silly manifold:  $\dim = 0$ ) ■

## TOPOLOGICAL PROPERTIES

### COMPACTNESS

recall compact means open covers always have finite subcovers.

- useful trick: 1) first embed  $G \subseteq \mathbb{R}^m$  (some large  $m$ )  
2) then use Heine-Borel theorem for  $\mathbb{R}^m$ :

compact  $\Leftrightarrow$  closed & bounded

that is: • check limits stay in  $G$

•  $\mathbb{R}^m$ -norm on  $G \subseteq \mathbb{R}^m$  is bounded

EXAMPLE  $S^1 \subseteq \mathbb{R}^2$  • if  $z_n \rightarrow z$  with  $|z_n|=1$  then  $|z|=1$  so  $z \in S^1$   
•  $|z| \leq 1$  on  $S^1$  (since  $|z|=1$   $\forall z \in S^1$ )

### CONNECTEDNESS

facts • manifolds are metric spaces (since can always embed  $M \subseteq \mathbb{R}^{high-d}$ )

- for manifolds a subset is a connected component  $\Leftrightarrow$  open and closed
- for manifolds: connected  $\Leftrightarrow$  path-connected

EXAMPLE

$GL(1, \mathbb{R}) = \mathbb{R} \setminus \{0\} = \mathbb{R}^+ \sqcup \mathbb{R}^-$  not connected

$GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$  connected since path-connected.

→ DON'T NEED TO MEMORIZE THIS!

Some Topological facts (some are tricky to prove)

$G$	connected ( $G = G_0$ )	# connected components if $\neq$ 1	compact
$\mathbb{R}^n$	✓		✓ (n>1)
$\mathbb{T}^n$			✓
$GL(n, \mathbb{R})$	X		X (n>2)
$SL(n, \mathbb{R})$	✓		
$O(n)$	X		
$SO(n)$	✓		
$GL(n, \mathbb{C})$			
$SL(n, \mathbb{C})$	✓		
$U(n)$			X (n>2)
$SU(n)$			
$GL(n, \mathbb{H})$			
$Sp(n)$			X (n>2)

## WHY ARE LIE GROUPS SUCH SPECIAL MANIFOLDS?

They have natural diffeomorphisms:  $\phi_g : G \rightarrow G$ ,  $h \mapsto gh$  (smooth with smooth inverse)

$$\text{NOTE: } \phi_g^{-1} = \phi_{g^{-1}}$$

Amazing consequences:

① Once you pick a parametrization near 1, say  $\psi: U \xrightarrow{\text{smooth}} V$  with  $\psi(0) = 1$ , get a parametrization near any  $g \in G$ :  $\phi_g \circ \psi: U \rightarrow g \cdot V$  with  $\phi_g \circ \psi(0) = g$

EXAMPLE  $\phi_g \circ \psi(x) = e^{i(c+x)}$  

② In these parametrizations,  $\phi_g$  is locally the identity map!

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\text{Id=Identity}} & \mathbb{R}^n \\ \psi & \xrightarrow{\phi_g} & \psi \circ \phi_g \end{array}$$

Which vector fields  $X$  on  $G$  have exactly the same local expression near any  $g$  in parametrizations  $\phi_g \circ \psi$ ?

$$\left. \begin{array}{l} \sum a_i \frac{\partial}{\partial x_i} : D\text{Id} = D\text{Id}, \quad \sum a_i(x) \frac{\partial}{\partial x_i} : D(\phi_g \circ \psi) = D\phi_g \circ D\psi \\ \text{hence need } X|_{g \cdot \psi(U)} = D\phi_g \cdot X|_{\psi(U)} \end{array} \right\} \text{chain rule}$$

$$X|_{\psi(U)} \xrightarrow{D\phi_g} X|_{g \cdot \psi(U)} \xrightarrow{\text{red}} \begin{array}{c} X \\ \nearrow g \\ g \end{array} \quad \text{Def A vector field } X \text{ on } G \text{ is left-invariant if}$$

$$D\phi_g \cdot X|_h = X|_{gh} \quad (\forall g, h \in G)$$

Def Lie  $G$  = {left-invariant vector fields on  $G$ }

Rmk this is a vector space, adding/scaling vector fields preserves left-invariance because  $D\phi_g$  is linear.

Theorem There is a natural isomorphism of vector spaces:

$$\begin{array}{ccc} \text{Lie } G & \xrightarrow{\quad} & T_1 G \\ X & \longmapsto & X|_1 \\ \left( X|_g = D\phi_g \cdot X \right) & \longmapsto & \begin{array}{c} X \\ \nearrow g \\ g \end{array} \end{array}$$

In particular  $\dim \text{Lie } G = \dim T_1 G = (\dim G \text{ as a manifold})$ .

## THE LIE BRACKET

For any vector fields  $X, Y$  on a manifold  $M$  there is a bracket operation  $[X, Y] =$  a new vector field, defined locally by

$$\left[ \sum a_i(x) \frac{\partial}{\partial x_i}, \sum b_i(x) \frac{\partial}{\partial x_i} \right] = \sum_j \left( a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

$$\text{EXAMPLE in } \mathbb{R}^2: \quad [x^2 y^3 \frac{\partial}{\partial x}, x^4 y^5 \frac{\partial}{\partial y}] = x^2 y^3 \frac{\partial}{\partial x} (x^4 y^5) \frac{\partial}{\partial y} - x^4 y^5 \frac{\partial}{\partial y} (x^2 y^3) \frac{\partial}{\partial x} = -3x^6 y^7 \frac{\partial}{\partial x} + 4x^5 y^8 \frac{\partial}{\partial y}$$

Recall: vector fields act on functions, and this locally determines the vector field since  $X \cdot x_i = a_i(x)$  if  $X = \sum a_i(x) \frac{\partial}{\partial x_i}$  locally. For brackets:

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f)$$

DETA:  $[X, Y]$  measures how badly  $X, Y$  fail to commute as differential operators e.g.  $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$  says partial derivatives commute on smooth functions:  $\frac{\partial^2}{\partial x_i \partial x_j} (f) = \frac{\partial^2}{\partial x_j \partial x_i} (f)$

### PROPERTIES

- i)  $[\cdot, \cdot]$  is bilinear:  $\mathbb{R}$ -linear in each entry.
- ii) antisymmetric:  $[X, Y] = -[Y, X]$ , so  $[X, X] = 0$
- iii) Jacobi's identity:  $[X, [Y, Z]] + [[Y, Z], X] + [[Z, X], Y] = 0$

exercise

## LIE ALGEBRAS

Def A Lie algebra is a vector space  $V$  together with a bilinear antisymmetric map  $[\cdot, \cdot]: V \times V \rightarrow V$  satisfying the Jacobi identity.

EXAMPLES •  $V = \text{Mat}_{nn}(\mathbb{R})$ ,  $[B, C] = BC - CB$  (matrix multiplication)

- $V = \mathbb{R}^3$ ,  $[\nu, \omega] = \nu \times \omega$  cross-product

• Abelian Lie algebras: any vector space  $V$  with  $[., .] \equiv 0$

Theorem Lie  $G$  is a Lie algebra of dimension  $\dim G$

Pf Need show:  $X, Y$  left-inv  $\Rightarrow [X, Y]$  left-inv.  
In parametrizations  $\phi_g$  of  $G$  the v.f.  $X$  has the same local expression near any  $g \in G$ , so does  $Y$ , hence so does  $[X, Y]$ , so  $[X, Y]$  is left-inv.

## EXAMPLE $\mathfrak{gl}(n, \mathbb{R}) = \text{Lie } GL(n, \mathbb{R})$

$$X = \sum_{i,j} X_{ij} \frac{\partial}{\partial x_{ij}} \quad \longleftrightarrow \quad B = X|_1 = \left( X_{ij}|_1 \right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \quad x \in GL(n, \mathbb{R})$$

left-inv:

$$X|_x = D_x \phi \cdot X|_1 = x \cdot B$$

(For linear maps  $\varphi$ , "D $\varphi$ " since D $\varphi$ .v =  $\lim_{t \rightarrow 0} \frac{\varphi(x+tv) - \varphi(x)}{t} = \varphi(v)$ ).

$$\begin{aligned} [X, Y] &= \sum_{i,j,k} \left( X_{ij} \frac{\partial Y_{jk}}{\partial x_{ij}} - Y_{ij} \frac{\partial X_{jk}}{\partial x_{ij}} \right) \overset{\exists}{\longleftrightarrow} \sum_{i,j,k} (B_{ij} C_{jk} - C_{ij} B_{jk}) \frac{\partial^2}{\partial x_{ij} \partial x_{jk}} \\ X|_x = x \cdot B \quad Y|_x = x \cdot C \quad (\text{so } Y_{\ell k} = \sum_j x_{ej} C_{jk} \text{ and } \frac{\partial Y_{\ell k}}{\partial x_{ij}} = C_{jk} \text{ for } \ell = i \text{ and zero otherwise}) \end{aligned}$$

Corollary  $\mathfrak{gl}(n, \mathbb{R}) \cong \text{Lie algebra Mat}_{n \times n}(\mathbb{R})$  with bracket  $[B, C] = BC - CB$

ISO of Lie algebras  $\longleftrightarrow$  ISO of v.s. (precise definition:  $\varphi: V \rightarrow W$  iso of v.s.  $\varphi([v_1, v_2]) = [\underbrace{\varphi(v_1), \varphi(v_2)}_{\text{in } V}]$  in  $W$ )

ISO of vector spaces preserving bracket (ISO of  $\varphi$  such that  $\varphi([v_1, v_2]) = [\varphi(v_1), \varphi(v_2)]$  all  $v_1, v_2 \in V$ )

Same calculation shows:

$O(n, \mathbb{R}) = \text{Lie } O(n, \mathbb{R}) \cong \{ B \in \text{Mat}_{n \times n}(\mathbb{R}) : B \text{ skew-symmetric} \}$

(ISO of Lie algebras using usual bracket for  $\text{Mat}_{n \times n}$ )

APPENDIX (Non-examinable – can ignore it)

Remarks about  $G_0$

• Not true in general that  $G \cong G_0 \times G/G_0$  as groups (not even  $G_0 \cong G/G_0$ ) because there is no reason homomorphisms  $G \rightarrow G_0$  and  $G/G_0 \rightarrow G$  should exist. It is true that  $G \cong G_0 \times G/G_0$  as manifolds since you just pick some representatives  $g$ : then  $G_0 \times \{g(G_0)\} \rightarrow G$  but can't make choices consistently with the group structure. Later in course we prove  $G \cong G_0 \times G/G_0$  works for abelian  $G$ .

• There may be bigger subgroups than  $G_0$ : for  $G = S^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ :  $G_0 = S^1 \times \{1\} \times \{1\} \subseteq G$ . But of course  $G_0$  is the largest connected subgroup of  $G$ .

About  $[X, Y]$  Why is  $[X, Y]$  a well-defined global vector field?

Answer: If for any function  $f$  defined near  $p \in M$  you define a new function  $Z \cdot f$  defined near  $p \in M$ , and you show the Leibniz rule holds  $Z \cdot (f_1 f_2) = (Z \cdot f_1) f_2 + f_1 (Z \cdot f_2)$  then  $Z$  is a vector field on  $M$  (try proving this using the Appendix of lecture 1). We defined  $[X, Y] \cdot f$  and we defined  $[X, Y]$  in local coordinates: the two definitions agree when compute  $[X_i, Y_j] \cdot f$  locally, and the local definition clearly satisfies Leibniz. ■

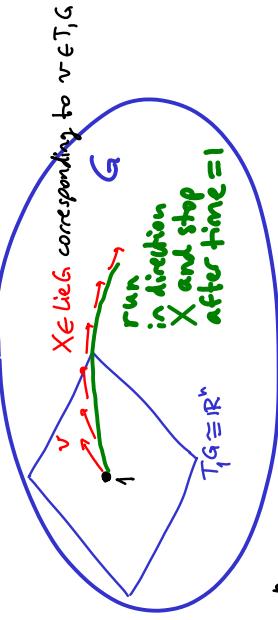
## LECTURE 3

LAST TIME: Parametrization  $\varphi$  near  $1 \in G \rightsquigarrow$  parametrizations  $\varphi_t$  near any  $t$

AIM:

Find the best parametrization  $\varphi$  near 1

IDEA:



$$\psi: \mathbb{R}^n \cong T_1 G \cong \text{Lie } G \longrightarrow G$$

choice of basis  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$

### FLOWS

Def for a vector field  $X$  on a manifold  $M$ , a flowline of  $X$  through  $p$  is a curve  $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$  ( $\varepsilon > 0$ )

$$\gamma(0) = p \quad \gamma'(t) = X|_{\gamma(t)}$$

If let  $p$  vary: flow of  $X$  near  $p$  is

$$F: (-\varepsilon, \varepsilon) \times U \xrightarrow{\text{small open set around } p} M \quad F(0, q) = q \quad \frac{\partial F}{\partial t} \Big|_{(t, q)} = X|_{F(t, q)}$$

$\Rightarrow F(\cdot, q)$  is flowline through  $q$ .

Rmk:  $\gamma'(t)$  is an abbreviation for  $D_t \gamma \cdot \frac{\partial}{\partial t}$ . Locally it's really  $\gamma'(t)$ .

The equation is locally a 1st order diff. eqn. on  $\mathbb{R}^n$ :

$$x'(t) = f(x(t)) \quad x(0) = p \quad f: \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^n$$

ODE theory  $\Rightarrow$   $x$  exists for small  $\varepsilon > 0$

$x$  is unique given  $p$   
 $x$  depends smoothly on the initial condition  $p$

So for small  $\varepsilon$  and  $U$ ,  $\gamma$  and  $F$  exist, unique, smooth.

Example  $M = (-1, 1) \subseteq \mathbb{R}$ ,  $X = \frac{d}{dt}$  then  $F(q, t) = q + t$  only defined if  $q + t \in (-1, 1)$ .

Abbreviate  $F_t(g) = F(t, g)$

Lemma 1  $F_s(F_t(g)) = F_{s+t}(g)$  (where defined)

pf for  $s=0$ : LHS = RHS =  $F(g)$   
 $\frac{\partial}{\partial s}|_{(LHS)} = X|_{F_s(F_t(g))} = \frac{\partial}{\partial s}|_{(RHS)} = X|_{F_{s+t}(g)}$

Now suppose  $M = G$  is a Lie group  
 $X \in \text{Lie}_G$  left-invariant

$$\text{If } \frac{\partial}{\partial t}|_t (g \cdot X(t)) = \frac{\partial}{\partial t} (\phi_\theta(g) \cdot X(\theta(t))) = D\phi_\theta \cdot X(\theta(t)) = D\phi_\theta \cdot X|_{g \cdot X(t)}$$

pf  $\frac{\partial}{\partial t}|_t (g \cdot X(t)) = D\phi_\theta \cdot X(\theta(t)) = D\phi_\theta \cdot X|_{g \cdot X(t)}$  chain rule  
 start being sloppy:  
 do it the derivative wherever it is relevant here  $X'(t)$  is a vector at  $\theta(t)$  so we take  $D\phi_\theta$

Crit flowlines of  $X \in \text{Lie}_G$  are defined for all time, and flow defined everywhere  
 pf  $\gamma: (-\varepsilon, \varepsilon) \rightarrow G$   $\Rightarrow$  can extend on  $(0, \varepsilon + \frac{\varepsilon}{2})$  using  $\gamma(\frac{\varepsilon}{2}) \gamma'(0)^{-1} \gamma(t - \frac{\varepsilon}{2})$

Explanation:  $t \mapsto \gamma(t - \text{constant})$  is a flowline by chain rule  $\frac{\partial}{\partial t} \gamma(t - c) = X(\gamma(t - c))$  then apply lemma 1 to flowline  $\gamma(t - \frac{\varepsilon}{2})$  with  $g = \gamma(\frac{\varepsilon}{2}) \gamma'(0)^{-1}$ . So  $\delta(t) = \gamma(\frac{\varepsilon}{2}) \gamma(t - \frac{\varepsilon}{2})$  is a flowline for  $X$ . We conclude it so that  $\delta(\frac{\varepsilon}{2}) = \gamma(\frac{\varepsilon}{2}) \gamma(t - \frac{\varepsilon}{2}) \gamma'(0) = \gamma(\frac{\varepsilon}{2})$ . But then  $\delta, \gamma$  are both flowlines for  $X$  and equal at  $t = \frac{\varepsilon}{2}$  so they equal on overlap by uniqueness. So  $\delta$  extends  $\gamma$  and  $\varepsilon$ . Similarly can extend beyond  $-\varepsilon$ . Finally argue by contradiction: if  $(-\varepsilon, \varepsilon)$  was the largest interval where  $\delta$  is defined, then we just showed that cannot be true since we can extend.

Key trick: let  $\gamma: \mathbb{R} \rightarrow G$  be the (unique) flowline of  $X$  with  $\gamma(0) = 1$  then by lemma 1 the flow of  $X$  is:

$$F_t(g) = g \cdot \gamma(t)$$

Theorem Let  $\gamma$  be the flowline of  $X \in \text{Lie}_G$  with  $\gamma(0) = 1$ . Then

$$\gamma(s) \gamma(t) = \gamma(s+t) \quad \forall s, t \in \mathbb{R} \quad (\text{in particular: } \gamma(t)s = \gamma(t)\gamma(s))$$

So  $\gamma: \mathbb{R} \rightarrow G$  is a Lie group homomorphism. Conversely, all lie group homomorphisms arise in this way for some  $X \in \text{Lie}_G$ .

Pf. By lemma 1 + cor 1:  $F_s(F_t(1)) = F_{s+t}(1) = 1 \cdot \gamma(s+t)$

$$\Downarrow F_s(1 \cdot \gamma(t)) = 1 \cdot \gamma(t) \cdot \gamma(s)$$

Conversely, if  $\gamma$  is a homomorphism  $\gamma(0) = 1$ , and we claim  $F(t, g) = g \cdot \gamma(t)$  is a flow for  $\frac{\partial}{\partial t}|_0 F(t, g) = D\phi_g \cdot \gamma'(0) = X|_g$  left-inv v.s. determined by  $\gamma'(0) \in T_1 G$ .

pf:  $F_{s+t}(g) = g \cdot \gamma(s+t) = g \cdot \gamma(t+s) = g \cdot \gamma(t) \gamma(s) = F_s(g) \gamma(t) = F_s F_t(g)$

and in general  $F_0 = id$ ,  $F_s F_t = F_{s+t}$  unless you are the flow for the v.f.  $X|_p = \frac{\partial}{\partial t}|_0 F_t(p)$  since:  
 (for manifold M)  
 $\frac{\partial}{\partial t}|_0 F_t(p) = \frac{\partial}{\partial s}|_0 F_{t+s}(p) = \frac{\partial}{\partial s}|_0 F_s(F_t(p)) = X|_{F_t(p)}$  chainrule  
 $F: \mathbb{R} \times M \rightarrow M$   
 $F_t: \mathbb{R} \rightarrow M$   
 $F_t F_s = F_{s+t}$   
 Call  $F_t = F(t, \cdot)$

Def The lie group morphisms  $\mathbb{R} \rightarrow G$  are called 1-parameter subgroups of  $G$

Cor  $\begin{array}{ccc} \text{Lie } G & \xleftarrow{\text{v.s.}} & T_1 G \\ X & \longleftarrow & \downarrow \circ = X|_1 = \gamma'(0) \end{array} \xleftarrow{\text{bijection}} \{ \text{1-parameter subgroups of } G \}$

$\frac{\partial}{\partial s}|(LHS) = X|_{F_s(F_t(g))} = \frac{\partial}{\partial s}|(RHS) = F_s|_{F_t(g)}$   
 Now suppose  $M = G$  is a Lie group  
 $X \in \text{Lie}_G$  left-invariant

Lemma 2  $\gamma$  flowline  $\Rightarrow g \cdot \gamma$  flowline  
 $\text{If } \frac{\partial}{\partial t}|_t (g \cdot X(t)) = D\phi_g \cdot X(\theta(t)) = D\phi_g \cdot X|_{g \cdot X(t)}$

start being sloppy:  
 do it the derivative wherever it is relevant here  $X'(t)$  is a vector at  $\theta(t)$  so we take  $D\phi_\theta$

EXAMPLE 1

Torus  $G = \mathbb{R}^n / \mathbb{Z}^n$ : obvious 1-param. subgrps  $\gamma_v(t) = t \cdot v \text{ mod } \mathbb{Z}^n$   
 Check condition ①:

$$\gamma_v(s+t) = (s+t)v = sv + tv \quad \text{hence } \gamma_v: \mathbb{R} \rightarrow G \text{ is homomorphism.}$$

This classifies all 1-param. subgrps since  $\gamma'(0) = v \in \mathbb{R}^n \cong T_1 G$  is general.

$$\Rightarrow \exp(v) = v \text{ mod } \mathbb{Z}^n$$

Alternative approach: check condition ② for  $\gamma_v(t) = tv \text{ mod } \mathbb{Z}^n$

$$\gamma_v(0) = 0, \quad \gamma'_v(t) = v, \quad \gamma'_v(t) = v$$

For the last two equalities we used the parametrization  $\pi: \mathbb{R}^n \rightarrow G$   
 $\pi(tv) = v \text{ mod } \mathbb{Z}^n$   
 Recall from Question sheet 1 that the left-inv v.f. on  $T^*$  in the parameterit.  $\pi$  are the constant vectors  $v \in \mathbb{R}^n$ .

Case n=1  $G = \mathbb{R}/\mathbb{Z} = S^1$  get  $\exp(x) = e^{2\pi i x} \in S^1$

Remark

here we identify  $T_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}$  via  $[t \mapsto tv \text{ mod } \mathbb{Z}] \hookrightarrow \mathbb{R}$   
 so we must get  $\exp(\mathbb{Z}) = 1$ . This corresponds to parametrizing  
 the circle with " $x \in [0, 1]$ " rather than with " $\theta \in [0, 2\pi]$ ".  
 If you instead parametrize with  $\theta \mapsto e^{i\theta}$  then  $\exp(\theta) = e^{i\theta} \in S^1$ .

## EXAMPLE 2

$$G = \text{GL}(n, \mathbb{R})$$

$$\gamma_B(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} B^n$$

$$\exp(B) = \sum \frac{1}{n!} B^n$$

CLAIM makes sense (converges) & can differentiate term by term

$$\text{proof} \Rightarrow \gamma_B(t) = I + tB + \frac{1}{2!} t^2 B^2 + \frac{1}{3!} t^3 B^3 + \dots$$

$$Y'_B(t) = B + tB^2 + \frac{1}{2!} t^2 B^3 + \dots$$

$$\Rightarrow Y_B(0) = I, \quad Y'_B(0) = B, \quad Y'_B(t) = \gamma_B(t) \cdot B = X|_{Y_B(t)} \quad \text{v.f. for } B \cdot X|_x = x \cdot B$$

$$\text{compare } \frac{\partial}{\partial t} e = e \cdot x$$

$$\text{finite dimension of all norms are equivalent}$$

$$\|B\| = n \cdot \max_{i,j} |B_{ij}|$$

Nice properties:

- (\*) •  $\|BC\| \leq \|B\| \cdot \|C\|$
- complete, i.e. Cauchy sequences converge (since finite dim!)
- $\text{Mat}_{n,n}(\mathbb{R})$  is an algebra (v.s.  $V$  with bilinear  $V \times V \rightarrow \mathbb{R}$  defining multiplication)

$$\begin{aligned} \|BC\| &= n \cdot \max_{i,k} \left| \sum_j B_{ij} C_{jk} \right| \\ &\leq n \cdot \max_{i,k} \underbrace{\sum_j |B_{ij}| \cdot |C_{jk}|}_{\leq \|B\|} \leq \|B\| \|C\| \end{aligned}$$

$$\leq \mu \cdot \sqrt{\frac{\|B\|}{n}} \|C\|$$

(v.s.  $V$  with bilinear  $V \times V \rightarrow \mathbb{R}$ )

Def Complete normed algebras satisfying (\*) are called Banach algebras

For Banach algebras can reprove all the usual results about series, absolute convergence, radius of convergence, etc.

If  $A \in \text{Banach} \Rightarrow \exp(tx) = 1 + tx + \frac{t^2}{2!} x^2 + \dots$

$\Rightarrow$  can differentiate in  $t$  term by term

$\Rightarrow$  local diffeomorphism near 0

Cor  $\psi: \mathbb{R}^n \cong T_x G \cong \text{Lie } G \xrightarrow{\exp} G$  is a parametrization near  $x \in G$

def Inverse Function Theorem  $\psi: M \rightarrow N$  smooth map of manifolds  $\psi \in G$  s.t.  $\exp|_U$  is a local diffeomorphism

def basis  $(GL(n, \mathbb{R}))$  choices (Hence get nice parametrizations  $\phi_g \circ \psi \dots$ )

## LECTURE 4

$$\beta \in \text{Mat}_{n,n}(\mathbb{R})$$

$$= T_1 G(n, \mathbb{R})$$

LAST TIME:

$$\begin{array}{ccc} \text{Lie } G & \cong & T_1 G & \cong & \{1\text{-parameter subgroups}\} \\ & \cong_{vs.} & & \text{bijection} & \\ X & \longleftrightarrow & \gamma & \longleftrightarrow & (\gamma|_{x=1} = \gamma_x : \mathbb{R} \rightarrow G) \end{array}$$

recall:  $\begin{cases} \gamma_v(0) = 1 & \text{also recall that} \\ \gamma'_v(0) = v = X|_1 & \text{IF } \gamma \text{ smooth} \\ \gamma'_v(t) = X|_{\gamma(t)} & \cdot \gamma: \mathbb{R} \rightarrow G \text{ hom} \\ (\gamma(s+t)) = \gamma(s)\gamma(t) & (\gamma(s+t)) = \gamma(s)\gamma(t) \end{cases}$  THEN  $\gamma = \gamma_x$  where  $v = \gamma'(0)$ .

THE EXPONENTIAL MAP

$$\text{Def exp: } \text{Lie } G \cong T_1 G \rightarrow G, \quad \exp(v) = \gamma_v(1).$$

$$\text{Lemma: } \exp(sv) = \gamma_v(s)$$

$\frac{\partial}{\partial t} \gamma_{sv}(t) = sv, \quad \frac{\partial}{\partial t} \gamma_{sv}(st) = s \cdot \gamma'_v(s \cdot 0) = sv$ . So by uniqueness  $\gamma_{sv}(st) = \gamma_v(st)$  claim, taking  $t=1$

side remark: vector spaces  $V$  are manifolds: just pick a basis to get a (global) parametrization  $\mathbb{R}^n \cong V$ . Their tangent spaces are:

$T_x V \cong V, \quad \gamma(t) = v + tw \mapsto w = \gamma'(0)$  Hence  $\text{Lie } G$  is a manifold.

Theorem  $\exp: \text{Lie } G \rightarrow G$  is smooth

If  $\exp$  is the composite of 3 smooth maps:  $\text{Lie } G \rightarrow \mathbb{R} \times (G \times \text{Lie } G) \rightarrow G \times \text{Lie } G \rightarrow G$

$$\begin{array}{ccc} X & \longmapsto & (1, 1, X) \\ & \longmapsto & (t, 1, X) \\ & \longmapsto & (t, g, X) \end{array} \quad \begin{array}{c} \uparrow \\ (g \cdot \gamma_X(t), X) \\ \uparrow \\ (g, X) \end{array} \quad \begin{array}{c} \uparrow \\ g \\ \uparrow \\ \text{flow of } \gamma \text{ on } G \times \text{Lie } G \text{ where } \gamma|_{(g, X)} = (X|_g, 0) \end{array}$$

Lemma  $D_0 \exp = \text{Id}$

$$\text{If } D_0(\exp) \cdot w = \frac{\partial}{\partial s} \Big|_{s=0} \exp(0+sw) = \frac{\partial}{\partial s} \Big|_{s=0} \gamma_{sw}(1) = \frac{\partial}{\partial s} \Big|_{s=0} \gamma'_w(0) = w$$

Cor  $\exp: \text{Lie } G \rightarrow G$  is a local diffeomorphism near 0.

pf FACT Inverse Function Theorem  $\psi: M \rightarrow N$  smooth map of manifolds  $\psi \in G$  defined on a nbhd of  $0 \in \mathbb{R}^n$  is a diffeomorph.

pf open sets  $U \subseteq G$  s.t.  $\exp|_U$  is a local diffeo near  $m$

Cor  $\psi: \mathbb{R}^n \cong T_x G \cong \text{Lie } G \xrightarrow{\exp} G$  is a parametrization near  $x \in G$

## EXAMPLES

1)  $\exp: \mathbb{R} \rightarrow S^1$ ,  $\exp(x) = e^{2\pi i x}$  local diffeo near 0, but not global (not injective!).  
a local inverse near  $e^{2\pi i 0} = 1$  is  $\frac{1}{2\pi i} \log(y) \leftarrow$  pick a branch of complex  $\log$ .

2)  $\exp: \text{Mat}_{nn}(\mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R})$ ,  $\exp(B) = \sum_{n!}^{\infty} B^n$ , then a local inverse near  $I$  is

$$A \mapsto \log(A) = \log(I + (A - I)) \quad (\text{for } \|A - I\| < 1)$$

EXPLANATION (non-examitable):  
For a Banach algebra with 1 define

$$\log(1+tx) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n \quad \text{for } \|x\| < 1$$

Remark we know  $\exp(\log(1+tx)) = 1+tx$  for  $t,x \in \mathbb{R}$  is an absolutely convergent series in  $tx$  (where defined). Hence same must be true for Banach algebras with 1. The reason is: take  $x = 1 \in \text{Banach algebra}$  and  $t \in \mathbb{R}$ . Then the coefficients of those series in  $t, 1$  must agree with the coefficients of the series you got when working with  $\mathbb{R}$ ! (consider the coefficients individually letting  $t \rightarrow 0$  allows you to ignore higher order terms).

Def  $\varphi: G \rightarrow H$  Lie group homomorphism means  
1)  $\varphi$  group homomorphism  
2)  $\varphi$  smooth

Theorem (Naturality of exp)

If  $\varphi: G \rightarrow H$  Lie group hom then:

$$\begin{array}{ccc} T_1 G & \xrightarrow{D\varphi} & T_1 H \\ \exp \downarrow & \downarrow \text{exp} & \downarrow \\ G & \xrightarrow{\varphi} & H \end{array}$$

(i.e. composing  $\xrightarrow{\varphi}$   
equals composing  $\downarrow$ )

Note  $\varphi_{t,t}(t)$  is a 1-parameter group since  $\varphi$  is a group hom.  
Hence  $\exp$  is evaluation at  $t=1$ .  
 $\gamma_V(v) = v$

$$\gamma_V(1) \longrightarrow \varphi \circ \gamma_V(1) \longrightarrow \varphi(1)$$

$$\text{EXAMPLES } ① \mathbb{R} \cong TS^1 \longrightarrow \mathbb{C} = \text{Mat}_{1 \times 1}(\mathbb{C})$$

$$\mathbb{R}/\mathbb{Z} = S^1 \longrightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\} = \text{GL}(1, \mathbb{C})$$

$$x \bmod \mathbb{Z} \longrightarrow e^z = e^{2\pi i x}$$

(Note:  $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$  is  $\exp(z) = e^z$  since  $= \sum \frac{1}{n!} z^n$  as  $1 \times 1$  matrix)

- ②  $\text{SkewSym}_{nn}(\mathbb{R}) \rightarrow \text{Mat}_{nn}(\mathbb{R})$  implies that  $\exp(B) = \sum \frac{1}{n!} B^n$  also for  $O(n)$   
 $\downarrow$   
 $O(n)$  inclusion  $\rightarrow \text{GL}(n, \mathbb{R})$

## LECTURE 5 HOMOMORPHISMS

Def Lie group homomorphism  $\varphi: G \rightarrow H$  means 1)  $\varphi$  hom of grp's  
2)  $\varphi$  smooth map

- When  $H = \text{GL}(n, \mathbb{R})$ ,  $\varphi$  is called a representation of  $G$
- Lie group isomorphism if  $\varphi$  bijective and  $\varphi^{-1}$  Lie grp hom

Warning Let  $\mathbb{R}^{\text{disc}} = \mathbb{R}$  with discrete topology (each point is an open set) is Lie grp using + identity:  $\mathbb{R}^{\text{disc}} \rightarrow \mathbb{R}$  is a bijective Lie grp hom, inverse not continuous!  
(Exp/naturality tells you nothing since  $T_0 \mathbb{R}^{\text{disc}} = \{0\}$ )

Rmk Local diffeo + bijective = diffeo

Lemma 1  $\varphi: G \rightarrow H$  bijective Lie grp hom and  $\varphi^{-1}$  continuous near 1  $\Rightarrow \varphi$  Lie grp iso

pf By naturality of exp:  $(\text{nbhd of } 0 \in T_1 G) \xrightarrow{\text{Lie }} (\text{nbhd of } 0 \in T_1 H)$

May need to pick smaller nbhd of  $1 \in G$   $\xrightarrow{\text{exp}} U = (\text{nbhd of } 1 \in H)$

replace  $U$  by  $\varphi(V)$   
 $\varphi(V) = (\varphi^{-1})^{-1}(V)$  open since  $\varphi^{-1}$ cts near 1.  
 This fails in warning above!

Lemma 2  $D_1 \varphi$  iso  $\Rightarrow D_g \varphi$  iso for any  $g$   $\xrightarrow{\text{since } \varphi \text{ maps are diffeos and } \varphi}$

Alternative proof:  $D_1 \varphi(g) = D_g \varphi \circ D_g \varphi^{-1}: T_g G \rightarrow T_1 H \rightarrow T_{\varphi(g)} H$

The proof also shows:  
 $\xrightarrow{\text{Lemma 3}} \varphi$  locally homeo near 1  $\Rightarrow \varphi$  local diffeo near any  $g$ .

Exercise  $\varphi$  bijective and  $\dim G = \dim H \Rightarrow \varphi$  Lie grp iso

(Hint: use injectivity of  $\varphi: V \rightarrow U$  above to get  $D_g \varphi$  injective, then use  $\dim G = \dim H$ )

Harder exercise Manifolds are usually required to be second countable ( $\exists$  countable basis for the topology). If we require Lie grp to be 2nd countable, is it true that Bijective Lie grp hom  $\Rightarrow$  Lie grp iso?

(Idea: above if  $D_g(T_1 G) \neq T_1 H$  then it is a strictly lower dimensional vector subspace, so  $\varphi(V) \subseteq U$  is a strictly lower dimensional submanifold in  $H$ , so you need uncountably many left translates  $\varphi_h(V)$  to get a disjoint cover of  $U$  (by non-open sets). Then  $\varphi^{-1}(\varphi_h(V))$  give uncountably many disjoint opens covering  $V$  contradicting  $G$  is 2nd cble

Def Lie algebra homomorphism  $\psi: (V, [\cdot, \cdot]_V) \rightarrow (W, [\cdot, \cdot]_W)$  means

- 1)  $\psi$  linear map (homomorphism of vector spaces)
  - 2)  $[\psi x_1, \psi x_2]_W = \psi [x_1, x_2]_V$  all  $x_1, x_2 \in V$
- When  $W = \text{Mat}_{nn}(\mathbb{R})$ ,  $[\psi C]_W = BC - CB$ ,  $\psi$  is called a representation of  $V$
  - Lie algebra isomorphism if  $\psi$  also bijective (hence  $\psi$  iso of v.s.)

## MORE ABSTRACTLY FOR REPRESENTATIONS CAN REPLACE :

For Lie groups:	$GL(n, \mathbb{R})$	$\text{Aut}(R)$
For Lie algs:	$\text{Mat}_{nn}(\mathbb{R})$	$\text{End}(R) = \text{Hom}(R, R)$

Often call  $R$  the representation and write  $\underline{g} \cdot r$  instead of  $\underline{\varphi}(g)(r)$

## EXAMPLES

- $\gamma : \mathbb{R} \rightarrow G$  1-param. subgroups are Lie grp homs  
example:  $\mathbb{R} \rightarrow S^1$ ,  $x \mapsto e^{2\pi i x}$  (or  $x \bmod \mathbb{Z}$  if view  $S^1 = \mathbb{R}/\mathbb{Z}$ )
- $SU(2) \rightarrow SO(3)$  on Q.sheet 2

$$3) A_g : G \xrightarrow{\quad} G$$

$$A_g(h) = g h g^{-1}$$

Lie group isomorphism  
(the inverse is  $A_g^{-1} = A_{g^{-1}}$ )

$$4) \text{Ad} : G \longrightarrow \text{Aut}(T_1 G) \cong \text{Aut}(\text{Lie } G)$$

$$\text{Ad}(g) = D_1 A_g$$

$\hookrightarrow D, A_g$  is an automorph since has inverse  $D, A_g^{-1}$  (chain rule)

ADJOINT REPRESENTATION OF  $\text{LIE}(G)$

Why is  $D, \text{Ad}$  a Lie alg hom?  $\begin{pmatrix} \text{in lecture } G \text{ will prove that} \\ (\text{ad } X)(Y) = [X, Y] \end{pmatrix}$

Theorem For  $\varphi : G \rightarrow H$  Lie group hom

$$\varphi : \text{Lie } G \cong T_1 G \xrightarrow{D_1 \varphi} T_1 H \cong \text{Lie } H \text{ a Lie algebra hom}$$

FROM NOW ON ABBREVIATE  $\underline{G} = \text{LIE}(G)$

Consequences By naturality of  $\exp$ :

$$\text{Cor} \quad \begin{array}{ccc} G & \xrightarrow{\text{ad}} & \text{End}(g) \\ \exp \downarrow & \downarrow \exp & \downarrow \exp \\ G & \xrightarrow{\text{Ad}(g)} & G \end{array}$$

$$g \cdot \exp(X) \cdot g^{-1} = \exp(\text{Ad}(g) \cdot X)$$

$\sqrt{=}$

$\sqrt{\text{not obvious}}$

this exp we know  
 $\exp(B) = I + B + \frac{B^2}{2!} + \dots$   
1st box says:  $A^B A^{-1} = e^{AB^{-1}}$  (holds because  $(ABA^{-1})^n = ABA^{-1}$ )

Exercise What does 2nd box say, using result from lecture 6 that  $\text{ad } X = [X, \cdot]$

## PROOF OF THEOREM

$\underline{Z \in \text{Lie } G \Rightarrow \tilde{Z}|_g = D\varphi|_g \cdot Z|_g}$

where  $R$  is a vector space

$\underline{\varphi(g)(\tilde{g}) = g \tilde{g}}$

$$1) \quad \tilde{Z} = \varphi(Z), \text{ so: } \tilde{Z}|_h = D_1 \varphi|_h \cdot (D_1 \varphi \cdot Z)|_h = (D\varphi \cdot Z)|_h$$

$$\Rightarrow \tilde{Z}|_g = D_1 (\underbrace{\varphi(g) \circ \varphi}_\text{chain rule} \cdot \underline{Z})|_g = D\varphi \cdot \underline{D\varphi|_g \cdot Z|_g} = (D\varphi \cdot Z)|_g$$

because  $\varphi$  hom:  
 $\varphi(g) \varphi(g \bullet) = \varphi(g \bullet)$

since left-int

In general such vector fields  $\tilde{Z}|_g$

$$\Rightarrow \tilde{Z}|_{g(1)} = D\varphi \cdot Z$$

Rmk If  $\varphi$  was a diffeo, this would say that  $\tilde{Z}$  is the pushforward of  $Z$ . If  $\varphi$  not diffeo, then  $D\varphi \cdot Z$  need not be a vector field

Proposition If  $X, \tilde{X}$  and  $Y, \tilde{Y}$  are  $\varphi$ -related then  $[X, Y] = [\tilde{X}, \tilde{Y}]$  are  $\varphi$ -related

(Proof later) continue:  $[\tilde{X}, \tilde{Y}]|_{g(1)} = D\varphi \cdot [X, Y] = \underline{[X, Y]}|_{\varphi(1)}$

Proposition If  $X, \tilde{X}$  and  $Y, \tilde{Y}$  are  $\varphi$ -related then  $\varphi([X, Y]) = \varphi([\tilde{X}, \tilde{Y}])$

$\Rightarrow [\psi(X), \psi(Y)]|_{\varphi(1)} = (\varphi[X, Y])|_{\varphi(1)}$

$\Rightarrow [\psi(X), \psi(Y)]|_g = \psi[X, Y]$

$\Rightarrow [\psi(X), \psi(Y)]|_{\varphi(g)} = (\varphi[X, Y])|_{\varphi(g)}$

PROOF OF PROPOSITION Given:  $\varphi : M \xrightarrow{\text{smooth map}} N$

$\tilde{X}|_{g(1)} = D\varphi \cdot X$  some v.f.  $X$  on  $M$

$\tilde{Y}|_{g(1)} = D\varphi \cdot Y$  some v.f.  $Y$  on  $N$

Need TRICK:  $M \xrightarrow{\varphi} N$

PROOF OF TRICK Locally:

$\sqrt{= \sum \alpha_j \frac{\partial}{\partial x_j}}$  and  $\varphi = (\varphi_1, \dots, \varphi_n) : \mathbb{R}^m \dashrightarrow \mathbb{R}^n$

$\nabla \cdot (f \circ \varphi) = \sum \alpha_j \frac{\partial}{\partial x_j} (f \circ \varphi)$

$\nabla \cdot (f \circ \varphi) = (D\varphi \cdot X) \cdot f$  for vector field  $X$  on  $M$ :

$\nabla \cdot (f \circ \varphi) = (D\varphi \cdot X) \cdot f$  as functions  $M \rightarrow \mathbb{R}$

means:  $\underline{(D\varphi \cdot X) \cdot f}(m) = \sum \alpha_j \frac{\partial f}{\partial x_j} \cdot \frac{\partial \varphi_i}{\partial x_j}$

$= (D\varphi \cdot \nabla) \cdot f$

matrix for  $D\varphi$

PROOF OF THEOREM

$Z \in \text{Lie } G \Rightarrow \tilde{Z}|_g = D\varphi|_g \cdot Z|_g$

call  $\tilde{Z} = \varphi(Z)$ , so:

$$\tilde{Z}|_h = D_1 \varphi|_h \cdot (D_1 \varphi \cdot Z)|_h = D\varphi \cdot \underline{D\varphi|_g \cdot Z|_g} = (D\varphi \cdot Z)|_h$$

because  $\varphi$  hom:  
 $\varphi(g) \varphi(g \bullet) = \varphi(g \bullet)$

since left-int

In general such vector fields  $\tilde{Z}|_g$

$$\Rightarrow \tilde{Z}|_{g(1)} = D\varphi \cdot Z$$

Rmk If  $\varphi$  was a diffeo, this would say that  $\tilde{Z}$  is the pushforward of  $Z$ . If  $\varphi$  not diffeo, then  $D\varphi \cdot Z$  need not be a vector field

Proposition If  $X, \tilde{X}$  and  $Y, \tilde{Y}$  are  $\varphi$ -related then  $[X, Y] = [\tilde{X}, \tilde{Y}]$  are  $\varphi$ -related

$\Rightarrow [\tilde{X}, \tilde{Y}]|_{g(1)} = D\varphi \cdot [X, Y] = \underline{[X, Y]}|_{\varphi(1)}$

Proposition If  $X, \tilde{X}$  and  $Y, \tilde{Y}$  are  $\varphi$ -related then  $\varphi([X, Y]) = \varphi([\tilde{X}, \tilde{Y}])$

$\Rightarrow [\psi(X), \psi(Y)]|_{g(1)} = (\varphi[X, Y])|_{g(1)}$

$\Rightarrow [\psi(X), \psi(Y)]|_g = \psi[X, Y]$

$\Rightarrow [\psi(X), \psi(Y)]|_{\varphi(g)} = (\varphi[X, Y])|_{\varphi(g)}$

PROOF OF PROPOSITION Given:  $\varphi : M \rightarrow N$  and  $f : N \rightarrow \mathbb{R}$

$\tilde{X}|_{g(1)} = D\varphi \cdot X$  some v.f.  $X$  on  $M$

$\tilde{Y}|_{g(1)} = D\varphi \cdot Y$  some v.f.  $Y$  on  $N$

For  $\nabla \in T_m M$  and  $f : N \rightarrow \mathbb{R}$

$\nabla \cdot (f \circ \varphi) = (D\varphi \cdot \nabla) \cdot f \in \mathbb{R}$

For vector field  $X$  on  $M$ :  $X \cdot (f \circ \varphi) = (D\varphi \cdot X) \cdot f$

as functions  $M \rightarrow \mathbb{R}$

$$\begin{aligned} (\mathcal{D}\varphi \cdot [x, y]) \cdot f &= [x, y] \cdot (f \circ \varphi) \\ \text{Trick} &= X \cdot (Y \cdot (f \circ \varphi)) - Y \cdot (X \cdot (f \circ \varphi)) \\ &= X \cdot ((\mathcal{D}\varphi \cdot Y) \cdot f) - \text{switch } XY \\ \text{Trick} &= X \cdot ((\tilde{Y} \cdot f) \circ \varphi) - " \\ &= (\mathcal{D}\varphi \cdot X) \cdot (\tilde{Y} \cdot f) - " \\ \text{Trick} &= (\tilde{X} \cdot (\tilde{Y} \cdot f)) \Big|_{\varphi(1)} - " \\ &= [\tilde{X}, \tilde{Y}] \cdot f \quad \blacksquare \end{aligned}$$

## LECTURE 6

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### ADJOINT REPRESENTATION

$$\begin{aligned} \text{LAST TIME} \quad \mathcal{A}_g: G \rightarrow G &\quad \text{conjugation by } g \quad \mathcal{A}g(h) = ghg^{-1} \\ \text{Ad}: G \rightarrow \text{Aut}(g), \quad \text{Ad}(g) &= D_g \mathcal{A}_g \\ \text{ad} = D_1 \text{Ad}: g \rightarrow \text{End}(g) & \end{aligned}$$

$$\begin{aligned} &= \mathcal{D}\varphi \cdot [x, y] \Big|_{\varphi(1)} \cdot f \\ &\Rightarrow \varphi = \text{Lie}(G) \end{aligned}$$

$$\begin{aligned} \text{Theorem} \quad \text{ad}(X) \cdot Y &= [X, Y] \quad X, Y \in \mathfrak{g} \\ \text{pf.} \quad \text{ad}(X) \cdot Y &\stackrel{\mathfrak{g} \in \text{End}(g)}{=} (D_1(\text{Ad}) \cdot X) \cdot Y \end{aligned}$$

Example question sheet 2:  $\varphi: \mathfrak{su}(2) \rightarrow SO(3)$  double cover (in particular a  $\varphi$  (diffeo near result))  
 $\varphi = D_1\varphi: \mathfrak{su}(2) \xrightarrow{\sim} \mathfrak{so}(3)$  Lie algebra isomorphism!  
 $\{2 \times 2 \text{ complex}\} = \{3 \times 3 \text{ real skew-symmetric}\}$   
 $\{ \text{skew Hermitian}\}$  (connected component of  $\text{SO}(3)$ )

Lemma A neighbourhood  $V \subseteq G_0$  of 1 generates  $G_0$  as a group

pf: Can assume  $V$  is open (interior( $V$ ) is smaller than  $V$ ,  $1 \in \text{int}(V)$ )  
 $\Rightarrow \langle V \rangle = \text{subgroup generated by } V \text{ in } G_0$   
 $\Rightarrow \langle V \rangle \subseteq G_0$  open subset (since  $v_1^{\pm 1} \dots v_k^{\pm 1} \cdot V^{\pm 1} \subseteq G_0$ )  
 $\Rightarrow$  cosets  $g \cdot \langle V \rangle$  are open (since  $\mathcal{D}g$  different)  
 $\Rightarrow \langle V \rangle$  closed subset (since complement of open set  $\bigcup g \cdot \langle V \rangle$ )  
 $\Rightarrow \langle V \rangle$  connected component (since open & closed)  
 $\Rightarrow \langle V \rangle = G_0 \quad \blacksquare$

Theorem Let  $G$  be connected

A Lie grp from  $\varphi: G \rightarrow H$  is uniquely determined by  $D_1\varphi: T_G \rightarrow T_H$

(meaning: if  $\varphi, \tilde{\varphi}: G \rightarrow H$  Lie groups with  $D_1\varphi = D_1\tilde{\varphi}$  then  $\varphi = \tilde{\varphi}$ )

It's naturality of exp: (small nbhd  $0 \in T_G$ )  $\xrightarrow{\text{exp}} \text{DIFFEO}$   $\xrightarrow{\text{DIF}}$   $\text{H}$

$V = \text{(small nbhd of } 1 \in G\text{)} \xrightarrow{\varphi} H$

$\Rightarrow \varphi$  determined by  $D_1\varphi$  on  $V$

& hom  $\Rightarrow \varphi$  determined by  $D_1\varphi$  on  $\langle V \rangle = G_0 = G$   $\blacksquare$

Warning ("not everything is determined at the identity")

$\text{su}(2) \cong \mathfrak{so}(3)$  but  $SU(2) \not\cong SO(3)$  : different topologically.

$\sum_{i=1}^3 \mathbb{R}^3$  is simply connected (all loops are contractible)

$\mathbb{RP}^3$  is not simply connected.

FACT (non-examinable)  $\mathbb{RP}^3$  is simply connected (all loops are contractible)

$\mathbb{RP}^3$  is not simply connected.

$$\begin{aligned} \text{We will need 2 tricks from lecture 1:} \quad & \text{① } \mathcal{D}\varphi \cdot v = \frac{\partial}{\partial s} \Big|_{s=0} \varphi(s) \\ \text{(where } \varphi: M \rightarrow N, f: N \rightarrow \mathbb{R}, v = [\text{curve } \gamma(s)]) \quad & \text{② } v \cdot f = \frac{\partial}{\partial s} \Big|_{s=0} f(\gamma(s)) \end{aligned}$$

$$\begin{aligned} \text{pf.} \quad \text{ad}(X) \cdot Y &\stackrel{\mathfrak{g} \in \text{End}(g)}{=} (D_1(\text{Ad}) \cdot X) \cdot Y \\ &= \frac{\partial}{\partial s} \Big|_0 \text{Ad}(\gamma_X(s)) \cdot Y \quad \leftarrow \text{recall } \gamma'_X(s) = X \text{ for 1-param. subgr. } \gamma_X(s) \\ &= \frac{\partial}{\partial s} \Big|_0 D \mathcal{A}_{\gamma_X(s)} \cdot Y \\ &= \frac{\partial}{\partial s} \Big|_0 \frac{\partial}{\partial t} \Big|_0 \mathcal{A}_{\gamma_X(t)} (\gamma_Y(t)) \\ &= \frac{\partial^2}{\partial s \partial t} \Big|_{(0,0)} \gamma_X(s) \gamma_Y(t) \underbrace{\gamma_X(s)^{-1}}_{\leqslant \gamma_X(-s)} \leftarrow \text{Question sheet 2} \\ &= \frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} \gamma_X(s) \gamma_Y(t) \gamma_X(t)^{-1} - \frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} \gamma_X(s) \gamma_Y(t) \gamma_X(s) \\ &= \frac{\partial}{\partial s} \Big|_0 \frac{\partial}{\partial t} \Big|_0 F^Y(t, \gamma_X(s)) - \frac{\partial^2}{\partial t \partial s} \Big|_0 F^X(s, \gamma_Y(t)) \\ &\quad \text{partial derivs} \quad \text{partial derivs} \\ &\quad \text{commute} \quad \text{and recall} \\ &\quad \gamma_X(s) = 1 \quad \text{and} \\ &\quad \gamma_Y(t) = 1 \quad \text{flow of } \gamma \\ &\quad \Rightarrow \frac{\partial}{\partial s} = \sum \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} \\ &\quad = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \end{aligned}$$

$= [X, Y]$ , by next Lemma  $\blacksquare$

Lemma  $X, Y \in \mathfrak{g} \Rightarrow [X, Y] = \frac{\partial^2}{\partial s^2} \Big|_0 Y - \frac{\partial^2}{\partial s^2} \Big|_0 X$

Pf locally  $Y = \sum b_i(x) \frac{\partial}{\partial x^i}$  so  $[Y, Y] = \sum b_i(\gamma_X(s)) \frac{\partial^2}{\partial s^2} \Big|_0 Y$

Proof of Thm also showed:

Corollary  $X, Y \in \mathfrak{g} \Rightarrow \frac{\partial^2}{\partial s^2} \Big|_0 \gamma_X(s) \gamma_Y(t) = \gamma_X(t) \gamma_Y(s)$

Def A matrix group is a closed subgroup of  $GL(n, \mathbb{R})$  or  $GL(n, \mathbb{C})$   
(or  $\text{Aut}(R)$  for v.s. R)

Rmk By LECTURE 7, this condition ensures they are Lie groups.

Examples  $O(n), SO(n), U(n), SU(n), SL(n, \mathbb{R}), \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}, \dots$

EXAMPLE 1  $[X, Y] = XY - YX$  for matrix groups  $(X, Y \in \text{Mat}_{n,n})$

$\gamma_X(s) = 1 + sX + \sigma(s)$  if "little-o-hes" if  $\lim_{s \rightarrow 0} \frac{\sigma(s)}{s} = 0$   
hence  $\frac{\partial^2}{\partial s^2} \Big|_0 f(s) = 0$

since  $\gamma_X(0) = I$  since  $\gamma'_X(0) = X$

$\Rightarrow [X, Y] = \frac{\partial^2}{\partial s^2} \Big|_0 (\gamma_X(s) \gamma_Y(t) \gamma_X(-s)) = \frac{\partial^2}{\partial s^2} \Big|_0 ((1+sX+\sigma(s))(1+tY+\sigma(t))((1-sX+\sigma(s)))$

$\left( \text{The } \frac{s^2 t^2}{2!} \text{ terms vanish if do } \frac{2!}{2! \cdot 2!} \right) = \frac{\partial^2}{\partial s^2} \Big|_0 (1+tY+sX + sXtY + \sigma(st)) = XY - YX$

EXAMPLE 2  $G$  abelian ( $S^1 = U(1), T^n, \mathbb{R}^n, \dots$ )

$\Rightarrow g(h) = hgh^{-1} = hh^{-1}g = g$

$\Rightarrow Ad(g) = D_g = Id$

$\Rightarrow ad = D_g Ad = 0$

$\Rightarrow [ \cdot, \cdot ] \equiv 0$

$\Rightarrow \mathfrak{g}$  abelian Lie algebra

USING EXP TO DETERMINE  $\mathfrak{g}$  AND  $G$

Lemma If  $H \xrightarrow{\text{Lie}} G$  embedding then the 1-param. subgs of  $H$  are precisely those  $\gamma_X(s) \subseteq G$  which lie in  $H$ .

Proof naturality  $\exp: \mathfrak{g} \rightarrow G$  since embedding can view  $H \subseteq G$  and  $\mathfrak{g} \subseteq \mathfrak{G}$   
converse: if  $\gamma_X^G(s) \subseteq H$  then  $\gamma_X^G: \mathbb{R} \rightarrow H$  is a Lie gp from hence 1-param. subgp. in  $H$

Consequences

- Can identify  $\mathfrak{g} = \text{Lie}(H)$  with a vector subspace of  $\mathfrak{g}$ :  $\mathfrak{g} \subseteq \{X \in \mathfrak{g} : \gamma_X(s) \subseteq H \text{ for small (hence all) } s \in \mathbb{R}\}$
- Lie alg. iso. (respect bracket by above Corollary)
- exp for  $H$  agrees with exp for  $G$ :  $\exp(X) = \gamma_X(1) \in H$  if  $X \in \mathfrak{g} \subseteq \mathfrak{G}$

EXAMPLE 3  $\sigma(n) = \{X \in \text{Mat}_{n,n}(\mathbb{R}) : X^T + X = 0\}$

Proof  $\gamma_X(s) \subseteq O(n) \Rightarrow 1 = \gamma_X(s)^T \gamma_X(s)$   
in  $GL(n)$   $= (1+sX)^T (1+sX) + \sigma(s)$   
 $= 1 + s(X^T + X) + \sigma(s)$  hence  $X^T + X = 0$ .

converse:

$X^T + X = 0 \Rightarrow \gamma_X(s)^T \gamma_X(s) = \exp(sX)^T \exp(sX) \stackrel{\text{exp series}}{=} \exp(sX^T) \exp(sX) = 1$   
 $\Rightarrow \gamma_X(s) \subseteq O(n)$  ■

SAME PROOF SHOWS  $u(n) = \{X \in \text{Mat}_{n,n}(\mathbb{C}) : X^* + X = 0\}$

EXAMPLE 4  $sl(n) = \{X \in \text{Mat}_{n,n} : \text{Trace}(X) = 0\}$  ( $sl(n) = \text{Lie } SL(n)$ , work over  $\mathbb{R}$  or  $\mathbb{C}$ )

Pf  $\det \gamma_X(s) = \det(1+sX) + \sigma(s) = 1 + s \cdot \text{Tr}(X) + \sigma(s)$

$\Rightarrow \frac{\partial^2}{\partial s^2} \Big|_0 \det \gamma_X(s) = \text{Tr}(X)$

$\Rightarrow \frac{\partial^2}{\partial t^2} \Big|_0 \det \gamma_X(t) = \frac{2}{\partial s} \Big|_0 \det \gamma_X(t+s)$

chain rule  $\gamma_X(t+s) = \gamma_X(t) \gamma_X(s)$

$= \det \gamma_X(t) \cdot \frac{\partial}{\partial s} \Big|_0 \det \gamma_X(s)$

$= \det \gamma_X(t) \cdot \text{Tr}_r(X)$

Now:  $\gamma_X(s) \in SL(n) \Rightarrow \det \gamma_X(s) = 1 \Rightarrow 1 + s \cdot \text{Tr}(X) + \sigma(s) = 0 \Rightarrow \text{Tr}(X) = 0$ .

converse:  $\text{Tr } X = 0 \Rightarrow \det \gamma_X(t) = \det \gamma_X(t) = 1 \Rightarrow \gamma_X(t) \in SL(n)$

SAME PROOF SHOWS:  $su(n) = \{X \in u(n) : \text{Tr}(X) = 0\}$

Theorem Let  $G$  be connected.

1)  $\exp: \mathfrak{g} \rightarrow G$  is a group hom  $\Leftrightarrow G$  abelian

2)  $G$  abelian  $\Leftrightarrow G \cong$  torus  $\times$  vector space

Cor 1  $G$  abelian  $\Rightarrow \exp: \mathfrak{g} \rightarrow G$  surjective hom. onto  $G_0 = \text{comp. of } \text{Lie } G$

Cor 2  $G$  compact connected abelian  $\Rightarrow G \cong T^n$

Cultural Rmk In non-connected case in 2) get torus  $\times$  vector space  $\times$  discrete abelian group  
Proofs Thm 2  $\Rightarrow$  Cor 2: because a vector space  $\cong \mathbb{R}^k$  is non-compact ( $k \neq 0$ ) ■

## LECTURE 7

Pf 1:  $\Rightarrow$ :  $(G, +)$  is an abelian group  $\Rightarrow \exp(g)$  abelian  
But  $\exp(g)$  generates  $G_0 = G$  (lecture 5:  $\exp(g) \supseteq \text{nbhd } V$  of  $1 \in G$ )  
Thm 1  $\Rightarrow$  Cor 1  $\langle \exp(g) \rangle = G_0$  and, since image of a hom is a subgp, get  $\exp(g) = \langle \exp(g) \rangle$

Pf 1:  $\Leftarrow$ :  $G$  abelian  $\Rightarrow$  multiplication  $\mu: G \times G \rightarrow G$  is a Lie grp hom

$$\begin{aligned} \text{natrality of } \exp: \\ D_{t,1} \xrightarrow{\partial_t} \partial_t & \quad (X, Y) \xrightarrow{\text{exp}} D_1 \mu \cdot (X, Y) \\ \exp \downarrow & \quad \downarrow \mu \quad \text{abelian} \\ G \times G \xrightarrow{\mu} G & \quad (\exp X, \exp Y) \xrightarrow{\text{exp}(X) \exp(Y)} \end{aligned}$$

Rmk general fact  $\text{Lie}(G, x, g_1, g_2) = g_1 \oplus g_2 \xrightarrow{\exp} G_1 \times G_2$  is just  $\exp$  in each entry.  
Indeed  $\delta_{x_1, x_2}(t) = (\delta_{x_1(t)}, \delta_{x_2(t)}) \in G_1 \times G_2$ , since solve flow equation in each entry.

$$D_1 \mu \cdot (X, Y) \stackrel{(1)}{=} \frac{\partial}{\partial t} \Big|_0 \mu(\delta_X(t), \delta_Y(t)) = \frac{\partial}{\partial x_1} \mu(\delta_X(x_1), \delta_Y(x_2)) = X + Y$$

$$\xrightarrow{\text{natrality}} \exp(X+Y) = \exp(X) \exp(Y) \quad \text{so } \exp \text{ is hom} \quad \left( \begin{array}{l} \varphi: G \rightarrow H \text{ gp hom} \\ \text{ker } \varphi \xrightarrow{\text{iso}} \text{Im } \varphi \end{array} \right)$$

Pf Thm 2:  $\Rightarrow$  Idea is to use the 1st isomorphism theorem for groups  
we already know  $\text{Im}(\exp: g \rightarrow G) = G_0 = G$  (by "Thm 1  $\Rightarrow$  Cor 1"). Need find  $\text{Ker}(\exp)$

Claim  $K := \text{Ker}(\exp: G \rightarrow G)$  is a discrete subgroup of the vector space  $G$ .

Proof  $\exp: U \rightarrow V$  differs, and for  $X \in K$ :  $\exp(X+U) = \exp(X) \cdot \exp(U)$

$$\begin{matrix} 0 & 1 \\ 0 & -1 \end{matrix} \quad \begin{matrix} \exp(X) & \text{hom} \\ \text{nonzero} \end{matrix} \quad \begin{matrix} \exp(U) & \text{hom} \\ \text{by Thm 1.} \end{matrix} \quad \begin{matrix} \uparrow & \downarrow \\ \text{only get} \\ \exp(u) = 1 \\ \text{if } u = 0 \end{matrix}$$

$\Rightarrow (X+U) \cap K = \{X\}$  (note:  $X+U$  is an open set around  $X$ )

Def Discrete subgroups of a vector space are called lattices.

FACT: discrete subgroups of a vector space are generated (as group, so over  $\mathbb{Z}$  not  $\mathbb{R}$ ) by a finite collection of linearly independent vectors.

(This is proved by induction on dim of the vector space. We take it as a fact for this course)

$$\Rightarrow K = \text{span}_{\mathbb{Z}}(X_1, \dots, X_k) \cong \mathbb{Z}^k \subseteq \mathbb{R}^k = \text{span}_{\mathbb{R}}(X_1, \dots, X_k)$$

$$\xrightarrow{\text{complete to a basis}} \frac{g}{K} = \text{span}_{\mathbb{R}}(X_1, \dots, X_n) \cong \mathbb{R}^k \times \mathbb{R}^{n-k} \xrightarrow{\text{Z}^k \times 0} \mathbb{Z}^k \times 0$$

$$\xrightarrow{\text{1st isomorph theorem}} \frac{g}{K} \cong T^k \times \mathbb{R}^{n-k} \xrightarrow{\exp} \text{Image}(\exp) = G \quad \text{by 1st iso. theorem}$$

Above proof showed in general: Lemma  $D_{(1,0)} \mu \cdot (X, Y) = X + Y$  for  $\mu: G \times G \xrightarrow{\text{mult}}, G$

COR

All closed normal Lie subgroups arise as kernels:  $N = \text{Ker}(H \xrightarrow{\text{quotient map}} G)$

## LIE SUBGROUPS

$$\begin{aligned} \text{Lie subgroup } H \subseteq G \text{ means} \\ & \begin{cases} H \subseteq G \text{ subgp} \\ \text{inclusion } H \hookrightarrow G \text{ is smooth} \end{cases} \end{aligned}$$

Equivalently: a Lie subgroup is an injective Lie group hom  $H \hookrightarrow G$

(identifies  $H \equiv j(H) \subseteq G$ )

### EXAMPLES

- $H = \mathbb{R} \cdot (1, 3) \subseteq T^2 = \mathbb{R}^2 / \mathbb{Z}^2$  Lie subgrp  $\cong S^1$ , namely  $S^1 \xrightarrow{j} S^1 \times S^1 \xrightarrow{j} (e^{2\pi i t}, e^{2\pi i t}) = (e^{2\pi i t}, e^{2\pi i t})$   
In this case,  $H$  is also a submanifold: a circle wrapping around first  $S^1$  factor once and wrapping 3 times around second  $S^1$  factor.
- $H = \mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2 = \mathbb{R}^2 / \mathbb{Z}^2$  Lie subgrp  $\cong \mathbb{R}$ , but not a submanifold (Q.sheet 3)
- $H = \text{Image } (\delta_X) \subseteq G$  Lie subgrp ( $\cong \mathbb{R}$  or  $S^1$ ) depending on whether  $\delta_X$  is injective or not (where  $X \neq 0 \in \text{Lie } G$ )

$$H = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} : \lambda_j \in S^1 \right\} \subseteq U(3)$$

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \subseteq SL(3, \mathbb{R})$$

$$G_0 \subseteq G \text{ for example } SO(2) \subseteq O(2)$$

- NON-EXAMPLE: Let  $\mathbb{R} \not\subseteq \mathbb{R}$  be a linear iso of vector spaces over  $\mathbb{Q}$  such that  $\sqrt{2}, \sqrt{3}, \sqrt{5}$  map to  $\sqrt{2}, \sqrt{3}, \sqrt{5}$  respectively. Then  $\varphi$  is a group hom (using addition on  $\mathbb{R}$ ) but  $\varphi$  is not smooth since not continuous (continuous bijections  $\mathbb{R} \rightarrow \mathbb{R}$  are either strictly increasing or strictly decreasing)

### GENERAL EXAMPLES

- $H \hookrightarrow G$  Lie grp hom  $\Rightarrow \text{Ker } \varphi \subseteq H$  is closed normal subgroup (Q.sheet 4)
- FACT 1  $N \subseteq H$  closed normal subgroup  $\Rightarrow H/N$  is a Lie grp in natural way (see lecture 8)
  - $H \hookrightarrow G$  Lie grp hom  $\xrightarrow{\text{1st iso thm + Fact 1}} H/\text{Ker } \varphi \cong \text{Im } \varphi$  Lie subgrp of  $G$
  - example  $\delta_X: \mathbb{R} \rightarrow G, X \neq 0 \Rightarrow \mathbb{R}/\text{Ker } \delta_X \cong \text{Im } \delta_X \subseteq G$  Lie subgrp (indeed  $\cong \mathbb{R}$  or  $S^1$ )

### WARNING : LIE SUBGROUPS MAY NOT BE SUBMANIFOLDS

Examples  $\mathbb{R} \rightarrow T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ ,  $x_1 \mapsto (x_1, \lambda x_2)$ , for  $\lambda$  irrational, is injective and has dense image, not a homeomorphism onto image so not a submfld (the subspace topology for the image is not the usual topology on  $\mathbb{R}^2$ )

Lemma 1  $H \xrightarrow{j} G$  Lie subgroup  $\Rightarrow D_\theta \varphi: T_\theta H \rightarrow T_{\varphi(\theta)} G$  injective for all  $\theta \in H$

Pf.  $D_\theta j$  is injective by naturality of  $\exp$  since  $j$  is injective nbhd  $\theta \in G$   $\Rightarrow$  nbhd  $\exp^{-1}(\theta) \in H$  injective, nbhd  $\theta \in G$  (linear map is injective if it is injective near 0)

To show  $D_\theta j$  injective:

$$D_\theta \varphi \cdot X_h = D_h \varphi \cdot D_\theta^\alpha X_1 \xrightarrow{\varphi \circ D_\theta^\alpha} D_\theta^\alpha \cdot D_\theta \varphi \cdot X_1 \xrightarrow{\text{injective}} \varphi(h) \circ \varphi(1) \xrightarrow{\text{injective}} \varphi(h) \circ \varphi(1) = \varphi(h) \circ \varphi$$

Rmk Maps  $N \xrightarrow{\varphi} M$  with  $D_\theta \varphi$  injective are called immersions.  $\varphi$  immersion  $\Leftrightarrow \varphi$  is a local embedding (i.e.  $\forall \theta \in N \exists \theta \in M$  such that  $\varphi(\theta) = \varphi(\theta')$  for some  $\theta' \in U \subset M$  (non-examinable: implicit function theorem argument))

Cor Lie subgroups are locally embedded.

Example locally  $\mathbb{R} \rightarrow \mathbb{T}^2$ ,  $x \mapsto (x, \lambda x)$  looks like  $\boxed{\quad}$

### LIE SUBALGEBRAS

Lie subalgebra  $W \subseteq (V, [\cdot, \cdot])$  means • vector subspace  $W \subseteq V$

Equivalently : a Lie subalgebra is an injective Lie algebra hom  $i: W \rightarrow V$  (identify  $W \equiv i(W) \subseteq V$ )

Lemma  $H \xrightarrow{i} G$  Lie subgp  $\Rightarrow \mathfrak{d}_{ij}$  of Lie subalg. By Lemma 1,  $\mathfrak{d}_{ij}$  injective. ■

### EXAMPLES

- $H = \gamma_x(\mathbb{R}) \subseteq G$  gives  $\mathfrak{g} = \text{span}(X) = \mathbb{R} \cdot X \subseteq \mathfrak{g}$  (abelian Lie subalg since  $[X, X] = 0$ )
- Q. sheet 4 :  $\text{Lie}(\ker(\varphi: H \rightarrow G)) = \text{Ker } D_\theta \varphi \subseteq \mathfrak{g}$  Lie subalg
- $\text{Lie} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{R} \right\} \subseteq \mathfrak{u}(3)$
- $\text{Lie} \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \subseteq \mathfrak{sl}(3, \mathbb{R})$

We defined submanifolds  $N \subseteq M$  by saying that the inclusion is an embedding, but that is not very practical. Better:

FACT A subset  $N \subseteq M$  is a submfld  $\Leftrightarrow \forall \text{OPEN } \exists \text{ product parametrization near } p$

Means : 

such that •  $\psi(0) = p$   
•  $(\psi|_{R^n \times 0}) \circ \psi = N \cap U = N \cap M$

Notice:  $\psi|_{R^n \times 0}$  gives a parametrization for  $N$   
 $R^m = \mathbb{R}^n \times \mathbb{R}^{m-n} \supseteq V \xrightarrow{\psi} U \subseteq M$   
so locally  $N$  is the subset defined by the equations  $\begin{cases} x_{n+1} = 0 \\ \dots \\ x_m = 0 \end{cases}$

### WHEN ARE LIE SUBGROUPS ALSO SUBMANIFOLDS?

Theorem Let  $G$  be a Lie group.  
A subgroup  $H \subseteq G$  is a submfld  $\Leftrightarrow H \subseteq G$  closed subset  
(and hence an embedded Lie subgroup)  
(not assuming that  $H$  is a Lie group)

Example Any matrix group (= closed subgp of  $GL(n)$ ) is a Lie group!

## LECTURE 8

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C3.5 LIE GROUPS, HT2015, Oxford.

### WHEN ARE LIE SUBGROUPS ALSO SUBMANIFOLDS?

Theorem Let  $G$  be a Lie group.

A subgroup  $H \subseteq G$  is a submfld  $\iff H \subseteq G$  closed subset

(Not assuming that  $H$ )  
(and hence an  
embedded Lie subgroup)  
 $\Rightarrow$  is a Lie group

Example Any matrix group (= closed subset of  $GL(n)$ ) is a Lie group!

Proof of Thm.  $\Rightarrow$ :  $H$  submfld  $\Rightarrow H$  locally closed in  $G$   $\Rightarrow \exists$  nbhd  $V$  of  $I$  with  $V \cap H$  closed

Let  $y \in \overline{H}$ , pick  $x \in H$  close to  $y$ :  $x \in yV^{-1}$   
(note:  $V$  contains open nbhd of  $I \in G$ , so  
 $V^{-1}$  does also since inversion is a diffeo.  
so  $yV \cap H \neq \emptyset$ )  
 $\Rightarrow y \in xV \cap \overline{H}$

$\Rightarrow x^{-1}y \in V \cap \overline{H} = V \cap H \Rightarrow y \in xH \subseteq H \Rightarrow \overline{H} \subseteq H$  hence equal.  $\checkmark$

$x^{-1}\overline{H} = \overline{x^{-1}H} = \overline{V \cap H}$  closed

$\Leftarrow$ :  $g = \text{Lie } G \geq V^{\circ \circ} \supseteq V' = \log(U')$   $\leftarrow$  (we will show this is Lie group)

$\exp$  differs from  $\log = \text{inverse of } \exp: V \rightarrow U$

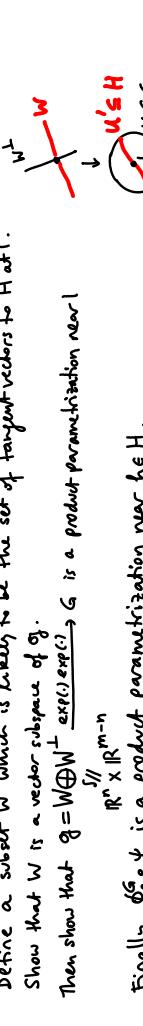
$G = U_{\psi} \supseteq U' = H \cap U = \text{nbhd of } I \in H$

NOTE  $V'$  is just a subset of  $G$ . We want to build a vector subspace  $W$  from  $V'$   
which is a likely candidate for  $W = \text{Lie } H \subseteq g$ .

STRATEGY: Pick an inner product on  $g$  ( $g \cong \mathbb{R}^m$  and use usual  $\langle \cdot, \cdot \rangle$ ).  
Define a subset  $W$  which is likely to be the set of tangent vectors to  $H$  at  $I$ .  
Show that  $g = W \oplus W^\perp$   $\exp|_{W^\perp}$  is a product parametrization near  $I$ .

Then show that  $g = W \oplus W^\perp$   $\exp|_{W^\perp}$  is a product parametrization near  $I$ .

Finally,  $\phi_G \circ \psi$  is a product parametrization near  $I \in H$ .



DEFINITION  $W = \{tX : X = \lim_{n \rightarrow \infty} \frac{v_n}{|v_n|} \text{ some } v_n' \in V', v_n' \rightarrow 0, t \in \mathbb{R}\}$

MOTIVATION: You are trying to understand which are the likely tangent directions to  $H$  at  $I$  if it really were a submanifold.  
So you consider sequences of vectors  $v_n' = \log(k_n)$  for  $k_n \in H$  close to  $I$ :  
this explains the conditions  $v_n' \in V'$  and  $v_n' \rightarrow 0$ .

- You are not interested in the zero vector, so you might as well normalize:  $\frac{v_n'}{|v_n'|}$
- The condition that  $v_n'/|v_n'|$  converges is not restrictive at all, since  $V \cap I \in \mathcal{G}$ ,  $|v_n'|/|v_n|$  lie in the compact unit sphere of  $\mathcal{G}$ , so passing to a subsequence you can always assume that it converges in  $\mathcal{G}$ .
- Morally it should be enough to take  $W = \mathbb{R}$ . ( $V'$  is small sphere in  $\mathcal{G}$ ) since this would be the  $\mathbb{R}$ -span of a small sphere in  $\text{Lie } H$  and hence is all of  $\text{Lie } H$ . The problem you would encounter is that it's not clear that  $V' \cap (\text{all boundary } S \text{ in } \mathcal{G}) \subseteq W$

SIMPLICITY CHECK  
If we knew  $H$  was an embedded Lie subgroup and we knew  $V' = \text{nbhd of } 0 \in \text{Lie } H$  then  $W = \text{Lie}(H)$ . Proof:  $X \in \text{Lie } H \Rightarrow v_n' = \frac{1}{|v_n|} X \in V'$  for large  $n$ ,  $v_n' \rightarrow 0$ ,  $\frac{v_n'}{|v_n|} = \frac{X}{|X|} \rightarrow \frac{X}{|X|} \in W$  hence  $t \cdot \frac{X}{|X|} \in W$  all  $t \in \mathbb{R}$ . Take  $t = |X|$  to get  $X \in W$ .

Claim 1  $\exp(W) \subseteq H$   
pf: We need to show  $\exp(tX) \in H$  (where  $X = \lim_{n \rightarrow \infty} \frac{v_n'}{|v_n|}$ ,  $v_n' \in V'$ ,  $v_n' \rightarrow 0$ )  
Note  $|v_n'| \rightarrow 0$  and  $\frac{t}{|v_n'|} \cdot v_n' \rightarrow tX$   
pick  $m_n \in \mathbb{Z}$  with  $m_n |v_n'| \rightarrow t$   $\leftarrow$  (idea: approximating  $\frac{t}{|v_n'|}$  by integers  $m_n$ )  
 $\Rightarrow \exp(m_n \cdot \frac{v_n'}{|v_n|}) = \exp\left(\frac{m_n \cdot v_n'}{|v_n|}\right) \rightarrow \exp(tX)\right\} \Rightarrow \exp(tX) \in \overline{H} = H$ .  $\square$

Claim 2  $W \subseteq \mathcal{G}$  is a vector subspace  
pf: scaling ✓ adding:  $X, Y \in W \Rightarrow \text{let } \gamma(t) = \underbrace{\log(\exp(tx))}_{\in H \text{ by claim 1, so } \gamma(t) \in V' \text{ (small } t\text{)}} - \underbrace{\log(\exp(tY))}_{\in H \text{ by claim 1, so } \gamma(t) = 0}$   
 $\Rightarrow \gamma'(t) = \lim_{t \rightarrow 0} \frac{\gamma(t)}{t} = X + Y$   
 $\Rightarrow \lim_{t \rightarrow 0} \frac{\gamma(t)}{t} = \lim_{t \rightarrow 0} \frac{\gamma(t)}{t} \cdot \frac{t}{|t|} = \frac{X+Y}{|X+Y|} \in W$   $\leftarrow$  (taking  $X \frac{1}{|X|} \in V'$  as the "nbhd" in the definition of  $W$ )  
 $\Rightarrow$  rescale  $X+Y \in W$   $\square$

Claim 3 Define  $\mathbb{R}^n \times \mathbb{R}^{m-n} \cong W \oplus W^\perp = \text{Lie } G \xrightarrow{\psi} \mathcal{G}$ ,  $\psi(w, \tilde{w}) = \exp(w) \cdot \exp(\tilde{w})$   
Then  $\psi$  is a product parametrization for  $H \subseteq G$  near  $I \in G$ .  
pf:  $D_0 \psi \cdot (X, Y) = X+Y \leftarrow$  q. sheet 3 (compare  $D_0 \psi$  from earlier 6)  
 $\Rightarrow D_0 \psi$  iso, so by inverse function theorem  $\psi$  is diffeo near  $0$  hence parametrization near  $I \in G$ .  
Remains to show  $\psi(w)$  is a neighbourhood of  $I \in H$  (Note by claim 1,  $\psi(w) \subseteq H$ ).  
Suppose not, by contradiction. Since  $\psi$  is surjective near  $I \in G$ , this implies:  
 $\exists w \in \text{nbhd}(w) \exp(\tilde{w}) \in H \setminus \psi(W)$  arbitrarily close to  $I$ , so:  $\tilde{w} \neq 0$ , and  $(w_n, \tilde{w}_n) \xrightarrow{\psi} 0$ .  
STRATEGY: build a non-zero  $\tilde{w} \in W \cap W^\perp$  for  $w_n \in H$  close to  $I$ :

$\bar{w}_n \in$  compact unit sphere in  $W^\perp$ , so passing to a subsequence can assume  $\frac{\bar{w}_n}{\|\bar{w}_n\|} \rightarrow \bar{w} \in W^\perp$ ,  $\|\bar{w}\|=1$ .  
 But  $\exp(\bar{w}_n) \in H$  (claim) so  $\exp(\bar{w}_n) = \frac{\exp(w_n)}{\|\bar{w}_n\|} \frac{\exp(w_n)}{\|\bar{w}_n\|} \in H$  since subgr.  
 $\Rightarrow \bar{w}_n = \log \exp \bar{w}_n \in V'$ ,  $\bar{w}_n \rightarrow 0$ ,  $\bar{w} = \lim \bar{w}_n$  so by definition of  $W$  get  $\bar{w} \in W$ . ■

## LECTURE 2

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THE SUBGROUP – SUBALGEBRA CORRESPONDENCE

$G$  Lie group,  $\mathfrak{g} = \text{Lie } G$ .

## QUOTIENT LIE GROUPS

For  $N$  closed normal subgr of  $G \Rightarrow G/N$  Lie group

- $G \xrightarrow{\pi} G/N$  smooth
- $\text{Lie}(G/N) = \mathfrak{g}/\eta$  where  $\eta = \text{Lie}(N)$ 
  - $[\mathbf{x} + \mathbf{n}, \mathbf{y} + \mathbf{n}] = [\mathbf{x}, \mathbf{y}] + \mathbf{n}$  (well-defined since  $\mathbf{n}$  ideal)  
 $\xrightarrow{\text{Q. sheet 4}} \mathbf{x} \cdot \mathbf{N}$
- Topology on  $G/N$  is quotient topology ( $U \subseteq G/N$  open  $\Leftrightarrow \pi^{-1}(U) \subseteq G$  open).
- Local parametrization near  $1 \in G/N$  is
- $\psi: (\text{nbhd of } 0 \in W^\perp) \xrightarrow{\text{smooth}} W \oplus W^\perp = \mathfrak{g} \xrightarrow{\exp, G \xrightarrow{\pi} G/N}$  (using notation of above proof for  $H = N$ )
- Near  $g \in G/N$  the local param. is  $\phi_g \circ \psi$ .

## CONTINUOUS LIE GRP HOMS ARE SMOOTH

Theorem  $\varphi: H \xrightarrow{\text{continuous}} G$  group homomorph  $\Rightarrow \varphi$  smooth  
 $H, G$  Lie grps  
 (hence Lie geom)

EXAMPLE Q.sheet 2:  $SU(2) \rightarrow SO(3)$  obviously ct<sub>3</sub> grp hom  $\Rightarrow$  smooth

p.f. graph of  $\varphi: \Gamma_\varphi = \{(h, \varphi(h)): h \in H\} \subseteq H \times G$  is a closed subgr since  $\varphi$  ct<sub>3</sub>  
 hence Lie subgr, so submfld, so  $\Gamma_\varphi \hookrightarrow H \times G$  smooth.

i)  $H \times G \xrightarrow{\pi} H$  projection (smooth of course)  
 $\Rightarrow \tilde{\pi} = \pi \circ \varphi: \Gamma_\varphi \rightarrow H$  smooth

ii)  $\tilde{\pi}$  is homeomorphism (inverse  $h \mapsto (h, \varphi(h))$ )  
 Lemma 3  
 Lecture 5  
 iii)  $\varphi$  grp hom  $\Rightarrow \tilde{\pi}$  grp hom

$\tilde{\pi}$  homeomorph + local diffeo  $\Rightarrow$  diffeo (see Lecture 5)  $\left( \begin{matrix} G, \mathfrak{g}, G, \text{Lie grp hom} \\ \varphi \text{ locally homeo near } 1 \end{matrix} \right) \Rightarrow \varphi$  local diffeo

$\Rightarrow \varphi: H \xrightarrow{\tilde{\pi}^{-1}} \Gamma_\varphi \hookrightarrow H \times G$  project  $G$  smooth ■

Chevalley's Theorem	
{Lie subalgebras $\mathfrak{g} \subseteq \mathfrak{g}$ } $\xleftarrow[1:1]{\mathfrak{g} = \text{Lie}(H)} \{$ connected Lie subgroups $H \subseteq G\}$	$\mathfrak{g} \subseteq \mathfrak{g}$

$\mathfrak{g} \subseteq \mathfrak{g}$  = subgr generated by exp  $\mathfrak{g}$

EXAMPLE	dim	$\mathfrak{g} \subseteq \mathfrak{g}$	$H \subseteq G$
	0	{0}	{1}
	1	(Lie subalgebra since $[X, X] = 0$ )	$\mathbb{R} \times \mathbb{R}$ (image of the 1-parameter subgr)

EXAMPLE  $G = T^n = \mathbb{R}^n / \mathbb{Z}^n \Rightarrow [\cdot, \cdot] = 0$

Recall $\exp: \mathbb{R}^n \rightarrow T^n$ is the homomorphism $\pi(v) = v \bmod \mathbb{Z}^n$	any vector subspace $\mathfrak{g} \subseteq \mathfrak{g}$ is a Lie subalgebra
$\Rightarrow \exp(\mathfrak{g}) = \exp(\mathfrak{g}) = \mathfrak{g} \bmod \mathbb{Z}^n \quad (\cong \mathfrak{g} / \mathfrak{g} \cap \mathbb{Z}^n)$	
$\Rightarrow$ correspondence is: $\left( \begin{matrix} \text{vector subspaces} \\ \mathfrak{g} \subseteq \mathbb{R}^n = \text{Lie}(T^n) \end{matrix} \right) \xleftrightarrow{1:1} \left( \begin{matrix} \text{abelian subgroups} \\ \mathfrak{g} \mod \mathbb{Z}^n \subseteq T^n \end{matrix} \right)$	have the form subgroups $x \in \mathbb{R}^n$ $\pi(x) = v \bmod \mathbb{Z}^n$

MORE EXAMPLES Q.sheet 4:  $SO(3), SL(2, \mathbb{Z})$ .

Remarks • It's because we want the above correspondence that we do not require Lie subgrps to be submflds (lecture 7, Q.sheet 3)

• The correspondence is difficult to prove because  $H = \langle \exp \mathfrak{g} \rangle$  need not be a submfld. In general we need to define a new topology on  $H$ , which may not be the subspace topology, in order to prove that  $H$  is a Lie group!

(Example from Lecture 7,  $\mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2$ : the subspace topology has less open sets than the usual topology on  $\mathbb{R}^2$ )

Connected is necessary:  $O(3), SO(3) \subseteq GL(3, \mathbb{R})$  have Lie algs  $\mathfrak{o}(3) = \mathfrak{so}(3) \subseteq gl(3, \mathbb{R})$ .

Proof of uniqueness:  $H$  connected  $\Rightarrow H$  generated by nbhd of  $1 \in H$

- $\Rightarrow \exp(\beta)$  generates  $H$  (since contains nbhd of 1)  
 $\Rightarrow H = \langle \exp(h) \rangle$  is the only possible choice if you want  $\text{Lie}_H = h$ .
- Proof of existence:
- ① Consider  $D = \text{span}(\beta) \subseteq TG$ . Pick basis  $X_1, \dots, X_d$  of  $\beta$ .  
Notice: at each  $g \in G$ ,  $T_g g = \text{span}(X_1|_g, \dots, X_d|_g) \subseteq T_g G$  is a  $d$ -dim'l v.s. and locally near  $g \in G$  there are vector fields  $Y_1, \dots, Y_d$  with  $D = \text{span}(Y_1, \dots, Y_d)$  e.g. take  $y_i = X_i$ . Such  $D$  are called a  $d$ -dim'l distribution on the manifold  $G$ .
- ② Say that a vector field  $X$  on  $G$  is in  $D$ , written  $X \in D$ , if  $X|_g \in D_g$  all  $g \in G$ .  
Claim 1  $D$  is integrable (or involutive), meaning:  $[X, Y] \in D \quad \forall X, Y \in D$   
Proof all  $X \in D$  are pointwise in  $\text{span}$  of  $X_1, \dots, X_d$  hence  
 $X = \sum a_{ij}(x) X_j \Rightarrow [X, Y] = \sum_{i,j} a_{ij} [X_i, X_j] + a_i(X_i \cdot b_j) X_j - b_j(X_j \cdot a_i) X_i \in D$  ■  
 $Y = \sum b_{ij}(x) X_j \Rightarrow [X, Y] = \sum_{i,j} b_{ij} [X_i, X_j] + a_i(X_i \cdot b_j) X_j - b_j(X_j \cdot a_i) X_i \in D$  ■  
Claim 2 since Lie subalg  $\mathfrak{g}$  is a Lie subalgebra of  $\mathfrak{g}$ .
- ③ Local Frobenius theorem  
d-dim'l integrable distributions are locally of the form:  
 $\exists x_1, \dots, x_m$  local coords for the manifold with  $D_x = \text{span}(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m})$
- proof idea (non-examinable)  
Locally  $D = \text{span}(Y_1, \dots, Y_d)$ . By integrating  $Y_1$  get local coordinate  $x_1 = \text{time coord}$ . Using the flow of  $Y_1$  one can build local coordinates  $y_1, y_2, \dots, y_m$  with  $Y_1 = \frac{\partial}{\partial y_1}$  and  $y_i = x_i$ . Then modify other v.f.  $Z_j = Y_j - (Y_j \cdot x_1) Y_1$  ( $j > 2$ ) - notice  $Z_j \cdot x_1 = 0$ . The slice  $\{x_1 = 0\}$  is locally a submfld  $S$  and  $D' = \text{span}(Z_2, \dots, Z_d)$  is a  $(d-1)$ -dim'l integrable distribution on  $S$  (since  $Z_j \cdot x_1 = 0$  have  $Z_j \in TS$ ). Then use an induction on dim of distribution to get local coords  $x_2, \dots, x_m$  on  $S$ . Extend  $x_2, \dots, x_m$  to local coords near  $S$  by projecting to  $S$  (in  $y$ -coord system). By construction  $Y_1 = \frac{\partial}{\partial x_1}$  but need to check  $Z_j \cdot x = 0$  for coords  $x = x_{d+1}, \dots, x_m$ . We know this on  $S$  (by induction) so we need to show it holds also near  $S$ . Observe:  
 $\frac{\partial}{\partial x_1} Z_j \cdot x = Y_j \cdot (Z_j \cdot x) = \underbrace{[Y_j, Z_j]}_{\in D} \cdot x = Y_j \cdot x = 0$  since  $Y_j \cdot x = 0$  (for  $x = x_{d+1}, \dots, x_m$ )  
 $\Rightarrow \frac{\partial}{\partial x_1} Z_j \cdot x = \sum f_{j,k} \cdot Z_k \cdot x = \sum f_{j,k} \cdot x$  some functions fik  
Now fix values of  $x_2, \dots, x_m$ , then  $Z_k \cdot x = y_k(x)$  and get system of ODE's.  
 $y'_i(x_1) = \sum f_{j,k} y_k$  hence unique solution given initial condition
- Initial condition is  $y_j(x_1) = Z_j \cdot x = 0$  on  $S = \{x_1 = 0\}$  (by induction) so  $y_j \equiv 0$  unique solution  
 $\Rightarrow D = \text{span}(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$  (since  $D \cdot x = 0$  all  $x = x_{d+1}, \dots, x_m$ ) (Note: we are not claiming that  $Z_j = \frac{\partial}{\partial x_j}$ )
- ④ locally can integrate  $D$  meaning  $\exists$  submfld  $S \subseteq G$  with  $T_s S = D_s$ , namely:  $S = \{x = (x_1, \dots, x_d, c_{d+1}, \dots, c_m)\}$  ← called slice

- By ③ these  $S$  are the only connected integral manifolds of  $D$  (meaning  $T_s S = D_s \forall s \in S$ )  
 $\Rightarrow H$  near  $1 \in G$  is the unique slice of  $D$  through 1.
- ⑤ piece together slices of  $D$  starting with this one to build the manifold  $H$ .  
Rmk 1: slices are embedded, so we simply define the topology and manifold structure as the subspace topology and submanifold structure of  $S \subseteq G$ .
- Definition A leaf  $L$  is a connected integral manifold meaning:  
a manifold  $L$  together with an injective immersion  $L \xrightarrow{\varphi} G$  such that  $D\varphi \cdot TL = D$ .  
Examples: • a slice is an embedded leaf.  
•  $\mathbb{R} \cdot (1, \sqrt{2})$  is a leaf in  $T^2$ .
- Recall (lecture 7)  $\varphi$  immersion  $\Leftrightarrow \varphi$  local embedding  
Claim 2 A leaf is a union of slices, and each slice is an open subset of the leaf.  
Pf Using  $\varphi$ cts, for small connected  $U \subseteq L$  have  $\varphi(U) \subseteq \text{local model } \mathbb{R}^d$  so  $\varphi(U) = \text{some slice}$ . Since  $\varphi$  immersion,  $\varphi: U \rightarrow \varphi(U) \subseteq G$  local embedding (for small  $U$ ) so  $U \cong \varphi(U)$  diffeo so the topology & manifold structure on  $U$  is the same as for slice (Rmk 1) ■
- ⑥ Rmk 1 + claim 2  $\Rightarrow$  topology and manifold structure on leaves is determined by finer topology on  $G$  called leaf topology given by taking leaves as basis of open sets!
- Example •  $\mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2$  ← all these segments are now open sets! (leaves)
- Let  $H = \text{the connected component of } 1 \text{ in the leaf topology on } G$   
= maximal connected leaf in  $G$  through 1
- Non-examinable FACT A map  $S \xrightarrow{\text{cts}} H$  is smooth  $\Leftrightarrow$  composition  $S \xrightarrow{\text{cts}} H \xrightarrow{\text{smooth}}$  (← requires some work to show & cts)  
⑦ Claim  $H$  is a subgr (a sheet 1)  
 $\frac{\partial}{\partial x_1} X \in X$  for all  $X \in \mathfrak{g} \Rightarrow (\phi_g^*)_* D = D \Rightarrow \phi_g^*$  permutes the maximal leaves (a sheet 1)
- For hth:  $\phi_{k-1}^* H$  leaf containing 1  $\Rightarrow \phi_{k-1}^* H \subseteq H \Rightarrow h^{-1} \in H$   
 $\Rightarrow \phi_{(k-1)-1}^* H \subseteq H$  so  $h, h' \in H$  all  $h' \in H$  ■
- ⑧ Claim group operations in  $H$  are smooth  
Pf Want inversion  $H \xrightarrow{\text{cts}} H$  smooth. By above FACT need show composite  $S = H \xrightarrow{\text{cts}} H \xrightarrow{\text{smooth}} G$  hence smooth. This composite equals the composition  $H \xrightarrow{\text{smooth}} G$  hence smooth  
Want multiplication  $H \times H \xrightarrow{\text{cts}} H$  smooth. By FACT need show  $S = H \times H \xrightarrow{\text{cts}} H \xrightarrow{\text{smooth}} G$  hence smooth. This equals the composition  $H \times H \xrightarrow{\text{cts}} G \times G \xrightarrow{\text{smooth}} G$  hence smooth ✓ ■

## LECTURE 10

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### COVERING MAPS

A smooth surjective map of manifolds  $\pi: M \rightarrow N$  is a covering map if there is an open set  $U$  around any point  $p \in N$  with:

- $\pi^{-1}(U) = \bigsqcup_{i=1}^n U_i$  disjoint union of open sets (the sheets over  $U$ )
- $U_i$  diffeo  $U$

The fibre over  $p \in N$  is  $\pi^{-1}(p) = \bigsqcup_{i=1}^n \tilde{U}_i$  (discrete set)

#### EXAMPLES

$\mathbb{R} \xrightarrow{e^{2\pi i x}} S^1$ , pictorially:

• Q. sheet 1:  $\exp: \mathbb{R}^n \rightarrow \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$  cover, fibre  $\cong \mathbb{Z}^n$

• Q. sheet 2:  $SU(2) \rightarrow SO(3)$  double cover, fibre  $\cong \{\pm I\}$

• NON-EXAMPLE:  $(0, 3) \xrightarrow{e^{2\pi i x}} S^1$  local diffeo but not covering

#### RmkS

1)  $\pi$  local diffeo  
(pick a path  $\gamma$  from  $p$  to  $p'$  in  $N \Rightarrow \pi^{-1}(\gamma)$  are paths from  $\tilde{p}_i$  to  $\tilde{p}'_i$ )

2) If  $N$  is connected, the fibres are all homeomorphic  
( $\pi^{-1}(N)$  covering  $\Leftrightarrow \pi$  local diffeo  $\Leftrightarrow D\pi$  surjective)

3) FACT If  $M, N$  compact mfd's of same dimension,  $N$  connected then  $M \xrightarrow{\pi} N$  covering  $\Leftrightarrow \pi$  local diffeo  $\Leftrightarrow D\pi$  surjective

Lemma For  $\pi: H \rightarrow G$  Lie group hom and covering, then

i)  $\ker \pi = \pi^{-1}(1)$  is a discrete closed normal subgp of  $H$

ii) fibres are homeomorphic to  $\ker \pi$

iii) for small enough nbhd  $V$  of  $1 \in H$ ,  $\bigsqcup_{k \in \ker \pi} k \cdot V \rightarrow U = \pi(V)$  are the sheets over  $U$

iv)  $H/\ker \pi \cong G$  (by pt iso theorem)

Pf for (ii):  $\pi^{-1}(g) = h \cdot \ker \pi$  if  $\pi(h) = g$

for (iii): pick  $U \ni 1$  as in definition of covering,  $V = \tilde{U}$  sheet containing  $1$ .  $\tilde{U}'$  another sheet  $\Rightarrow \tilde{U}' \cap \ker \pi = \{k\}$  some  $k \Rightarrow \tilde{U}' = kV$

Theorem 1  $H \xrightarrow{\pi} G$  Lie gp hom,  $G$  connected, then

$\pi$  covering  $\Leftrightarrow D\pi: \mathfrak{h} \rightarrow \mathfrak{g}$  isomorphism

Pf of " $\Rightarrow$ ":  $\pi$  covering  $\Rightarrow \pi$  local diffeo near  $1 \Rightarrow D\pi: T_1 H \rightarrow T_1 G$  iso

Pf of " $\Leftarrow$ ":  $D\pi$  iso  $\Rightarrow D\pi$  is  $\text{locally$  iso (lecture 5)

think of this as a continuous family of paths  $\gamma_s = F(\cdot, s)$

$\Rightarrow \pi$  local diffeo  $\Leftrightarrow \pi^{-1}(1) = \ker \pi$  discrete

inverse function  $\Leftrightarrow \text{image}(\pi)$  is subgp of  $G$  containing nbhd of  $1$  ( $G$  connected)  $\Leftrightarrow \pi$  surjective (2)

Trick  $H \times H \xrightarrow{(h, \ell)} H \cdot \ell$  smooth  $\Rightarrow \exists$  nbhd  $V$  of  $1 \in H$  with  $(V^{-1} \cdot V) \cap \ker \pi = \{1\}$

Claim  $\bigsqcup_{h \in \ker \pi} h \cdot V \xrightarrow{\pi} g \cdot \pi(V) = \text{nbhd of } 1 \in G$  whenever  $\pi(h) = g$  (use (2))

Pf similar to pf of Lemma, in particular:  
 $h \cdot V \xrightarrow{\pi} g \cdot \pi(V)$  local diffeo ✓  
surjective ✓  
injective since:  $\pi(v_1) = \pi(v_2) \Rightarrow \pi(\underbrace{v_1 - v_2}_{\in V^{-1} \cdot V}) = 1$  so  $v_1 - v_2 = 1$

✓ SIMPLY-CONNECTED GROUPS

A path-connected manifold is simply-connected if continuous maps  $S^1 \xrightarrow{f} M$  are contractible (connected mfd)  
(see Lecture 2)

meaning:  $\exists$  continuous  $F: S^1 \times [0, 1] \rightarrow M$

EXAMPLES

$\mathbb{R}^n: F(x, t) = (1-t)x$   
• convex subsets of  $\mathbb{R}^n$

$\mathbb{R}^n: F(x, t) = t x_0 + (1-t)f(x)$   
•  $x_0$

$\mathbb{R}^n: F(x, t) = \text{constant}$

•  $S^n \subseteq \mathbb{R}^{n+1}$  spheres ( $n \geq 2$ )  $\leftarrow$  FACT it's simply connected

•  $SU(2) \cong S^3$   
•  $SU(n), SL(n, \mathbb{C})$   $\leftarrow$  FACT simply connected.

NON-EXAMPLES  $S^1, \mathbb{T}^n, SO(n), U(n), SL(n, \mathbb{R})$  ( $n > 2$ ),  $GL(n, \mathbb{C})$

A covering  $M \rightarrow N$  is called universal cover if  $M$  is simply connected.

FACT Universal covers exist and are unique up to diffeomorphism.

sketch of existence (Non-examable)

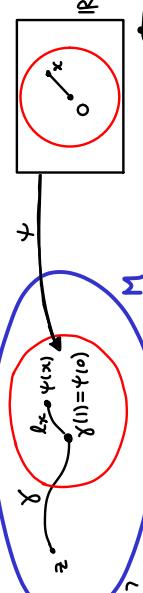
Fix  $z \in N$   $M = \{[\gamma]: \gamma: [0, 1] \rightarrow N \text{ continuous path, } \gamma(0) = z\}$

equivalence class: identify paths  $\gamma_0, \gamma_1$  if can continuously deform  $\gamma_0$  to  $\gamma_1$  keeping endpoints fixed

$F: [0, 1] \times [0, 1] \xrightarrow{\text{cont}} N$ ,  $F(\cdot, 0) = \gamma_0, F(\cdot, 1) = \gamma_1, F(0, \cdot) = z, F(1, \cdot) = \gamma_1(1) = \gamma_0(1) = \gamma_0(1)$

$M \xrightarrow{\pi} N, \pi([x]) = y(1) = \text{expoint of } y$

Rmk A local parametrization  $\mathbb{R}^n \xrightarrow{\psi} N$  near  $\psi(0)=y(1)$  also parametrizes  $M$  near  $[x]$  via  $[\gamma \# \ell_x] \xrightarrow{\psi} \text{straight line segment } \gamma(tx)_{0 \leq t \leq 1}$  to  $y$  so  $\pi[\gamma \# \ell_x] = \psi(x)$ .



Why  $M$  simply-connected?

IDEA: a loop  $S' \rightarrow M, s \mapsto [\delta_s]$  corresponds to a picture of form so just contract it down to  $\pi$  by  $F(s, r) = [\text{path } t \mapsto \delta_s((1-r)t)]$ .

Lemma For  $G$  Lie group, the universal cover  $\tilde{G}$  is a Lie group in a natural way so that  $\tilde{G} \xrightarrow{\pi} G$  is surj. Lie group hom, hence  $\text{Lie}(\tilde{G}) \xrightarrow{\text{D}, \pi} \text{Lie } G$  Lie algebra iso also  $\tilde{G}/\ker \pi \cong G$ .

Proof In the construction of  $\tilde{G}$  above pick  $\underline{z} = 1$ .

Unit:  $\tilde{1} = [\text{constant path at 1}]$  notice via  $\pi$  there are the multiplication:  $[\gamma_1] \cdot [\gamma_2] = [\text{path } t \mapsto \gamma_1(t) \cdot \gamma_2(t)]$  operations in  $N: \gamma_1(1) \cdot \gamma_2(1)$  and  $\gamma_1(1)^{-1}$   $\Rightarrow$  locally, in above parametrizations the operations are smooth since smooth in  $N$ .

EXAMPLES OF  $\tilde{G} \rightarrow G$

- $\mathbb{R} \xrightarrow{\exp} S'$  and  $\mathbb{R}^n \xrightarrow{\exp} T^n$

$SU(2) \rightarrow SO(3)$  Q.sheet 2

FACT  $\widetilde{SO(n)}$  definition of spin group for  $n \geq 3$  (example:  $\text{Spin}(3) \cong SU(2)$ )  
 $\text{Spin}(n) = \widetilde{SO(n)}$  (FACT  $\text{Spin}(n) \rightarrow SO(n)$  is a double cover)

FACT "Can't make universal covers any larger":

$\pi: \tilde{M} \rightarrow M$  covering,  $M$  simply conn.  $\Rightarrow \pi$  diffeo

Non-examinable proof idea: suffices to show fibre  $\pi^{-1}(m)$  is a point.

By contradiction, if  $\tilde{m}_1, \tilde{m}_2 \in \pi^{-1}(m)$ , a curve connecting  $\tilde{m}_1, \tilde{m}_2$  would give a contractible loop in  $M$  via  $\pi$ . Lifting this contraction to  $\tilde{M}$  then shows  $\tilde{m}_1 = \tilde{m}_2$ .

Cor  $\tilde{G} \xrightarrow{\pi} G$  Lie grp hom + covering  $\Rightarrow \pi$  Lie group iso connected simply connected (and connected)

CORRESPONDENCE BETWEEN LIE ALGS. & LIE GP. HOMOMORPHISMS

Theorem If  $H$  is simply-connected (and connected),  
 $\begin{cases} \text{Lie algebra homs} \\ \beta \xrightarrow{\psi} g \end{cases} \xleftrightarrow{1:1} \begin{cases} \text{Lie group homs} \\ H \xrightarrow{\varphi} G \end{cases}$

Proof Recall Lecture 5: a Lie gp hom  $H \xrightarrow{\varphi} G$  is uniquely determined by  $D_\varphi$  remains to show existence.

Remains to show existence.

$\psi: \mathfrak{g} \rightarrow g \Rightarrow \text{graph } \Gamma = \{(x, \psi(x)) : x \in \mathfrak{g}\} \subseteq \mathfrak{g} \times g$  is a Lie subalgebra since  $\psi$  Lie alg hom.

(Chevalley) corresponds to a connected Lie subgroup  $S \subseteq H \times G$ ,  $\text{Lie}(S) = \Gamma$ .

Consider the projection  $H \times G \rightarrow H$  Lie grp hom  $\Rightarrow \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{g}$  Lie alg hom

Observe:  $\pi: S \subseteq H \times G \rightarrow H$  has  $D_\pi: \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{g}$  isomorphism  $(x, \psi(x)) \xrightarrow{\pi} x$

(Theorem 1)

$\Rightarrow \pi: S \rightarrow H$  covering (Corollary)  
 $\Rightarrow \mathfrak{g} \xrightarrow{\pi^{-1}} S \subseteq H \times G$  project  $\hookrightarrow G$  Lie grp hom inducing  $\varphi$  since:

$$\begin{aligned} x &\xrightarrow{\pi} (\pi(x), \psi(x)) \xrightarrow{\pi^{-1}} \mathfrak{g} \\ &\xrightarrow{\psi(x)} \mathfrak{g} \end{aligned}$$

Cor  $H, G$  simply-connected Lie groups with  $\mathfrak{g} \cong \mathfrak{g}$   $\Rightarrow H \cong G$  iso Lie groups

Pf  $\mathfrak{g} \xrightarrow{\varphi} \mathfrak{g}$  gives a unique  $H \xrightarrow{\varphi} G$   
 $\mathfrak{g} \xrightarrow{\varphi^{-1}} \mathfrak{g}$  " " $G \xrightarrow{\varphi^{-1}} H$   
 $\mathfrak{g} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\varphi^{-1}} \mathfrak{g}$  " " $H \rightarrow H$  which must be both  $\varphi' \circ \varphi = \text{id}$  and identity  $\} \Rightarrow \varphi' \circ \varphi = \text{id} \Rightarrow \varphi \circ \varphi^{-1} = \text{id}$  ■

FACT Ado's theorem For any Lie algebra  $V$ , there is an injective Lie algebra hom  $V \rightarrow gl(m, \mathbb{R})$ , some  $m$ .

Lie's third theorem

There is a 1-to-1 correspondence  
 $\begin{cases} \text{Lie algebras} \\ \text{isos} \end{cases} \xleftrightarrow{1:1} \begin{cases} \text{simply-connected} \\ \text{Lie groups} \end{cases} \xleftrightarrow{\text{Lie grp isos}}$

Pf By Cor, get uniqueness of  $G$  (up to iso) for given  $\mathfrak{g}$  (up to iso)  
Remains to show existence of  $G$  given Lie algebra  $V$ .

Ado's thm  $\Rightarrow V \subseteq gl(m, \mathbb{R})$  Lie subalg

(chevallier)  $\exists$  connected Lie subgp  $H \subseteq GL(m, \mathbb{R})$  with  $\mathfrak{g} = V$   
 $\Rightarrow$  take  $G = \tilde{H}$  universal cover: simply connected and  $\mathfrak{g} \cong \mathfrak{g} = V$  ■

## LECTURE 11

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### REPRESENTATION THEORY

Lecture 5: Representation  $V$  of Lie group  $G$  means :

Continuous hom  $G \rightarrow \text{Aut}(V)$  where  $V$  vector space

- Rmk 5.1
- we work over field  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$
  - always assume  $V$  finite-dimensional v.s. /  $\mathbb{F}$ ,  $d = \dim_{\mathbb{F}}(V)$
  - 2)  $\text{Aut}(V) = \{\text{linear bijections } V \rightarrow V\}$ . If pick basis of  $V$ ,  $\text{Aut}(V) \cong \text{GL}(d, \mathbb{F})$
  - 3) Recall continuous hom  $\Rightarrow$  smooth hom

Lemma Equivalent definition of rep :

Group action  $G \times V \rightarrow V$  continuous in  $G$ , linear in  $V$

Explicitly:

- $1 \cdot v = v$  all  $v \in V$
- $(g_1 g_2) \cdot v = g_1 \cdot (g_2 \cdot v)$  all  $g_1, g_2 \in G, v \in V$

For this reason, we often call  $V$  a  $G$ -module or  $G$ -mod

Pf Action determines  $G \xrightarrow{g} \text{Bijections}(V, V)$ ,  $g \mapsto (v \mapsto g \cdot v)$

continuous in  $G$  and linear in  $V$ , hence  $G \xrightarrow{g} \text{Aut}(V)$  rep.

Conversely, for rep  $G \xrightarrow{\varphi} \text{Aut}(V)$  define action  $g \cdot v = \varphi(g)(v)$

Lecture 5: Representation  $V$  of Lie algebra  $\mathfrak{g}$  means :

Lie algebra hom  $\mathfrak{g} \rightarrow \text{End}(V)$  where  $V$  vector space

using the bracket  $[\alpha, \beta] = \alpha \beta - \beta \alpha$  on  $\text{End}(V)$

Rmk  $\text{End}(V) = \{\text{linear maps } V \rightarrow V\}$ . If pick basis of  $V$ ,  $\text{End}(V) \cong \text{Mat}_{d \times d}(\mathbb{F})$

Lecture 10  $\Rightarrow$  For  $G$  simply-connected (and connected) Lie gr:

$\{\text{Lie gr reps } G \rightarrow \text{Aut}(V)\} \xleftarrow{1:1} \{\text{Lie alg reps of } \text{End}(V)\}$

Q.5 Questionsheet 4 (case  $SU(2) \rightarrow SO(3)$ ) generalizes to universal covers  $\tilde{G} \xrightarrow{\pi} G$ :

For  $G$  connected Lie gr:

$\{\text{Lie gr reps } \tilde{G} \xrightarrow{\pi} \text{Aut}(V)\} \xleftarrow{1:1} \{\text{Lie alg reps of } \text{End}(V)\}$

For rep  $G \xrightarrow{\varphi} \text{Aut}(V) \Rightarrow$  get rep  $\tilde{\varphi} = \varphi \circ \pi: \tilde{G} \xrightarrow{\pi} G \xrightarrow{\varphi} \text{Aut}(V)$

$\Rightarrow$  forces  $\ker \pi \subseteq \ker \tilde{\varphi}$  (i.e.  $\ker \pi \subseteq \tilde{G}$  acts by Id on  $V$ )

$\Rightarrow \{\text{Lie gr reps } \tilde{G} \rightarrow \text{Aut}(V)\} \xleftarrow{1:1} \{\text{Lie gr reps } G \rightarrow \text{Aut}(V) \text{ with } \ker \pi \subseteq \ker \tilde{\varphi}\}$

### EXAMPLES

- Trivial representation  $G \rightarrow \text{Aut}(V)$ ,  $g \mapsto \text{Id}$
- Adjoint representation  $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$

Def faithful rep means injective rep.

In practice it means you can identify  $G$  or  $\mathfrak{g}$  with a subset of matrices.

### EXAMPLES

#### standard representations

$$\begin{cases} O(n) \rightarrow \text{Aut}(\mathbb{R}^n) \\ U(n) \rightarrow \text{Aut}(\mathbb{C}^n) \\ \dots \end{cases} \quad \begin{cases} \text{given by left-multiplication:} \\ A \longmapsto (\mathbf{v} \mapsto A \cdot \mathbf{v}) \end{cases}$$

Def  $V, W$   $G$ -mod, a  $G$ -linear map (or  $G$ -mod homomorphism) means

- $G$ -linear map  $f: V \rightarrow W$
- $f$  commutes with  $G$ -action:  $f(g \cdot v) = g \cdot f(v)$

$\text{Hom}_G(V, W) = \{G\text{-linear maps } f: V \rightarrow W\}$

Def  $V, W$  equivalent reps if  $\exists$   $G$ -isomorphism  $f: V \rightarrow W$   
 $\Downarrow$  (bijective  $G$ -linear map)

Explicitly:

$$f: G \rightarrow \text{Aut}(\mathbb{R}^n) = GL(n, \mathbb{R}) \quad \text{reps}$$

$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  iso  $\Rightarrow$  given by invertible matrix  $F$

$f: \text{equivalent} \Leftrightarrow f(\varphi_1(g)v) = \varphi_2(g)f(v) \Leftrightarrow \varphi_1(g) = F^{-1} \cdot \varphi_2(g) \cdot F$

$\Leftrightarrow \varphi_1(g)$  are conjugate via an iso  $F$ , for all  $g$

### INVARIANT INNER-PRODUCTS

Def For a rep  $V$ , an innerproduct (Hermitian i.p. if  $\mathbb{F} = \mathbb{C}$ )  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$  is  $G$ -invariant if  $\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle$  all  $g \in G, v, w \in V$

and call  $V$  an orthogonal rep ( $\mathbb{F} = \mathbb{R}$ ) or unitary rep ( $\mathbb{F} = \mathbb{C}$ ).

Explicitly If pick o.n./unitary basis  $e_i$  for  $V$

$$\begin{aligned} &\Leftrightarrow \langle \sum a_i e_i, \sum b_j e_j \rangle = a_i^* b_j \quad (a_i, b_j \in \mathbb{F}^n \text{ and } * = \text{conjugate transpose}) \\ &\Leftrightarrow \langle \varphi(g)^* \varphi(g)w, w \rangle = \langle \varphi(g)v, \varphi(g)v \rangle \quad \forall g \in G, w \in V \\ &\Leftrightarrow \varphi(g)^* \varphi(g) = \text{Id} \quad \text{so } \varphi(g) \text{ is an orthogonal/unitary matrix} \\ &\Leftrightarrow \varphi: G \rightarrow O(n) \subseteq GL(n, \mathbb{R}) \cong \text{Aut}(V) \end{aligned}$$

$\Rightarrow \mathbb{F} = \mathbb{R}: \varphi: G \rightarrow O(n) \subseteq GL(n, \mathbb{R}) \cong \text{Aut}(V)$

$\Rightarrow \mathbb{F} = \mathbb{C}: \varphi: G \rightarrow U(n) \subseteq GL(n, \mathbb{C}) \cong \text{Aut}(V)$

## NEW FROM OLD

Let  $V, W$  be  $G$ -mods : we want to build new  $G$ -mods from  $V, W$

- 1) direct sum  $V \oplus W$
- 2) tensor product  $V \otimes_W W$

Recall :  $V \otimes W$  is a vector space of dimension  $\dim V \cdot \dim W$ .

We use the symbols  $v_i \otimes w_j$  to denote a basis of  $V \otimes W$ , whenever  $\{v_i\}$  is a basis of  $V$

Example  $R^n \otimes R^m \cong R^{nm}$

It is convenient to extend the symbol  $\otimes$  to any vectors by declaring that:  $\sum v_i \otimes w = 0$  if  $v_i, w = 0$

Warning: not all vectors in  $V \otimes W$  arise as  $v \otimes w$ ! you need to allow sums  $\sum v_i \otimes w_i$ :

For example in  $R^2 \otimes R^2$ ,  $e_1 \otimes e_2 + e_2 \otimes e_1 \neq v \otimes w$  for any  $v, w \in R^2$ .

A linear map  $\varphi: V \otimes W \rightarrow U$  is determined by its values on generators,  $\varphi(v \otimes w)$ , since that determines  $\varphi$  on the basis  $v_i \otimes w_j$ . Conversely, if you define  $\varphi(v \otimes w)$  in a way that is linear in  $v$  and in  $w$ , then  $\varphi$  extends to a well-defined linear map  $\otimes: W \rightarrow U$

3)  $f \in \text{Hom}_G(V, W) \Rightarrow$  get  $G$ -mods :  $\text{Ker } f = \{v \in V \mid f(v) = 0\}$

$$\text{Im } f = \{w \in W \mid \exists v \in V \text{ s.t. } f(v) = w\}$$

$$\text{Coker } f = W / \text{Im } f = \{w \in W \mid \exists v \in V \text{ s.t. } f(v) = w\}$$

4) conjugate space  $\bar{V} := V$ , as sets, use same  $G$ -action, but change the  $\mathbb{C}$ -action:

$$x \cdot v = \bar{x}v \text{ for } x \in G \quad (\text{for } \mathbb{F} = \mathbb{C})$$

5) dual space  $V^* = \{\mathbb{F}\text{-linear } V \rightarrow \mathbb{F}\}$

Rmk : Need inverse  $g^{-1}$  to make it a left-action (check axiom (iii))

Another reason is that you want the following diagram to commute:

$$\begin{array}{ccc} V & \xrightarrow{\psi} & W \\ \downarrow g^{-1} & \nearrow \text{id} & \downarrow f \\ V & \xrightarrow{\psi} & W \end{array}$$

6) hom space  $\text{Hom}_{\mathbb{F}}(V, W) = \{\mathbb{F}\text{-linear } V \xrightarrow{\psi} W\}$

This ensures that the following diagram commutes :

$$\begin{array}{ccc} V & \xrightarrow{\psi} & W \\ \downarrow g^{-1} & \nearrow \text{id} & \downarrow f \\ V & \xrightarrow{\psi} & W \end{array}$$

Lemma  $V^* \otimes_{\mathbb{F}} W \cong \text{Hom}_{\mathbb{F}}(V, W)$   $G$ -iso

Pf  $\psi \otimes w \mapsto \left( \varphi: V \rightarrow W \right) \text{ and extend bilinearly in } \varphi \text{ and } w.$

Linear by construction.  $\checkmark$

• Bijective: pick basis  $v_i$  of  $V$ , get dual basis  $v_i^*$  so  $v_i^*(v_j) = \delta_{ij}$ . Pick basis  $w_i$  of  $W$ .

Then  $v_i^* \otimes w_j$  is a basis for  $V^* \otimes W$ . The corresponding  $\varphi$  maps are:  $\varphi_{ij}: V \rightarrow W$  with  $\varphi_{ij}(v_k) = v_i^*(v_k) w_j$  if  $i=k$  so  $\varphi_{ij}$  is the matrix with  $\{1\}$  in position  $(i, j)$  and  $0$  in all other positions. These matrices are a basis for  $\text{Hom}_{\mathbb{F}}(V, W)$ .  $\checkmark$

• preserves  $G$ -action:  $(g \cdot (\psi \otimes w)) \cdot v = ((g\psi) \otimes (gw)) \cdot v = (g\psi)(v) \cdot gw = g(\psi(g^{-1}v)) \cdot gw = (g \cdot \varphi)(v) \cdot gw$ .  $\checkmark$

## REDUCIBILITY

Def A  $G$ -submodule or subrepresentation of  $V$  is a  $G$ -invariant vector subspace  $W \subseteq V$ . Call  $V$  reducible if  $\exists$  subrep  $W \neq 0, V$  irreducible if  $W = 0, V$  are the only subreps.  $\leftarrow$  call  $V$  irreducible

### EXAMPLES

$G = S^1$  acts on  $\mathbb{C}^2$  by  $e^{i\theta}: (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$ . This is reducible:  $\mathcal{C} \cdot \{1\} \subseteq \mathbb{C}^2$  is invariant

$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R}, a, c \neq 0 \right\}$  acts naturally on  $\mathbb{R}^2$  ( $v \mapsto Av$  for  $v \in \mathbb{C}^2, A \in G$ )

But  $\mathbb{R}^2$  is reducible :  $W = \mathbb{R} \cdot \{0\}$  is a  $G$ -submod since  $G \cdot W \subseteq W$ .

$G = U(2)$  acting on  $\mathbb{C}^2$  is irreducible : if  $V \otimes \mathbb{C}^2$ , extend  $u_i = \frac{v_i}{\|v_i\|}$  to a unitary basis  $u_1, u_2$ . Then  $A = (u_1, u_2) \in U(2)$ , and  $Ae_1 = u_1 = \frac{v_1}{\|v_1\|}e_1$ . Similarly for  $B = (u_2, u_1)$ ,  $B^{-1} \cdot v = \frac{1}{\|v\|}e_2$ . So if  $W \subseteq \mathbb{C}^2$  is a subrep and  $V \otimes W \in W$ , then  $G \cdot W \subseteq W$ , in particular  $A^{-1} \cdot v, B^{-1} \cdot v \in W$ , so  $e_1, e_2 \in W$ , so  $W = \mathbb{C}^2$  (since it is a vector subspace,  $\text{span}(e_1, e_2) \subseteq W$ )

Schur's Lemma  $f \in \text{Hom}_G(V, W)$  for  $V, W$  irreps  $\Rightarrow f \text{ iso or } f = 0$

Schur's Lemma over  $\mathbb{C}$   $f \in \text{Hom}_{\mathbb{C}}(V, V)$  for  $V$  irrep  $\Rightarrow f = \lambda \cdot \text{Id}$  some  $\lambda \in \mathbb{C}$

Pf  $\mathbb{F} = \mathbb{C} \Rightarrow \exists \lambda \text{ eigenvalue of } f$

$\Rightarrow f - \lambda \cdot \text{Id} \in \text{Hom}_{\mathbb{C}}(V, W)$  with non-zero kernel

Schur  $f - \lambda \cdot \text{Id} = 0$   $\blacksquare$

Car  $V, W$  irreps  $\Rightarrow \left\{ \text{Hom}_G(V, W) = 0 \text{ if } V, W \text{ not equivalent} \right.$

$\left. \dim \text{Hom}_G(V, W) \geq 1 \text{ if } V, W \text{ equivalent} \right)$

Abbreviate  $nV = V \underset{n \text{ copies}}{\oplus} V$

Theorem  $V_i$  non-equivalent irreps, then  $\oplus m_i V_i \cong \oplus n_i V_i \iff m_i = n_i$  (we assume only finitely many  $m_i, n_i$  are non-zero)

Pf  $\Rightarrow: \text{Hom}(V_k, \oplus m_i V_i) \cong \text{Hom}(V_k, \oplus n_i V_i)$

Schur  $\oplus m_i \text{Hom}_G(V_k, V_i) \cong \oplus n_i \text{Hom}_G(V_k, V_i)$

IDEA: compare prime factorizations over  $\mathbb{N}$ .

$$\begin{array}{c} \oplus m_i \text{Hom}_G(V_k, V_i) = 0 \quad \text{unless } i=j \\ \text{unless } i=j \quad m_k = n_k \end{array}$$

Take dimensions  $\Rightarrow m_k = n_k \blacksquare$

COMPLETE REDUCIBILITY

Def A rep  $V$  is completely reducible if  $V = \bigoplus_i V_i$  is a sum of irreps.

Question Is every rep a sum of irreps?

Answer No:  $G = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \alpha \in \mathbb{C}$  (abelian!).

$V = \mathbb{C}^2$  is reducible, and subreps are:

- $O, W = \{(x) : x \in \mathbb{C}\}, V$
- but  $V \not\cong W \oplus \text{irrep}$

More details:  
 If  $\begin{pmatrix} x \\ y \end{pmatrix} \in \text{subrep } W$  then:  
 $\begin{pmatrix} 1 & -\frac{x}{y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} \in W$   
 $\text{so } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} \in W$   
 but  $W$  is subspace, so  
 $\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 1-y \end{pmatrix} \in W$   
 $\text{span } \mathbb{C}^2 \text{ so } W = \mathbb{C}^2$

Lemma For  $\mathbb{F} = \mathbb{C}$ ,  $G$  abelian  $\Rightarrow$  irreps are 1-dimensional

Pf For  $V$  irrep,  $G$  abelian, the multiplication  $\phi_g: V \rightarrow V, \phi_g(v) = gv$  is  $G$ -linear:  
 $\phi_g(g'v) = g'gv = g'g v = g' \phi_g(v)$

$$\begin{aligned} \phi_g &= \lambda_g \cdot \text{Id} \quad \text{some } \lambda_g \in \mathbb{C}. \\ \Rightarrow \phi_g(\mathbb{C}v) &= \lambda_g \cdot \mathbb{C}v = \mathbb{C}v \quad \text{for all } g \in G \\ \Rightarrow \mathbb{C}v &\text{ subrep of } V, \text{ so } V = \mathbb{C}v \text{ (since irrep)} \end{aligned}$$

AIM: prove that for compact  $G$ , reps are completely reducible.  
 The proof for finite groups  $G$  is as follows:

① Given an inner product  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$  can produce a  $G$ -invariant inner product  $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$  by averaging

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} (g \cdot v, g \cdot w).$$

② If  $V$  irrep, done ✓  
 If  $V$  reducible, let  $W \subseteq V$  be a subrep  $\neq 0, V$ . Then  $W^\perp$  is a subrep

$$\begin{aligned} \text{proof: } \langle g v, w \rangle &= \langle g v, g g^{-1} w \rangle = \langle v, g^{-1} w \rangle \\ \Rightarrow g v \in W^\perp &\quad \text{G-invar} \quad \text{G-invar} \quad \text{G-invar} \end{aligned}$$

③ Induction on  $\dim V \Rightarrow$  can completely reduce  $W, W^\perp \Rightarrow$  can completely reduce  $V = W \oplus W^\perp$ .

Theorem If  $V$  admits a  $G$ -inv'tl inner product, then  $V$  is completely reducible

Proof By steps ② & ③ above. ■

↙ PROOF NON-EXAMINABLE - SEE NON-EXAMINABLE HAND-OUT (are limits of Riemann sums)

Theorem  $G$  compact Lie group  
 $\Rightarrow \exists$  unique normalized left-invariant integral over  $G$ , meaning:  
 To any continuous function  $f: G \rightarrow \mathbb{R}$  it associates a value  $\int_G f = \int_G f(g) dg \in \mathbb{R}$

such that •  $\int_G 1 = 1$   
 • If  $f > 0$  then  $\int_G f > 0$   
 •  $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2$  for  $\lambda \in \mathbb{R}$   
 •  $\int_G f \circ \phi_g = \int_G f(hg) = \int_G f$  (left-invariance)

Rmks 1) linearity + positivity  $\Rightarrow$  monotonicity:  
 • if  $f > g$  then  $\int_G f > \int_G g$

(using normalization, this also holds with " $\geq$ "). Proof: if  $f > -\epsilon$  so  $\int_G f > -\epsilon \int_G 1 = -\epsilon$ )

2) In fact the integral is bi-invariant, i.e. left and right invariant. Indeed it satisfies (right-invariance)

•  $\int_G f(gh) = \int_G f$   
 •  $\int_G f(g^{-1}) = \int_G f$  (inversion-invariance)  
 •  $\int_G f \circ \phi_g = \int_{G_1} f$  for ch.  $f: G_2 \rightarrow \mathbb{R}$  and Lie gp iso  $\varphi: G_1 \rightarrow G_2$   
 (isomorphism-invariance)

3) Can integrate continuous maps  $f: G \rightarrow \mathbb{F}^d$  by integrating in each entry so  $\int_G f$  exists:  
 $\Rightarrow$  " " " "  $f: G \rightarrow V$  (pick basis, so  $V \cong \mathbb{F}^d$ ) so  $\int_G f \in V$

Exercise: LINEARITY  $\varphi: V \rightarrow V$  linear  $\Rightarrow \int_G \varphi \circ f = \varphi \left( \int_G f \right)$   
 linearity  $\Rightarrow$  (V any v.s.)  $f: G \rightarrow V$

Corollary  $G$  compact  $\Rightarrow$  for any rep  $V$  there is a  $G$ -inv'tl inner product  $\langle \cdot, \cdot \rangle$ , all reps are completely reducible

Pf Pick any inner product  $\langle \cdot, \cdot \rangle$  on  $V$  (e.g. standard i.p. on  $\mathbb{F}^d \cong V$ )  
 Define  $\langle v, w \rangle = \int_G (g \cdot v, g \cdot w)$

Linear in each entry since  $G$ -action linear, (.) bilinear, and  $\int_G$  is linear.  
 Positive definite since  $\langle v, v \rangle = \int_G (g \cdot v, g \cdot v) > 0$  using positivity of  $\int_G$ .

G-invariant: for  $h \in G$ ,  
 $\langle h \cdot v, h \cdot w \rangle = \int_G (gh \cdot v, gh \cdot w) = \int_G f(gh) = \int_G f$  (for  $v \neq 0$ )

Define  $f: G \rightarrow \mathbb{R}$   
 $f(g) = \langle gv, gw \rangle$  (fixed  $v, w$ )  
 $f(g) = \langle gv, gw \rangle$  (right-invariance)

## CHARACTERS

Def The character  $\chi_v = \chi_{\rho: G \rightarrow \mathbb{F}}$  of a rep  $\rho: G \rightarrow \text{Aut } V$  is

$$\chi_v(g) = \text{Tr}(\rho(g))$$

Q. sheet 5 :  $\chi_v$  smooth.

- $\chi_v(1) = \dim V$
- $\chi_v(h^{-1}gh) = \chi_v(g)$
- $V \cong W \Rightarrow \chi_v = \chi_w$
- $\chi_v \oplus \chi_w(g) = \chi_v(g) + \chi_w(g)$
- $\chi_{V \otimes W}(g) = \chi_v(g) \cdot \chi_w(g)$
- $\chi_{V^*}(g) = \chi_v(g^{-1})$
- $\chi_{\overline{V}}(g) = \overline{\chi_v(g)}$

Lemma 1 ( $\mathbb{F} = \mathbb{C}$ )  $G$  compact  $\Rightarrow \chi_v(g^{-1}) = \overline{\chi_v(g)}$

Pf  $\chi_v(g^{-1}) = \chi_{v^*}(g) = \chi_{\overline{V}}(g) = \overline{\chi_v(g)}.$   
 $v^* \approx \overline{V}$  Q.sheet 5, using  $G$  compact. ■

## FIXED POINTS

Def  $v \in V$  is a fixed point of the  $G$ -action if  $g \cdot v = v$  for all  $g \in G$ .

$$\Rightarrow V^G = \{\text{fixed points}\} \subseteq V \quad \text{subrep}$$

For finite groups  $G$  you build fixed points by averaging :

$$V^G = \left\{ \frac{1}{|G|} \sum_{g \in G} g \cdot w : w \in V \right\}$$

Proof:  $h \in G \Rightarrow h \cdot \left( \frac{1}{|G|} \sum_{g \in G} g \cdot w \right) = \frac{1}{|G|} \sum_{g \in G} hg \cdot w = \frac{1}{|G|} \sum_{g \in G} gw$  since  $\frac{1}{|G|} \sum_{g \in G} g \mapsto G$  bijection.

Conversely:  $v \in V^G \Rightarrow \frac{1}{|G|} \sum_{g \in G} g \cdot v = \frac{1}{|G|} \sum_{g \in G} v = \underbrace{\left( \frac{1}{|G|} \sum_{g \in G} 1 \right)}_{=1} \cdot v = v$  ■

Thm For compact Lie group  $G$ ,  $V^G = \left\{ \int_{g \in G} g \cdot w : w \in V \right\} \subseteq V$

If: For  $h \in G$ ,  $v \mapsto hv$  is linear  $\Rightarrow h \int_{g \in G} g \cdot w = \int_{g \in G} hg \cdot w = \int_{g \in G} g \cdot w$ .  
 $\subseteq: v \in V^G \Rightarrow \int_{g \in G} g \cdot w = \left( \int_{g \in G} 1 \right) \cdot v = v$  ■

Lemma 2  $\dim V^G = \int_{g \in G} \chi_v(g)$

$$\underline{\text{Pf}} \quad \int \chi_v(g) = \int \text{Tr}(\rho(g)) = \int_{g \in G} \chi_v(g)$$

LINEARITY  $\uparrow$  integrate each matrix entry  
(Rmk 3)

TRICK: averaging operator  $\varphi: V \rightarrow V$ ,  $\varphi(v) = \int g \cdot v$  is a projection onto  $V^G$  meaning  $\varphi^2 = \varphi$  (indeed if  $v \in V^G = \text{Image}(\varphi)$  then  $\varphi(v) = \int_{g \in G} g \cdot v = v$ )  
For projection maps,  $\text{Tr}(\varphi) = \dim_{\mathbb{F}} \text{Im}(\varphi)$ , since  $V = \text{Im}(\varphi) \oplus \text{Ker}(\varphi)$   
 $\Rightarrow \text{Tr}(\varphi) = \text{Tr}(\varphi) = \dim_{\mathbb{F}} \text{Im}(\varphi) = \dim_{\mathbb{F}} V^G$  ■

## ORTHOGONALITY RELATIONS

Theorem For compact Lie grp  $G$ ,  
 $\langle \chi_v, \chi_w \rangle > \underset{\text{define}}{\overline{\int_{g \in G} \chi_v(g) \chi_w(g)}} = \dim \text{Hom}_{\mathbb{C}}(V, W)$

Cor  $\langle \chi_v, \chi_w \rangle$  defines an inner product on  $\text{span}_{\mathbb{F}} \{ \chi_v : V \text{ rep} \}$

- $V_i$  non-equivalent irreps  $\Rightarrow \chi_{V_i}$  are orthogonal, so linearly independent

Lecture 11 :  $V \cong W$  irreps  $\Rightarrow \int \chi_v \cdot \chi_w = \begin{cases} 1 & \text{if } \mathbb{F} = \mathbb{C} \\ 0 & \text{if } \mathbb{F} = \mathbb{R} \end{cases}$   
(Cor of Schur)

Pf Then Work over  $\mathbb{F} = \mathbb{C}$  (if  $\mathbb{F} = \mathbb{R}$  just think of  $G \rightarrow \text{Aut}(\mathbb{C}^n) \subseteq \text{Aut}(\mathbb{C}^n)$ )

TRICK  $\text{Hom}_G(V, W) = \text{Hom}_{\mathbb{C}}(V, W)^G$   
recall  $(g\varphi)(v) = (g \circ \varphi \circ g^{-1})(v) \Rightarrow g\varphi = \varphi \Leftrightarrow \varphi(gv) = g\varphi(v) \quad \forall v \in V$   
 $\Rightarrow \dim \text{Hom}_G(V, W) = \int \chi_{\text{Hom}_{\mathbb{C}}(V, W)}(g)$  (Lemma 2)  
 $= \int \chi_{V^*}(g) \chi_W(g)$  (Lecture 11 :  $\text{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$   
and use properties of characters)  
 $= \int \overline{\chi_V(g)} \chi_W(g)$  (Lemma 1) ■

Then  $G$  compact  $\Rightarrow$  any rep is determined uniquely (up to equivalence) by character

Pf  $V \cong \bigoplus n_i V_i \Rightarrow \chi_v = \sum n_i \chi_{V_i} \Rightarrow n_i = \frac{\langle \chi_v, \chi_{V_i} \rangle}{\langle \chi_{V_i}, \chi_{V_i} \rangle} = \frac{\langle \chi_v, \chi_{V_i} \rangle}{\dim V_i}$   
complete reducibility  $\Rightarrow$  irreps orthogonality relns  $\dim V_i = 1 \quad \mathbb{F} = \mathbb{R}$

## LECTURE 12

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### NON-EXAMINABLE HANDOUT

Def An  $n$ -form  $\omega$  on an  $n$ -dimensional manifold  $M$  is a function

$$\omega|_P : \underbrace{T_p M \times \dots \times T_p M}_{n \text{ copies}} \rightarrow \mathbb{R} \quad \begin{matrix} \text{eats } n \text{ vectors at } p \\ \text{and splits out a number} \end{matrix}$$

(1) MULTI-LINEAR: linear in each entry

(2) ALTERNATING: switches sign if you transpose two entries

(3) SMOOTH in  $p$ : if  $X_1, \dots, X_n$  are smooth v.f. then  $M \rightarrow \mathbb{R}$ ,  $P \mapsto \omega(X_1, \dots, X_n)|_P$  is smooth.

Lemma 1 Locally, in coordinates  $x_1, \dots, x_n$  after identifying  $T_x \mathbb{R}^n \cong \mathbb{R}^n$ :

$\omega = g(x) dx_1 \wedge \dots \wedge dx_n$  where  $g(x) = \omega\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$  is a smooth function and  $dx_1 \wedge \dots \wedge dx_n : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $(v_1, \dots, v_n) \mapsto \det(v_1, \dots, v_n)$

If Recall  $\exists$  unique  $\delta : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{=n \times n \text{ matrix}} \rightarrow \mathbb{R}$  satisfying (1), (2),  $\delta(e_i, e_2, \dots, e_n) = 1$  (namely  $\delta = \det$ )

Why that notation? ( $\leftarrow$  more on this in the Appendix)

$$\mathbb{R}^n \equiv T_x \mathbb{R} \quad \text{dual vector space} \quad (\mathbb{R}^n)^* \equiv T_x^* \mathbb{R} \quad e_i^* \equiv dx_i$$

$$\Rightarrow dx_i : \left(\frac{\partial}{\partial x_j}\right) = e_i^*(e_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\Rightarrow X = \sum \underbrace{dx_i(X)}_{\text{coefficients}} \cdot \frac{\partial}{\partial x_i} \quad \text{for any vector field } X$$

$$\stackrel{(1)}{\Rightarrow} (dx_1 \wedge \dots \wedge dx_n)(X_1, \dots, X_n) = \det(dx_i(X_j))$$

Example  $\mathbb{R}^2$ :  $dx_1 \wedge dx_2 \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) = 1$  (identity matrix)  
 $dx_1 \wedge dx_2 \left( \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$  (as expected by (2))

Def If  $\varphi : M \rightarrow N$  smooth then can pull-back  $n$ -forms from  $N$  to  $M$ :

$$(\varphi^* \omega_N)|_{(v_1, \dots, v_n)} = \omega_N \Big|_{\underbrace{v_i \in T_x M}_{\varphi(x)}}$$

Rmk  $\varphi_1^* \circ \varphi_2^* = (\varphi_2 \circ \varphi_1)^*$  by the chain rule.

Lemma 2 Locally  $\varphi^* (g(y) dy_1 \wedge \dots \wedge dy_n)|_x = g(\varphi(x)) \det D\varphi \cdot dx_1 \wedge \dots \wedge dx_n$

$$\text{Pf} \quad (\varphi^* g(y) dy_1 \wedge \dots \wedge dy_n) \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) = g(\varphi(x)) \det D\varphi \cdot \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

$$\stackrel{\text{Def}}{=} g(\varphi(x)) \det D\varphi \cdot \underbrace{\sum_k (\det D\varphi)_{jk} \frac{\partial}{\partial y_k}}_{(D\varphi)_{jj}} = g(\varphi(x)) \cdot \det D\varphi$$

Def A volume form  $\Omega$  on  $M^n$  is an  $n$ -form with  $\Omega_p \neq 0 \forall p \in M$   
 $\Rightarrow$  locally,  $g(x) \neq 0 \forall x$  in Lemma

Consequences:

4) either  $g > 0 \forall x$  or  $g < 0 \forall x$ .

$\Rightarrow$  can always ensure  $g > 0$  by composing the parametrisation with  $\mathbb{R}^n \cong \mathbb{R}^n$   
 $(x_1, x_2, \dots) \mapsto (-x_1, x_2, \dots)$

only use parametrisations for which  $g > 0$

5) All  $n$ -forms are given by smooth function  $f : M \rightarrow \mathbb{R}$   
 $\omega = f \cdot \Omega$   
 $\text{locally: } f(x) = \omega\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$

Lemma 3 transition  $\tau = \varphi^{-1} \circ \varphi$  between parametrisations as in (4)  $\Rightarrow \det(D\tau) > 0$

Pf  $\tau^* = \varphi^* \circ (\varphi^{-1})^* : dx_1 \wedge \dots \wedge dx_n \mapsto \frac{\Omega}{g(x)}$

$\det(D\tau) > 0$   $\downarrow$

$\det(D\varphi) \neq 0$  (Lemma 2)

Cor (4)  $\Rightarrow$  Locally can integrate  $n$ -forms

Pf For a parametrisation  $\varphi : V \xrightarrow{\sim} U \subseteq M$  as in (4),  
 $\varphi^* \omega = g(x) dx_1 \wedge \dots \wedge dx_n$

$$\Rightarrow \text{define } \int_U \omega = \int_V \psi^* \omega := \int_{V \subseteq \mathbb{R}^n} g(x) dx_1 \dots dx_n \quad \text{known by multivariable calculus}$$

## INTEGRATION ON COMPACT GROUPS

$\textcircled{*}$  Check independent of choice of  $\psi$  by comparing with  $\varphi: \tilde{U} \xrightarrow{\cong} U$ :

$$\int_{\tilde{U}} \tilde{g}(\tilde{x}) d\tilde{x}_1 \dots d\tilde{x}_n = \int_{U \subseteq \mathbb{R}^n} g(x) |\det \tau| dx_1 \dots dx_n$$

multivariable calculus: change of variables:  $\tilde{x} \mapsto x = \tau(\tilde{x})$

Problem:  $\tau^*(\tilde{g}(\tilde{x})) d\tilde{x}_1 \dots d\tilde{x}_n = g(x) \cdot \det \tau. dx_1 \dots dx_n$   
solution: Lemma 3  $\Rightarrow \det \tau = |\det \tau|$ , so O.K.

Cor (4) & M compact  $\Rightarrow$  Globally can integrate n-form

IDEA: add up local contributions:

$$\int_M \omega = \sum_i \int_{U_i} \omega \quad \left( \text{see Appendix for details: only want to count integral on overlaps } U_i \cap U_j \text{ once} \right)$$

Corollary If a manifold M has a volume form (such mfs are called orientable) and M compact then one can integrate functions  $f: M \rightarrow \mathbb{R}$ :

$$\int_M f = \int_M f \cdot \mathcal{R}$$

(sometimes write  $\left( \int_P f(p) \cdot \mathcal{R}|_P \right)$ )

Satisfies: LINEARITY:  $\int_M \lambda_1 f_1 + \lambda_2 f_2 = \lambda_1 \int_M f_1 + \lambda_2 \int_M f_2$

POSITIVITY:  $f \geq 0 \Rightarrow \int_M f \geq 0$  (and similarly with " $>$ ")

MONOTONE:  $f_1 \geq f_2 \Rightarrow \int_M f_1 \geq \int_M f_2$

CHANGE OF VARIABLES:  $\int_M \varphi^*(f \cdot \mathcal{R}) = \pm \int_{M'} f \cdot \mathcal{R}$  for M connected and  $M' = M$  differ

Proof  $\int_M \varphi^*(f \cdot \mathcal{R}) = \sum_i \int_{U_i} \varphi^*(f \cdot \mathcal{R}) = \sum_i (\varphi \circ \psi_i)^*(f \cdot \mathcal{R}) = \pm \int_{M'} f \cdot \mathcal{R}$

because:  $\varphi \circ \psi_i: V_i \rightarrow U_i$  are parametrizations and since  $M'$  is connected either

$(\varphi \circ \psi_i)^* \mathcal{R} = F_i(x) dx_1 \dots dx_n$  for  $F_i > 0 \forall i$  or  $F_i < 0 \forall i$ . In the first case call  $\varphi$  orientation-preserving: by  $\oplus$  can use  $\varphi \circ \psi_i$ : instead of  $\psi_i$  to compute integral. So get "+" in change of variable formula. In second case, get "-" since applying  $\varphi$  has the same effect as changing  $\mathcal{R}$  to  $-\mathcal{R}$ .

$\varphi$  called orientation-reversing

$$\begin{array}{c} \text{IDEA: a tiny cube} \\ \text{ee, } \boxed{\text{EE, EE}} \text{ of volume } \varepsilon^n \\ \text{maps approximately to} \\ \text{the parallelepiped} \\ \text{EE, EE, EE, EE} \\ \text{volume: } \varepsilon \cdot \varepsilon \cdot \varepsilon \cdot \varepsilon \end{array}$$

Lemma 2  
Problem:  $\tau^*(\tilde{g}(\tilde{x})) d\tilde{x}_1 \dots d\tilde{x}_n = g(x) \cdot \det \tau. dx_1 \dots dx_n$   
solution: Lemma 3  $\Rightarrow \det \tau = |\det \tau|$ , so O.K.

Cor (4) & M compact  $\Rightarrow$  Globally can integrate n-form  
IDEA: add up local contributions:

$$\int_M \omega = \sum_i \int_{U_i} \omega \quad \left( \text{see Appendix for details: only want to count integral on overlaps } U_i \cap U_j \text{ once} \right)$$

Corollary If a manifold M has a volume form (such mfs are called orientable) and M compact then one can integrate functions  $f: M \rightarrow \mathbb{R}$ :

(sometimes write  $\left( \int_P f(p) \cdot \mathcal{R}|_P \right)$ )

Example For finite groups G,  $\int_G f = \frac{1}{|G|} \sum_{g \in G} f(g)$ .

FROM NOW ON: G COMPACT,  $\mathcal{R}$  LEFT-INvariant & NORMALIZED.

Lemma The integral is left-invariant  $\int_G f = \int_G f \circ \phi_h^{-1} \left( \int_{g \in G} f(g) = \int_{g \in G} f(g) \forall h \in H \right)$

Pf  $\int_G f = \int_G f \circ \phi_h^{-1} (f \circ \mathcal{R}) = \int_G f \circ \phi_h^{-1} \cdot \underbrace{\phi_h^* \mathcal{R}}_{\text{since left-invariant}} = \int_G f \circ \phi_h^{-1} = \int_G f$

change of variables

FACT Also right-invariant:  $\int f = \int_{g \in G} f(g \cdot h)$

iso-invariant: for  $\varphi: G_1 \rightarrow G_2$  Lie group iso,  $\int_{G_2} f = \int_{G_1} f \circ \varphi$

inversion invariant:  $\int_{G \backslash G} f(g) = \int_{G/G} f(g^{-1})$ .

## APPENDIX

Def An  $k$ -form  $\omega$  on  $M^n$  is a map  $\omega_p : T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$  which is:

1) **MULTI-LINEAR**: linear in each entry  $\underbrace{k \text{ copies}}$

2) **ALTERNATING**: switches sign if you transpose two entries

3) **SMOOTH** in  $p$ : if  $X_1, \dots, X_k$  are smooth v.f. then

Examples  $\omega : M \rightarrow \mathbb{R}$ ,  $p \mapsto \omega(X_1, \dots, X_k)|_p$  is smooth.

1) In local coordinates  $(x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $dx_i$  is a 1-form:

$$dx_i : T\mathbb{R}^n \rightarrow \mathbb{R}, \quad dx_i\left(\frac{\partial}{\partial x_j}\right) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

2) Locally,  $dx_1 \wedge \dots \wedge dx_k$  is a  $k$ -form defined by:

$$(dx_1 \wedge \dots \wedge dx_k)(X_1, \dots, X_k) = \det(dx_i(X_j)) = \sum_{\text{permutations}} \text{sign}(\sigma) dx_1(X_{\sigma(1)}) \dots dx_k(X_{\sigma(k)})$$

$$\text{For example: } dx_1 \wedge dx_3 \left( \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \right) = | \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} | = -1$$

3) Forms can be added / scaled, so get vector space.  
 $\Rightarrow$  locally, any  $k$ -form is:  $\underbrace{\text{smooth functions}}$

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1 \dots i_k}(x) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Note: can reorder  
 $dx_i \wedge dx_j = -dx_j \wedge dx_i$   
 (transposing 2 mrs switches sign of det)  
 also  $dx_i \wedge dx_i = 0$   
 $(dx_i = 0 \text{ if 2 mrs are equal})$

PULL-BACK: smooth maps  $\varphi : N \rightarrow M$  pull-back forms:

$$\varphi^* : \Omega^k N \rightarrow \Omega^k M, \quad \varphi^* \omega_N(X_1, \dots, X_k) = \omega_M(D\varphi \cdot X_1, \dots, D\varphi \cdot X_k)$$

$\uparrow$   $\uparrow$   
 $\text{k-forms}$   $\text{vector fields on } M$

## GLOBAL INTEGRATION

Given a countable cover  $U_1, U_2, \dots$  of  $M$  and parametrizations  $\psi_i : V_i \xrightarrow{\cong} U_i$  one can construct functions  $\varphi : M \rightarrow [0, 1]$  (called partition of unity) such that  $\varphi_i = \begin{cases} 0 & \text{outside } U_i \\ 1 & \text{on } U_i \text{ except near boundary points} \end{cases}$

such that  $\sum_i \varphi_i \equiv 1$ . Then:  $\int_M \omega = \int \sum_i \varphi_i \omega = \sum_i \int_{U_i} \varphi_i^* \omega = \sum_i \int_{V_i} \varphi_i^* (\varphi_i \omega)$

## OPTIONAL NON-EXAMINABLE EXERCISES

- 1) Find the left-invariant volume form for  $S^1$  such that  $\int_{S^1} 1 = 1$ , and state an easy formula to compute  $\int_S f$  in practice. Do the same for  $T^n$ .

Hints For  $S^1$ ,  $\omega = g(\theta) d\theta$  but left-inverse forces  $g = \text{constant}$  and normalization forces  $g = \frac{1}{2\pi}$  so  $\int_{S^1} f = \frac{1}{2\pi} \int_{2\pi} f(\theta) d\theta$ .

- 2)  $G$  compact Lie group with the normalized left-invariant volume form.  
 Show the following properties of the integral:

- i) right-invariance:  $\int f = \int_{g \in G} f(g h) \quad \forall h \in H$
- ii) isomorphism-invariance:  $\int f = \int_{G_1} f \circ \varphi$  if  $\varphi : G_1 \rightarrow G_2$  is a Lie group isomorphism
- iii) inversion-invariance:  $\int f(g) = \int f(g^{-1})$

Hints Use the change of variables for integrals to reduce the problem to whether or not the new volume form you get is left-invariant and normalized.

More Hints call  $\psi_k : G \rightarrow G$ ,  $\psi_k(g) = gh$  (right-translation)  
 $\int f(g) \Delta = \int \underbrace{\psi_k \cdot \psi_k^*(f \cdot \Delta)}_{\substack{\text{change} \\ \text{of vars}}} = \int f(g h) \cdot \psi_k^*\Delta$

Now check that  $\psi_k^* \Delta$  is left-invariant and normalized, hence by uniqueness  $\psi_k^* \Delta = \Delta$ , thus get (i).

## LECTURE 13

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This lecture : G COMPACT LIE GROUP,  $\mathbb{F} = \mathbb{C}$   
REPRESENTATION RING = CHARACTER RING

Def Representation ring  $R(G) = \left\{ \sum n_i V_i : n_i \in \mathbb{Z}, \text{ finitely many } n_i \neq 0 \right\}$

using + and  $\otimes$   $V_i = \text{non-equivalent irreps of } G$ .

$(V_i \otimes V_j = \sum m_k V_k \text{ in } R(G) \text{ if } V_i \otimes_{\mathbb{C}} V_j \simeq \bigoplus m_k V_k)$  virtual reps or virtual G-mods  
"honest" reps:  $\left\{ \sum n_i V_i \in R(G) : n_i > 0 \right\} \xleftrightarrow{\text{!}} \left\{ \text{equivalence classes of reps} \right\}$

(complete reducibility + last thm of lecture 11)  
Def  $C\ell(G) = \text{ring of class functions : continuous } G \rightarrow \mathbb{C} \text{ satisfying } f(h^{-1}gh) = f(g)$   
pointwise addition & multiplication  $\hookrightarrow$  Example characters  $f = \chi_V$ .

Thm  $\chi : R(G) \rightarrow C\ell(G), \chi(\sum n_i V_i) = \sum n_i \chi_{V_i}$  is an injective hom of rings

Pf injective by orthogonality relns, hom since  $\chi_{V_i} \circ \chi_{V_j} = \chi_{V_i} \cdot \chi_{V_j}$  ■

Def Often identify  $R(G)$  with  $X(R(G)) \hookrightarrow \text{character ring}$

Thm 1 Any class function is uniformly approximated by  $\sum z_i \chi_{V_i}$  ( $z_i \in \mathbb{C}$ )  
(Q. Sheet 6) That is:  $\text{span}_{\mathbb{C}}(\text{image } \chi) \subseteq C\ell(G)$  dense.

Rmk fails for  $\mathbb{F} = \mathbb{R}$ : holds if we use  $\{f : G \rightarrow \mathbb{R} \text{ in } C\ell(G) \text{ with } f(g) = f(g^{-1})\}$

EXAMPLE  $G = S^1 = \mathbb{R}/\mathbb{Z}$   $\rho_1 : S^1 \rightarrow GL(1, \mathbb{C})$

(here  $a \in \mathbb{Z}$ ,  $\rho_a = \rho_1 \otimes \text{Id}$  if  $a < 0 \in \mathbb{Z}$ )

$\Rightarrow R(S^1) = \left\{ \sum_{a \in \mathbb{Z}} n_a \rho_a : \text{finite sum, } n_a \in \mathbb{Z} \right\} \cong \mathbb{Z}[[t, t^{-1}]] = \text{Laurent polys in } t$

Q.sheet 5  $\rho_a : R(S^1) \rightarrow C\ell(S^1), \chi(\sum n_a \rho_a) = \sum n_a e^{2\pi i a x} = \text{trigonometric polys with integer coeffs.}$

EXAMPLE  $G = T^n = \mathbb{R}^n/\mathbb{Z}^n$  Q.sheet 6 using  $T^n = \text{span}_{\mathbb{C}} x_1 \dots x_n$  (or directly by Q.sheet 5):

$R(G) = \left\{ \sum_{a \in \mathbb{Z}^n} n_a \rho_a : \text{finite sum, } n_a \in \mathbb{Z} \right\} \cong \mathbb{Z}[[t_1^{\pm 1}, \dots, t_n^{\pm 1}]] = \text{Laurent polys in } t_1, \dots, t_n$

$\rho_a : T^n \rightarrow GL(1, \mathbb{C}), \rho_a(x) = e^{2\pi i \langle a, x \rangle \cdot \text{Id}}, \chi_a(x) = e^{2\pi i \langle a, x \rangle}$   
where  $\langle a, x \rangle = a_1 x_1 + \dots + a_n x_n$

PETER-WEYL THEOREM

$f : G \rightarrow \text{Aut}(\mathbb{C}^n)$  gives a subset of  $C(G) = \{ \text{continuous functions } f : G \rightarrow \mathbb{C} \}$

called matrix entries :  $f(g) = ((i,j)\text{-entry of the matrix } f(g)) = \text{Trace}(f(g)) = \text{Trace}(\varphi \circ f(g))$

where  $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  linear map with  $\varphi(e_i) = e_j$  and  $\varphi(e_k) = 0$  all  $k \neq j$ .  
( $\varphi = \text{matrix with 1 in position } (j,i) \text{ and 0 elsewhere}$ )

EXAMPLE  $G = S^1 \subset \mathbb{C}^2, \rho(x) = \begin{pmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{pmatrix} \in \text{Aut}(\mathbb{C}^2)$ . Take  $(i,j) = (1,2)$

$\varphi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \varphi \circ \rho(x) = \begin{pmatrix} 0 & 0 \\ 0 & \cos 2\pi x - \sin 2\pi x \end{pmatrix} \Rightarrow \text{Trace}(\varphi \circ \rho) = -\sin 2\pi x = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ entry} \right) \text{ of } \rho(x)$

FACT (Schur) matrix entries of irreps are orthogonal using  $\langle f_1, f_2 \rangle = \int_{\mathbb{C}} f_1(g) \overline{f_2(g)}$ .

Rmk holds for  $\mathbb{F} = \mathbb{R}$  for two non-equivalent irreps, otherwise fails (e.g. example above).

EXAMPLE  $G = S^1$  dim(irreps) = 1  $\Rightarrow$  matrix entry  $= \chi_a = e^{2\pi i ax}$  for  $a \in \mathbb{Z}$  are orthogonal.  
(indeed orthogonality relns:  $\langle \chi_a, \chi_b \rangle = 0$  for  $a \neq b$ ). Above example fails since reducible!

Def Representative function means any linear combination of matrix entries

They can always be written as  $L \circ \varphi$  where  $\varphi : G \rightarrow \text{Aut}(\mathbb{V}), L \in \text{Hom}_{\mathbb{C}}(\mathbb{V}/\mathbb{V})^*$

Let  $\mathcal{F}(G) = \{ \text{representative fns} \}$

①  $L_1 \circ \varphi_1 + \lambda L_2 \circ \varphi_2 = (L_1 + \lambda L_2) \circ (\varphi_1 + \varphi_2)$  using  $V = V_1 \oplus V_2 \Rightarrow \mathcal{F}(G)$  is v.s./ $\mathbb{C}$

② Product of two matrix entries from  $\varphi_1, \varphi_2$  is a matrix entry of  $\varphi_1 \otimes \varphi_2$  (Q.1 Q.Sheet 5)  
 $\Rightarrow \mathcal{F}(G) \subseteq C(G)$  is subring and  $\frac{v.s.}{\mathbb{C}}$ , so it's  $\mathbb{C}$ -algebra.

③ If only allow rep  $V_j$  get vector subspace  $\mathcal{F}_V(G) \subseteq \mathcal{F}(G)$  of  $\dim \leq (\dim V)^2 = \#(\text{matrix entries})$

Above Fact  $\Rightarrow \mathcal{F}(G) = \bigoplus \mathcal{F}_{V_i}(G)$  orthogonal direct sum over the irreps  $V_i$

Peter-Weyl Theorem (version 1)  $\mathcal{F}(G) \subseteq C(G)$  is dense (will not prove it.)

Rmk also holds for  $\mathbb{F} = \mathbb{R}$ .  
 $\mathcal{F} \rightarrow \mathbb{C}$  ctgs  $\Rightarrow$  can uniformly approximate f by representative fns

(given  $\varepsilon > 0$ ,  $\exists \varphi : G \rightarrow \text{Aut}(\mathbb{V}), \exists \psi \in \text{Hom}_{\mathbb{C}}(\mathbb{V}, \mathbb{V})$  with  $\sup_{g \in G} |f - \text{Tr}(\varphi(g))| < \varepsilon$ )

Fact  $(f : G \rightarrow \mathbb{C}) \in \mathcal{C}(G) \Rightarrow$  can choose  $\varphi \in \text{Hom}_{\mathbb{C}}(\mathbb{V}, \mathbb{V}) \subseteq \text{Hom}_{\mathbb{C}}(\mathbb{V}/\mathbb{V})$

Stone-Weierstrass theorem (NON-EXAMINABLE) (Really mostly functional analysis)

$\mathcal{F}$  also holds for  $\mathbb{F} = \mathbb{R}$ .

M compact mfd (more generally a compact Hausdorff topological space)

S  $\subseteq C(M) = \{ \text{cts } M \rightarrow \mathbb{C} \}$  is \*-subalgebra separating points with  $1 \in S$

Then  $S \subseteq C(M)$  is dense!  $f \in S \subseteq C(M)$  subtracting  $m \in M \Rightarrow \exists f_m \in S$  and vector subspace with  $f(m) + f(m')$

Rmk also holds for  $\mathbb{F} = \mathbb{R}$  so  $S \subseteq C(M, \mathbb{R})$ .

EXAMPLE  $S = \text{span}\{\rho_a(x) = e^{2\pi i a x} : a \in \mathbb{Z}\} \subseteq C(S^1)$  where  $x \in S^1 = \mathbb{R}/\mathbb{Z}$

S is v.s. since it's a span, and a subalgebra since  $f a b = f a b \in S$

$f_0 \in S \Rightarrow \overline{f_0} \in S$   $\overline{f_0} = e^{-2\pi i a x} = f_{-a} \in S$   $x \neq y \pmod{\mathbb{Z}} \Rightarrow e^{2\pi i x} \neq e^{2\pi i y} \Rightarrow f_1(x) \neq f_1(y)$

$\Rightarrow S = \{ \text{"trig polynomials"} \} \subseteq C(S^1)$  dense!

EXAMPLE for  $\mathbb{F} = \mathbb{R}$  use  $\text{span}\{\cos 2\pi x, \sin 2\pi x\} \subseteq C(S^1, \mathbb{R})$ .

## CONSEQUENCE: FOURIER ANALYSIS

Recall  $C(S) \subseteq L^2(S') = \{\text{square-integrable fns } f: S' \rightarrow \mathbb{C}\}$  dense (easy fact)  
 $\Rightarrow$  can  $L^2$ -approximate any  $L^2$ -function by trig polys. We knew this:

- $f = (\text{Fourier series}) \sum_{a \in \mathbb{Z}} z_a e^{2\pi i ax} \approx \sum_{a=-N}^N z_a f_a \text{ for } N \gg 0.$
- $z_a = \int_0^1 e^{-2\pi i ax} f(x) dx = \int_0^1 \overline{f_a(x)} f(x) dx = \langle f_a, f \rangle \in \mathbb{C}$  if parametrize by  $\theta \in [0, 2\pi]$

### PROOF OF PETER-WEYL FOR MATRIX GROUP

CLAIM  $G = \text{compact matrix group} \Rightarrow F(G) \subseteq C(G)$  dense  
Pf. Already showed  $S = F(G)$  is  $\mathbb{C}$ -algebra.

$1 \in S$  since  $(1, 1)$  entry of trivial rep  $G \rightarrow \text{Aut}(\mathbb{C}), g \mapsto \text{Id.}$   
 $f = L \circ g \in S \Rightarrow \overline{f} = L \circ \overline{g} \in S$  using dual rep  $V^* \simeq \overline{V}$  (Q.sheet 5, G compact)  
separate points: use standard rep  $G \subseteq GL(n, \mathbb{R})$  acting on  $\mathbb{C}^n$  by matrix mult,  
so  $g: G \subseteq GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{C})$  inclusion, and if  $g_1 \neq g_2$  are different  
matrices then some entry must be different  $\Rightarrow f(g_1) \neq f(g_2)$  some  $f = \text{Tr}(g \circ \rho)$   $\rho \in S$ .

### COMPACT LIE GPS ARE MATRIX GROUPS

#### Peter-Weyl theorem (version 2)

$G$  compact Lie gp  $\Rightarrow \exists$  faithful rep  $G \longrightarrow U(m)$  Some  $m$   
 $\hookrightarrow$  (injective)

We proved version 2  $\Rightarrow$  version 1, Q.sheet 6: version 1  $\Rightarrow$  version 2!

### REGULAR REP, FOURIER ANALYSIS, $\infty$ -DIM REPS (NON-EXAMINABLE!)

Peter-Weyl Theorem (version 3)  
Any unitary rep  $G \rightarrow U(H) \subseteq \text{Aut}(H)$  on a Hilbert space  $H$   
is an orthogonal direct sum of finite dim'l unitary subreps

$$H = \bigoplus W_i \leftarrow \begin{array}{l} \text{(allow infinite sums if cgt in } H \\ \text{(i.e. closure of the usual direct sum)} \end{array}$$

Recall for finite gp  $G$  the regular rep is  $H = \text{Functions } (G, \mathbb{C}).$   
 $H$  is a v.s. of dim  $= |G|$  with basis  $e_h$  indexed by  $h \in G$  given by:  
 $e_h = (\text{function } h \mapsto 1 \text{ and all other } g \mapsto 0)$

G-action:  $g \cdot e_h = e_{gh}$  (since on functions  $(g \cdot e_h)(gh) = e_h(g^{-1}gh) = 1$ )  
 $\Rightarrow G$  faithful  $\{\text{permutation matrices}\} \subseteq \text{Aut } (\mathbb{C}^{|G|})$ .

FACT  $G$  finite  $\Rightarrow$  reg rep.  $H \cong \bigoplus (\dim V_i) V_i$  summing over all irreps  
So in principle can find all irreps of  $G$ !

Can't work for compact Lie  $G$  unless allow  $\infty$ -dim reps since for infinite  $G$  (compact) there are countably infinitely many finite dim'l irreps (for example for  $SL(2, \mathbb{Q})$ . Q.sheet 5). This will follow from PW version 4 below.

Regular rep  $H = L^2(G) = \{\text{square-integrable } f: G \rightarrow \mathbb{C}\}$

Hilbert space with  $\langle f, f_2 \rangle = \int_G \overline{f(g)} f_2(g) \text{ and } G\text{-action: } (h \cdot f)(g) = f(h^{-1}g)$

Peter-Weyl Theorem (version 4)  $L^2(G) = \bigoplus W_i$   
where  $W_i \simeq (\dim V_i) \cdot V_i$  and all finite-dim irreps  $V_i$  arise

FACT An orthonormal analysis-basis for  $L^2(G)$  is  $\sqrt{\dim V_i} \cdot \rho_i^{(jk)}$   
where  $\rho_i^{(jk)}(g) = \langle f(g), v_j, v_k \rangle$  is the  $(j, k)$  matrix entry in o.n. basis  $v$ . For  $v_i$ : Analysis-basis  $e_n$  means linear combos can be infinite convergent sums so each  $\int \rho \in H$  is uniquely  $\int = \sum_{n \in \mathbb{Z}} \int e_n \dim L^2(G)$  countable Q.sheet 6

EXAMPLE  $L^2(S')$   
Reg. rep.  $\rho: G \longrightarrow \text{Aut}(L^2(S'))$ ,  $(\rho(x) \cdot f)(y) = f(y - x) \quad (x, y \in S')$

Claim  $L^2(S') = \bigoplus_{a \in \mathbb{Z}} \mathbb{C} \cdot e^{-2\pi i ax}$

Pf. By P.W. thm:  $L^2(S') = \bigoplus_{a \in \mathbb{Z}} V_a$  (indep of  $S'$  have  $\dim = 1, V_a = e^{2\pi i ax}$ )  
if parametrize  $S'$  by  $e^{i\theta}$   
then  $(\rho(e^{i\theta}) f)(e^{i\psi}) = f(e^{-i(\theta-\psi)})$   
 $= f(e^{i(\psi-\theta)})$

hence  $V_a = \{f \in L^2(S'): \rho(x) \cdot f = e^{2\pi i ax} \cdot f\}$   
 $= \{f \in L^2(S'): f(y - x) = e^{2\pi i ax} f(y)\}$   
put  $y=0$  replace  $x$  by  $-x$   
 $f(x) = f(0) \cdot e^{-2\pi i ax}$

Rmk  $F(G) \subseteq C(G) \subseteq L^2(G)$  are dense inclusions  
 $\text{span}(x) \subseteq \rho(G) = C(G)^G \subseteq L^2(G)^G$  are dense inclusions.  $\hookrightarrow$  by Thm 1  
What is character of  $L^2(S')$ ?

$$\chi_{L^2(S')} = \sum_{a \in \mathbb{Z}} e^{2\pi i ax} \text{ does not converge to a function.}$$

It does however converge to a distribution (i.e. linear functional  $C^\infty(S') \rightarrow \mathbb{C}$ )  
 $f \in C^\infty(S') \mapsto \int_N e^{2\pi i ax} f(x) dx = \sum_{a=-N}^N z_a \rightarrow \sum_{a=-N}^N z_a = f(0)$   
 $f \mapsto f(0)$  is the delta-function!  
General  $G: \infty$ -dim rep  $\rho$ , the character is distribution  $\chi_H(f) = \text{Tr} \left( \int_{G \times H} f(g) \rho(g) \right)$

For reg. rep  $L^2(G): \chi_{L^2(G)}(f) = f(1)$  is delta-function at 1  
 $\Rightarrow$  recover  $f: \chi_{L^2(G)}(f \circ \phi_h) = f(\phi_h(1)) = f(h)$ !

## L14 HANDOUT REP. THEORY OF $S^1$ AND $T^n$

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Over  $\mathbb{F} = \mathbb{C}$

- $G$  compact  $\Rightarrow$  Reps decompose  $\rho = \bigoplus n_i \rho_i$ : into irreps
- $G$  abelian  $\Rightarrow$  irreps are 1-dimensional,  $G \rightarrow GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$
- $G$  compact & abelian  $\Rightarrow$  irreps must land in  $S' \subseteq \mathcal{C} \setminus \{0\}$

Proof If  $\rho(g) = \lambda \in \mathcal{C} \setminus \{0\}$  with  $|\lambda| > 1$  then  $\rho(g^n) = \lambda^n \rightarrow \infty$   
 $"$       "       $|\lambda| < 1$  then  $\rho(g^n) = \lambda^{-n} \rightarrow \infty$   
 But image  $(\rho: G \rightarrow \mathcal{C} \setminus \{0\})$  is compact hence bounded, so  $|\lambda| = 1$  ■

Hence:

- $G$  compact & abelian  $\Rightarrow$   $\{\text{irreps}\} / \text{equivalence} \cong \{G \rightarrow S'\}$

$\mathbb{F} = \mathbb{C}, G = S'$  We know Lie grp homs  $G = S' \rightarrow S'$  so irreps are:  
 $\rho_\alpha: S' = \mathbb{R}/\mathbb{Z} \rightarrow GL(1, \mathbb{C}), x \mapsto e^{2\pi i ax} \cdot \text{Id}$  ( $a \in \mathbb{Z}$ )  
 $\mathbb{F} = \mathbb{C}, G = T^n$  We know Lie grp homs  $G = T^n \rightarrow S'$  so irreps are:  
 $\rho_\alpha: T^n = \mathbb{R}/\mathbb{Z}^n \rightarrow GL(1, \mathbb{C}), \rho_\alpha(x) = e^{2\pi i \langle \alpha, x \rangle} \cdot \text{Id}$  ( $\langle \alpha, x \rangle = \alpha_1 x_1 + \dots + \alpha_n x_n$ )

$\mathbb{F} = \mathbb{R}$

$G$  compact  $\Rightarrow$  Reps decompose  $\rho = \bigoplus n_i \rho_i$ : into irreps

$G$  abelian  $\Rightarrow$  irreps have  $\dim_{\mathbb{R}} = 1$  or 2

Proof  $G \rightarrow GL(n, \mathbb{R}) \subseteq GL(n, \mathbb{C})$   
 So as a complex rep, it has a complex subrep  $\mathbb{C} \cdot v$  of  $\dim_{\mathbb{C}} = 1$

Let  $x = Re(v) = \frac{1}{2}(v + \bar{v})$  and  $y = Im(v) = \frac{1}{2i}(v - \bar{v})$  then  
 $\text{span}_{\mathbb{R}}(x, y) = Re(\mathbb{C}v) + Im(\mathbb{C}v) \subseteq \mathbb{R}^n$  is real subrep of  $\dim_{\mathbb{R}} = 2$  or 1  
 Indeed: for  $A = \rho(g): Ax = \frac{1}{2}(Av + A\bar{v}) = \underbrace{\text{Re}(Av)}_{\in \mathbb{C}v}$   
 similarly  $Ay = \text{Im}(Av) \in \text{Im}(\mathbb{C}v)$ . ■

$\mathbb{F} = \mathbb{R}, G = S'$  Nontrivial irreps have  $\dim_{\mathbb{R}} = 2$

Proof  $\dim_{\mathbb{R}} = 1 \Rightarrow \dim_{\mathbb{C}} = 1$  irrep  $\Rightarrow \rho = \chi_\alpha \cdot \text{Id} \not\in GL(1, \mathbb{R})$   
 except for  $\alpha = 0$  get trivial rep

Claim  $\rho: G = S' \rightarrow GL(2, \mathbb{R}), \rho_\alpha(x) = \begin{pmatrix} \cos 2\pi \alpha & -\sin 2\pi \alpha \\ \sin 2\pi \alpha & \cos 2\pi \alpha \end{pmatrix}$  ( $\alpha \neq 0 \in \mathbb{Z}$ ) is irreducible.

Pf 1  $\forall \neq 0 \in \mathbb{R}^2 \Rightarrow v, \rho_\alpha(\frac{1}{4\alpha})v = (v \text{ rotated by } \frac{\pi}{2})$  are lin. indep  $\mathbb{R}^2$   
 $\Rightarrow$  no subreps except  $0, \mathbb{R}^2$  ■

Pf 2  $A \dim_{\mathbb{R}} = 1$  subrep would be a common eigenspace of all  $\rho_\alpha(x)$  ■

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Claim  $\rho|_{\mathbb{R}} \cong \rho'|_{\mathbb{R}} \iff \alpha = -b$   
 $\rho_f \text{ "}" \Rightarrow: \chi_{\rho_\alpha}(x) = 2 \cos(2\pi \alpha x)$  and recall  $\rho \cong \rho' \Rightarrow \chi_\rho = \chi_{\rho'}$   
 $\iff: s^{-1} \circ \rho_\alpha \circ s = \rho'_b$  if  $s = \text{reflection in } x\text{-axis}$  ■

Claim The irreps/ $\mathbb{R}$  of  $S^1$  are  $\begin{cases} \rho_\alpha^\mathbb{R} \text{ for } \alpha = 1, 2, 3, \dots \in \mathbb{N} \\ \text{trivial irrep } S^1 \rightarrow GL(1, \mathbb{R}), x \mapsto \text{Id} \end{cases}$

Pf 1  $\rho$  irrep of  $\dim_{\mathbb{R}} = 2 \Rightarrow \text{irrep of } \mathbb{C} \text{ of } \dim_{\mathbb{C}} = 2 \Rightarrow \rho \cong \rho_a \oplus \rho_b$  over  $\mathbb{C}$   
 $\rho \text{ real} \Rightarrow \rho = \overline{\rho} \Rightarrow \rho_a \oplus \rho_b \cong \overline{\rho_a \oplus \rho_b} = \rho_{-a} \oplus \rho_{-b}$   
 But  $\alpha = -a, b = -b$  would imply  $\alpha = 0, b = 0$  so  $\rho = \text{trivial}$ . So  $a = -b$ .  
 $\Rightarrow \rho \cong \rho_a \oplus \rho_{-a}$ . So  $\exists$  subrep  $\mathbb{C}v, v \in \mathbb{C}v$ , with  $\rho(x)v = \rho(x)v = e^{2\pi i \alpha x}v$ .  
 Then  $\rho(x)\bar{v} \neq \overline{\rho(x)v} = e^{-2\pi i \alpha x}\bar{v} = e^{2\pi i \alpha x}v$  ( $\Rightarrow \bar{v} = \text{subrep } \rho_{-a}$ )

Let  $\alpha = \frac{1}{2i}(v - \bar{v}) = \text{Im}(v), \beta = \frac{1}{2}(v + \bar{v}) = \text{Re}(v) \in \mathbb{R}^2$ . As above, for  $A = \rho(x):$   
 $A\alpha = \text{Im}(Av) = \text{Im}(e^{2\pi i \alpha x}v) = \cos(2\pi \alpha x) \cdot \text{Im}(v) + \sin(2\pi \alpha x) \cdot \text{Re}(v)$   
 $A\beta = \text{Re}(Av) = \text{Re}(e^{2\pi i \alpha x}v) = \cos(2\pi \alpha x) \cdot \text{Re}(v) - \sin(2\pi \alpha x) \cdot \text{Im}(v)$   
 so in basis  $\alpha, \beta \in \mathbb{R}^2$  get  $A = \rho(x) = (\begin{smallmatrix} \cos 2\pi \alpha x & -\sin 2\pi \alpha x \\ \sin 2\pi \alpha x & \cos 2\pi \alpha x \end{smallmatrix}) = \rho_\alpha^\mathbb{R}(x)$ . ■

Pf 2 Pick a  $G$ -invariant inner product on  $\mathbb{R}^n \Rightarrow$  (up to an equivalence since change to an orthonormal basis of  $G$ -invt. i.p.) can assume  $\rho: G = S^1 \rightarrow O(n) \subseteq GL(2, \mathbb{R})$  indeed (and in  $SO(2)$  since  $S^1$  connected)  
 $\Rightarrow$  if identify  $\mathbb{R}^2 = \mathbb{C}$ , (rotation by  $\theta$  on  $\mathbb{R}^2$ ) = (multiplication by  $e^{i\theta}$  on  $\mathbb{C}$ )  
 $\Rightarrow$  get  $\dim_{\mathbb{R}} = 1$   $\times$  irrep (some  $\alpha \in \mathbb{Z}$ )  $\rho_\alpha(x) = \rho_\alpha^\mathbb{R}(x) = (\begin{smallmatrix} \cos 2\pi \alpha x & -\sin 2\pi \alpha x \\ \sin 2\pi \alpha x & \cos 2\pi \alpha x \end{smallmatrix}) = \rho_\alpha^\mathbb{R}(x)$

Pf 3 (using Peter-Weyl) matrix entries of  $\rho_\alpha$  are  $\cos(2\pi \alpha x), \sin(2\pi \alpha x)$   
 As vary  $\alpha \in \mathbb{N}$ , these generate a dense subalgebra of  $C(S^1, \mathbb{R})$  (Fourier analysis)  
 The matrix entries of two non-equivalent irreps are orthogonal, so if  $\rho$  is an irrep different from  $\rho_\alpha$ , then  $\langle X_\rho, f \rangle = 0$  for  $f \in$  (that subalgebra). But such  $f$  are dense in  $C(S^1)$  so  $\langle X_\rho, f \rangle = 0$  for  $f \in C(S^1)$  so  $X_\rho = 0$  contradicting the orthogonality relation  $\langle X_\rho, X_\rho \rangle \geq 1$  (over  $\mathbb{R}$ ) ■

Real irreps of  $T^n$ : similar argument, get trivial irrep  $\mathbb{R}$  and 2-dim irreps:  
 $\rho_\alpha: T^n \rightarrow GL(2, \mathbb{R}), \rho_\alpha(x) = \begin{pmatrix} \cos 2\pi \alpha x & -\sin 2\pi \alpha x \\ \sin 2\pi \alpha x & \cos 2\pi \alpha x \end{pmatrix}$

for  $\alpha \in \mathbb{Z} \setminus \{0\}$  and  $\rho_\alpha \cong \rho_b \iff \alpha = -b$ .



③ Lie subalgebra  $t = \text{Lie } T \subseteq \mathfrak{g}$  (after  $\otimes_{\mathbb{R}} \mathbb{C}$ ) is called Cartan subalgebra since  $t$  abelian and  $\text{ad } t$  acts diagonally (i.e.  $\mathfrak{g} \otimes \mathbb{C} = \oplus$  weight spaces) and maximal ( $\mathfrak{g}_0 = t$ )

Lemma  $T$  maximal  $\Leftrightarrow t = \mathfrak{g}_0 \subseteq \mathfrak{g}$  ( $t = \text{Lie}(T)$ )

Pf "=>":  $T$  abelian  $\Rightarrow \text{Ad}(g) = \text{Id}$  on  $T \Rightarrow \text{Ad}(x) = \text{Id}$ ,  $\text{Ad}(g) = \text{Id}$  on  $t \Rightarrow t \subseteq \mathfrak{g}_0$ . If  $y \in \mathfrak{g}_0 \Rightarrow \text{Ad}(x)y = y \Rightarrow \exp(y) = x \exp(\text{Ad}(x) \cdot y) = x \exp(y)x^{-1}$  all  $x \in T$ .

Same holds for  $sy, s \in \mathbb{R}$  so  $T$  commutes with subgp  $H = \{\exp(sy) : s \in \mathbb{R}\}$  so  $\langle T \cup H \rangle$  is a connected abelian subgp larger than  $T$  unless  $y \in T$

"=>":  $T \subseteq T' \subseteq G$  larger torus  $\Rightarrow t \subseteq \text{Lie } T' \subseteq \mathfrak{g}' \subseteq \mathfrak{g}_0$

Assumption  $t = \mathfrak{g}_0 \Rightarrow t = \text{Lie } T' \Rightarrow T = T'$   $\begin{matrix} \uparrow \\ \text{1st part} \\ \text{of proof} \end{matrix}$   $\begin{matrix} \uparrow \\ \text{so also} \\ \text{for } x = x \in T' \end{matrix}$  decompose of using  $T'$

EXAMPLE Recall (Q. sheet 3) for matrix groups  $G$ ,  $\text{Ad}(A) \cdot B = A B A^{-1}$ .

$$G = U(2) = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} : \lambda_k = e^{2\pi i x_k}, (x_1, x_2) \in \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \right\}$$

$$\begin{aligned} \mathfrak{n}(2) &= \left\{ B \in \text{Mat}_{2 \times 2}(\mathbb{C}) : \overline{B}^T = -B \right\} = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & id \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\} \\ &= \mathbb{R} \cdot \underbrace{\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{R} \cdot \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}}_{\mathfrak{g}_0} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} : z \in \mathbb{C} \right\}}_{\mathfrak{g}_\alpha} \end{aligned}$$

$$\begin{aligned} \text{Proof: } \text{Ad} \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \right) &= \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \left( \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right) = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \\ \text{Ad} \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \right) &= \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \end{aligned}$$

Note:  $\mathfrak{g}_0 = \left\{ \begin{pmatrix} ia & 0 \\ 0 & id \end{pmatrix} \right\} = t$  hence  $T$  maximal.

Now find  $\alpha \in \mathbb{Z}^2$ :

$$\begin{aligned} \text{Ad} \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right) &= \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} \left( \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right) \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right) = \begin{pmatrix} 0 & \lambda_1 \lambda_2^{-1} z \\ -\lambda_1^{-1} \lambda_2^{-1} \bar{z} & 0 \end{pmatrix} \\ \Rightarrow \mathfrak{g}_\alpha \cong \mathbb{C} &\cong \mathbb{R}^2 \text{ is the rep } z \mapsto \lambda_1 \lambda_2^{-1} z = e^{2\pi i(x_1 - x_2)} z \\ (\text{rot}) &\mapsto z \mapsto (\text{rot}) \end{aligned}$$

$$\Rightarrow \theta_\alpha(x) = \alpha_1 x_1 + \alpha_2 x_2 = x_1 - x_2 \quad \text{root}$$

$$\Rightarrow \alpha = (1, -1) \in \mathbb{Z}^2$$

Typically write  $g_{x_1, -x_2}$  instead of  $g_{(1, -1)}$ .

$$\Rightarrow \mathcal{U}(2) \cong \mathfrak{g}_0 \oplus g_{x_1, -x_2} \text{ has one root: } x_1 - x_2.$$

Rmk Lie algebra approach is: abbreviate  $y_1 = 2\pi i x_1, y_2 = 2\pi i x_2$  then:  $\text{ad} \left( \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right) = \left[ \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right] = \left( \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} - \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} - \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right) = \left( \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} - \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right)^2 = 0$

## CONJUGATES OF MAX T COVER G

Theorem Any  $h \in G$  lies in a conjugate of  $T$  ( $h = g x g^{-1}$  some  $x \in T$ )

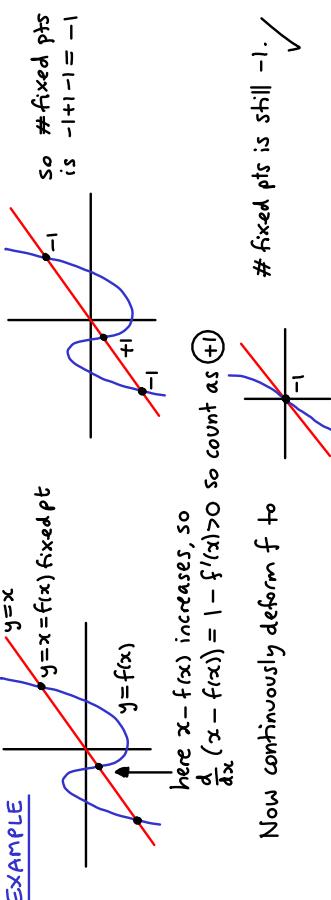
Example Any  $A \in \text{U}(n)$  is diagonalizable, so conjugate to some  $(\lambda_1, \dots, \lambda_n) \in T$ . Rmk Uses  $G$  connected since  $g T g^{-1} \subseteq G_0$  cannot reach  $h \in G \setminus G_0$ . Proof (Non-examinable)

Trick  $f = \phi_h : G/T \rightarrow G/T$ ,  $f(gT) = hgT$  (just a smooth map,  $G/T$  manifold)  $gT$  fixed point of  $f \Leftrightarrow f(gT) = gT \Leftrightarrow f(gt) = gt \Leftrightarrow g^{-1}hg \in T$

Remains to show  $\exists$  fixed point.

Trick 2 For a smooth map  $f: M \rightarrow M$  of a manifold, the number of fixed points (counted with multiplicity) does not change if we continuously deform  $f$ .  $f(x) = x$  count  $x$  as  $+1$  if  $\det(Id - D_x f) > 0$  (more complicated if  $\det = 0$ : have multiplicity)

(Lefschetz fixed pt thm)



Trick 3 Q.2 sheet 3: almost any  $\tau \in T$  generates  $\overline{\tau} = T$ . Pick such a  $\tau$ . (closure of subgroup gen by  $\tau$ ) Since  $G$  connected and it is path-connected, so deform  $f$  by moving  $\tau$  to  $\overline{\tau}$ . Remains to show:  $f: G/T \rightarrow G/T$ ,  $f(gT) = \tau gT$  has #fixed pts  $\neq 0$ .

Claim {fixed points} =  $\{nT : n \in N(T)\}$

If  $nT$  fixed  $\Leftrightarrow n^{-1} \tau n \in T \Leftrightarrow \overline{n^{-1} \tau n} = n^{-1} \tau n = n^{-1} T n \subseteq T$

Rmk  $T \subseteq N(T) \subseteq G$  closed subgroup so compact Lie subgroup.  $N(T) = \{g \in G : g T g^{-1} = T\}$  (equality since  $n^{-1} T n \subseteq T$  torus)  $\Rightarrow N(T)_0 = T$  (closure of subgroup gen by  $\tau$ ) Pf  $n \in N(T) \Rightarrow T \rightarrow T$ ,  $g \mapsto ngn^{-1}$  is a Lie gp from depending on  $\alpha$  continuous parameter  $n$ . But homs  $T \rightarrow T$  are parametrized by a discrete parameter in  $\mathbb{Z}^n \times \mathbb{Z}^n$  (Q.2 sheet 3). So  $ng^{-1}$  independent of  $n$  up to deforming. So move  $n \mapsto 1 \Rightarrow ng^{-1} = g \Rightarrow ng = g$ . If  $T \not\subseteq N(T) \Rightarrow$  could create connected abelian subgp larger than  $T$  by taking  $\subset T \cup \{exp(sy) : s \in \mathbb{R}\} > T \cup \{exp(sy) : s \in \mathbb{R}\} \setminus T$

Consequence fixed points are cosets  $n\tau$  of  $\tau = N(\tau)$  in  $N(\tau)$   
 $\Rightarrow$  finitely many since  $N(\tau)/N(\tau)_0$  is discrete + compact.

Trick 4  $\det(I - D_{n\tau} f)$  is independent of  $n$ .

PF Consider  $\psi_n^{-1} \circ f \circ \psi_n$  where  $\psi_n$  = right-multiplication by  $n$  on  $G/\tau$  (NOTE  $\tau g n = g \tau n$ )  
 $n(\tau)_0 \subseteq N(\tau)$  open and cosets cover  $N(\tau)$

$$\Rightarrow \psi_n^{-1} \circ f \circ \psi_n = f$$

$$\Rightarrow \det(I - D_\tau f) = \det(I - D_{n\tau} \psi_n^{-1} \circ D_\tau f \circ D_{n\tau} \psi_n) = \det(D\psi_n^{-1}(I - D_{n\tau} f)D\psi_n)$$

$$\stackrel{\text{chain rule}}{=} \det(I - D_{n\tau} f) = \det(I - D_{n\tau} f) \quad \blacksquare$$

Remains to calculate  $\text{sign}(\det(I - D_\tau f))$ .

Trick 5  $A_\tau : G \rightarrow G/\tau$ ,  $A_\tau(g) = \tau g \tau^{-1}$

$$\Rightarrow A_\tau : G/\tau \rightarrow G/\tau, \quad A_\tau(\tau g) = \tau g \tau^{-1} = \tau g \tau = f(g\tau)$$

Since tangent space  $T_\tau G \cong T_\tau \tau \oplus T_\tau(G/\tau)$ ,

$$D_\tau f = D(A_\tau \text{ restricted to a vector space complement of } T_\tau \tau \subseteq T_\tau G)$$

$$= D_1 A_\tau \Big|_{\oplus g_\alpha} = \text{Ad}(\tau) \Big|_{\oplus g_\alpha} = \text{Ad}(\tau) \left( \begin{array}{c} \text{rotation by } \tau \\ \text{by } 2\pi \theta_\alpha(\tau) \text{ on } g_\alpha \end{array} \right)$$

$$(omit \overline{g_\alpha} = t \in T_\tau \tau)$$

$$\det(I - D_\tau f) = \prod_\alpha \det \begin{pmatrix} 1 - \cos 2\pi \theta_\alpha(\tau) & \sin 2\pi \theta_\alpha(\tau) \\ -\sin 2\pi \theta_\alpha(\tau) & 1 - \cos 2\pi \theta_\alpha(\tau) \end{pmatrix}$$

$$= \prod_\alpha 2 \underbrace{(1 - \cos 2\pi \theta_\alpha(\tau))}_{>0 \text{ since } \theta_\alpha(\tau) \neq 0 \text{ mod } \mathbb{Z}}$$

$$\text{otherwise } \theta_\alpha(\tau) \neq 0 \text{ mod } \mathbb{Z}$$

$$\text{but } \theta_\alpha \neq 0 \text{ for } \alpha \neq 0.$$

$\Rightarrow$  multiplicity of all  $n\tau$  is +1.

$\Rightarrow$  # fixed points  $> 0$  (if there is no  $g_\alpha$ , then  $g=t$ , so  $g=\tau$ ) ■

Corollaries

- ① All maximal tori are conjugate:  $\tau, \tau' \text{ max} \Rightarrow \tau' = g\tau g^{-1} \text{ some } g$
- ② Every element  $h \in G$  lies in some max torus
- ③ Decomposition  $\mathfrak{g} \approx \text{no } \mathfrak{t} \oplus \mathfrak{n}_{\text{Va}}$  is independent of choice of max  $\tau$ .  
 so roots  $\theta_\alpha$  do not depend on choice and  $\dim(\text{max } \tau) = \dim(\mathfrak{g}_0)$  called rank( $G$ ).

PF ①  $\tau' = \overline{\tau'}$  some  $\tau'$ . By Thm,  $\tau' = g\tau g^{-1}$  some  $g$ , by Thm.

②  $h \in g\tau g^{-1}$  some  $g$ , by Thm.

③  $\tau' = g\tau g^{-1} \Rightarrow \underline{\text{Claim }} \overline{g_\alpha} = \text{Ad}(g) \cdot g_\alpha \cdot \text{Ad}(g)^{-1}$  so  $\overline{g_\alpha} \approx g_\alpha$  equivalent reps.

PF  $\text{Ad}(g\tau g^{-1}) \cdot \text{Ad}(g) \cdot y \cdot \text{Ad}(g)^{-1} = \text{Ad}(g) \text{Ad}(\tau) y \text{ Ad}(g^{-1})$  and  $\text{Ad}(x) y \in \mathfrak{g}_0$  iff  $y \in \mathfrak{g}_0$ . ■  
 Similarly for  $g_\alpha$ . Since  $\overline{g_\alpha} \approx g_\alpha$  also  $\overline{h_\alpha} = h_\alpha$  so characters  $= 2\cos(2\pi \theta_\alpha(x))$  same  
 so  $\theta_\alpha = \pm \theta_\alpha$  (and recall we don't distinguish  $\theta_\alpha, -\theta_\alpha$ ). ■

## LECTURE 15

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 C3.5 LIE GROUPS, HT2015, Oxford

ASSUME:  $\tau$  MAX TORUS  
 $G$  COMPACT CONNECTED

WEYL GROUP

$$W(\tau) = \left\{ \underline{A_n}|_\tau : \tau \rightarrow \tau \text{ via } x \mapsto nxn^{-1} \text{ where } n \in N(\tau) \right\} \subseteq \text{Aut}(\tau)$$

Recall  $N(\tau) = \text{normalizer} = \{ n \in G : n\tau n^{-1} = \tau \}$

Rmk Independent (up to isomorphism) of choice of max torus  $\tau$ :  
 $N(g\tau g^{-1}) = gN(\tau)g^{-1}$  and  $W(g\tau g^{-1}) = \underline{g} \circ W(\tau) \circ \underline{g}^{-1}$

Lemma  $W(\tau) \cong N(\tau)/\tau$  via  $\underline{A_n}|_\tau \mapsto n\tau$

PF  $N(\tau) \rightarrow W(\tau)$  surjective b/c so done by 1st iso thm  
 $n \mapsto \underline{A_n}|_\tau$  if can show kernel =  $\tau$ .

kernel:  $n \in N(\tau)$  with  $\underline{A_n}|_\tau = \text{id}$  so  $n\tau n^{-1} = \tau$  all  $x \in \tau$   
 $\Rightarrow n \in \tau$  otherwise  $\tau \cup \{n\}$  larger abelian subgp than  $\tau$ .  
 (reproving 2(t)=T see Lecture 14) ■

Rmk By Lecture 14  $\tau = N(\tau)_0$  so  $N(\tau)/\tau = N(\tau)/N(\tau)_0$

is discrete and compact, hence finite. So  $W(\tau)$  is a finite group

Cor Characters  $\chi$  of  $G$  are determined by their restrictions  $\chi|_\tau$   
 and the  $\chi|_\tau$  are invariant under the Weyl group

PF  $\chi(h) = \chi(g\tau g^{-1}) = \chi(x) = \chi|_\tau(x)$  ✓  
 (recall thm: any  $h \in G$  lies in a conjugate of  $\tau$ , say  $h = gxg^{-1}$ ,  $x \in \tau$ )

$(\underline{A_n} \cdot \chi|_\tau)(x) = (\chi|_\tau \circ \underline{A_n}^{-1})(x) = \chi|_\tau(n^{-1}xn) = \chi|_\tau(x)$  ✓  
 (recall trace is conjugation invariant)

Theorem  $R(G) \rightarrow R(\tau)^W = \{\text{Weyl invariant virtual characters}\}$   
 $\chi \mapsto \chi|_\tau$  is an isomorphism!

Cor above shows injective b/c surjectivity is harder (won't prove it). In practice, you don't use surjectivity: first you find reps of  $G$  giving all possible characters in  $R(\tau)$  then by injectivity you know you have found all reps.

### Example Representation theory of $U(n)$

$T = \{\text{diagonal matrices}\} \subseteq U(n)$

Claim 1  $W(T) = S_n$  = symmetric group acting on  $T$  by permuting diagonal entries.  
Pf  $A_n(x) = n!x^{n-1}$  does not change the eigenvalues of  $x \in T$  (diagonal entries).

Recall (Q.sheet 3)  $T = \overline{ZxZ}$  if  $x = \begin{pmatrix} e^{2\pi i x_1} & & \\ & \ddots & \\ & & e^{2\pi i x_n} \end{pmatrix}$  with  $1, x_1, \dots, x_n$  lin. indep.  $\Leftrightarrow$

$A_n$  permutes the distinct entries  $e^{2\pi i x_j}$   $\square$

But  $A_n(x)$  determines  $A_n$  on  $T = \overline{ZxZ}$  by continuity

$\Rightarrow A_n = \text{a permutation of the diagonal entries}$

Conversely: all permutations arise because all transpositions arise:

$$\text{2x2 case: } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{so } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N(T)$$

nxn case: use matrix with  $\begin{cases} 1 & \text{on diagonal except in positions } (i,i), (j,j) \\ \text{transposes } \lambda_i \lambda_j & \text{in entries } (i,j), (j,i) \\ 0 & \text{else} \end{cases}$  (diagonal entries of  $T$ )

claim 2  $R(T)^{W(T)} = \mathbb{Z}[[t_1^{\pm 1}, \dots, t_n^{\pm 1}]]^S_n = \mathbb{Z}[[P_1, \dots, P_n, P_n^{-1}]]$  where

the  $P_j$  are the elementary symmetric polynomials in  $n$  variables:

$$P_1 = \sum t_j, \quad P_2 = \sum_{i < j} t_i t_j, \quad \dots, \quad P_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} X_{i_1} X_{i_2} \dots X_{i_k}, \quad \dots, \quad P_n = X_1 X_2 \dots X_n$$

Pf  $t_j$  corresponds to rep:  $(\lambda_1, \dots, \lambda_n) \in T \mapsto \lambda_j = e^{2\pi i x_j}$

$\Rightarrow S_n$  acts by permuting the  $t_1, \dots, t_n$ .

Fact from algebra:  $\mathbb{Z}[[t_1, \dots, t_n]]^S_n = \mathbb{Z}[[P_1, \dots, P_n]]$  poly ring in elem-sym. polys  
 If  $f \in \mathbb{Z}[[t_1^{\pm 1}, \dots, t_n^{\pm 1}]]^S_n$  then (large power of  $P_n$ )  $f$  will not have negative powers of  $t_j$ , so  $f \in P_n^-$ .  $\mathbb{Z}[[P_1, \dots, P_n]] \subseteq \mathbb{Z}[[P_1, \dots, P_n, P_n^{-1}]]$ .

Conversely  $\mathbb{Z}[[P_1, \dots, P_n, P_n^{-1}]] \subseteq \mathbb{Z}[[t_1^{\pm 1}, \dots, t_n^{\pm 1}]]^S_n$ , hence equality.  $\square$

Claim 3 There exists a rep  $V_k$  of  $U(n)$  with character  $\chi_{V_k} = P_k$  if  $U(n)$  acts on  $V \otimes V$  (standard rep:  $f(g) = g \in U(n) \subseteq GL(n, \mathbb{C}) \Rightarrow U(n)$  acts on  $V \otimes V$ ). On generators:  $g: (V_1 \otimes V_2) = gV_1 \otimes gV_2$ . But not irreps: has a subrep:

$\Lambda^2 V = \left\{ \text{tensors } \sum v_i \otimes w_i \text{ such that symmetric group } S_2 \text{ acting by permuting factors } v_i, w_i \text{ acts by sign(permuation). } Id \right\}$

$\Rightarrow \Lambda^2 V = \text{span} \{ V \otimes V - V \otimes V : v, w \in V \}$

(transposition (12) acts b $\gamma$   $w \otimes v - v \otimes w$  so by  $-Id = \text{sign}.Id$ )

Convenient to abbreviate  $v \wedge w = v \otimes w - w \otimes v$

$\Rightarrow \Lambda^2 V = \text{vector space over } \mathbb{C} \text{ with basis } (e_i \wedge e_j) \quad 1 \leq i < j \leq n$

With  $U(n)$ -action  $g: (e_i \wedge e_j) = g e_i \wedge g e_j$  is linear in each entry and

Rmk we stipulate that the symbol  $\wedge$  is antisymmetric:  $e_j \wedge e_i = -e_i \wedge e_j$ , so  $g(e_i \wedge e_j) = g e_i \wedge g e_j$  makes sense.

alternating product  $\wedge^k V = \{z \in V^{\otimes k} : \sigma \cdot z = \text{sign}(\sigma) \cdot z \text{ for all } \sigma \in S_k\}$

Rmk by permuting tensor factors

$\cong$  vector space over  $\mathbb{C}$  with basis the symbols  $(e_{i_1} \wedge \dots \wedge e_{i_k})_{1 \leq i_1 < \dots < i_k}$

$U(n)$ -action  $g: (e_{i_1} \wedge \dots \wedge e_{i_k}) = g e_{i_1} \wedge \dots \wedge g e_{i_k}$

$(v_1 \wedge \dots \wedge v_k) \equiv \sum_{\sigma \in S_k} \text{sign}(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}$

In particular,  $\wedge^n V = \mathbb{C} \cdot e_1 \wedge e_2 \dots \wedge e_n$  is 1-dimensional.

Character restricted to  $T$ :  $\text{trace} = \Sigma \text{ evaluates:}$

$$g = (\lambda_1 \dots \lambda_n) \in T \Rightarrow g \cdot e_i = \lambda_i e_i \Rightarrow \chi_{V_i} = \sum \lambda_i$$

Also  $g \cdot (e_i \wedge e_j) = g e_i \wedge g e_j = \lambda_i e_i \wedge \lambda_j e_j \stackrel{\text{if } \lambda_i \neq \lambda_j}{=} \lambda_i \lambda_j (e_i \wedge e_j) \Rightarrow \chi_{V_2} = \sum \lambda_i \lambda_j$

$\lambda_i \lambda_j = (\lambda_{i_1} \dots \lambda_{i_k}) e_{i_1} \wedge \dots \wedge e_{i_k} \Rightarrow \chi_{V_k} = \sum \lambda_{i_1} \dots \lambda_{i_k}$

In particular:  $\chi_{V_n}(\lambda_1 \dots \lambda_n) = \lambda_1 \lambda_2 \dots \lambda_n = \det(g)$  (1-dimensional rep)  $\square$

$$R(U(n)) = \mathbb{C}[\chi_V, \chi_{V^2}, \dots, \chi_{V^n}, \chi_{V^n}^{-1}]$$

Claim 4

$$\det(\det)^{-1} = \det$$

Notice: we only use that  $R(U(n)) \rightarrow R(T)^{W(T)}$  is injective (which we proved in general) and we deduced surjectivity in this example by claim 3.

Claim 5 The  $V, V^2, \dots, V^n, \overline{V^n}$  are irreps ( $\overline{V^n}$  obviously since 1-dimensional)

Pf Suppose not:  $\wedge^k V = U_1 \oplus U_2$  sum of subreps. If we knew that  $e_1, e_2, \dots, e_k \in U_1$ , then in fact  $U_1 = \wedge^k V, U_2 = \{0\}$  because the matrix  $g \in U(n)$  with columns  $(e_{i_1} \wedge \dots \wedge e_{i_k})$  acts by  $g \cdot e_m = e_{i_m}$  so  $g \cdot (e_1 \wedge \dots \wedge e_k) = e_{i_1} \wedge \dots \wedge e_{i_k}$  which is a basis for  $\wedge^k V$ .

Trick Consider action of max tons  $T \subseteq U(n) \rightarrow \text{Aut}(\wedge^k V)$ . Then  $\wedge^k V = \bigoplus V_\alpha$  where  $V_\alpha = \{v \in \wedge^k V : g(v) = \chi_\alpha(g)v\}$  are weight spaces ( $\alpha \in \mathbb{Z}^n$ ):

$V_\alpha = \text{sum of 1-dim irreps } c_\alpha \text{ each with character } \chi'_\alpha(g) = e^{2\pi i \langle \alpha, g \rangle}$ . Similarly  $U_1 = \bigoplus V_{\alpha_1}, U_2 = \bigoplus V_{\alpha_2}$  with  $V_{\alpha_1}, V_{\alpha_2} \subseteq V_\alpha$  (unique decomposition into irreps!) But we know  $V_\alpha = \{v \in \wedge^k V : g(v) = \chi_\alpha(g)v\}$  are weight spaces ( $\alpha \in \mathbb{Z}^n$ ):

(Indeed  $V_\alpha = \mathbb{C} e_{i_1} \wedge \dots \wedge e_{i_k}$  where  $a$  has 1 in entries  $i_1, \dots, i_k$  and 0 elsewhere)  $\Rightarrow V_{\alpha_1} = V_\alpha$  or  $V_{\alpha_2} = V_\alpha$  so  $e_1, \dots, e_k \in U_1$  or  $U_2$  (as opposed to being  $U_1 \cup U_2$ )  $\square$

Rmk There is a more systematic approach to finding irreps for  $G$  by looking for "highest weight vectors" in terms of a certain ordering of weights  $\alpha \in \mathbb{Z}^n$ . For  $U(n)$ , the lexicographic order works so  $(1, \dots, 1, 0, \dots, 0)$  is the highest weight arising in  $\wedge^k V$  and  $(0, \dots, 0, 1)$  is the highest weight vector.



## KILLING FORM

$G$  compact connected

Recall (Q. sheet 2) :

$$\langle x, y \rangle = \text{Trace}(\text{ad}(x)\text{ad}(y)) \quad \text{recall } \text{ad}(x) \in \text{End}(V)$$

$\text{ad}(x)(y) = [x, y]$

is bilinear map  $V \times V \rightarrow F$

any Lie algebra  $V$ .

Thm For  $G$  compact,

$$\langle x, x \rangle \leq 0 \quad \text{all } x \in \mathfrak{g}$$

if and only if  $x \in \text{Centre}(G)$

Pf Using  $\text{Ad}(G)$ -invariant metric  $\langle \cdot, \cdot \rangle$  from above,  $\mathfrak{g} \cong \mathbb{R}^m$  and

$$\begin{aligned} \text{Ad}(\mathfrak{g}) &\in O(m) \quad \text{so } \text{ad}(\mathfrak{g}) \in \sigma(m) \text{ by naturality: } \mathfrak{g} \xrightarrow{\text{ad}} \sigma(m) \\ \Rightarrow A &= \text{ad}(x) = \text{skew-symmetric matrix} \\ \Rightarrow \langle x, x \rangle &= \text{Tr}(AA) = \text{Tr}(-A^T A) = \sum_i -A_{ii}^2 \leq 0 \end{aligned}$$

iff all  $A_{ij} = 0$ , that is  $A = \text{ad}(x) = 0$ . ■

Cor For  $G$  compact,  $\mathfrak{g} = \text{Centre}(G) \oplus \mathfrak{g}'$  where  $\mathfrak{g}' \subseteq \mathfrak{g}$  is an ideal on which the Killing form is negative definite.

$$\begin{aligned} \text{Ad}(\mathfrak{g}) &\text{ acts by orthogonal matrices so sends } \mathfrak{g}' \rightarrow \mathfrak{g}', \text{ so } \text{ad} = \text{Ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}') \\ \Rightarrow \text{ad}(x)(y) &= [x, y] \in \mathfrak{g}' \quad \text{for } x \in \mathfrak{g}, y \in \mathfrak{g}' \text{ so } \mathfrak{g}' \text{ is an ideal.} \end{aligned}$$

By above Thm,  $\langle y, y \rangle > 0$  for  $y \neq 0 \in \mathfrak{g}'$  since  $y \notin \text{Centre}(G)$  ■

## CLASSIFICATION OF COMPACT LIE GROUPS

Recall (Q. Sheet 4)

Lie algebra  $V$  called simple if  $V$  not abelian and only ideals are  $0, V$

$V$  semi-simple if  $V = \bigoplus$  simple Lie algebras.

Q. Sheet 4 : for connected Lie group  $G$ ,

$\mathfrak{g}$  simple  $\Leftrightarrow G$  simple (meaning: not abelian and  $\{1\}$  is the only non-trivial connected normal Lie subgroup)

Fact Killing form on  $V$  is non-degenerate  $\Rightarrow V$  semi-simple

Cor  $G$  compact  $\Rightarrow \mathfrak{g} = (\text{abelian summand}) \oplus (\text{semi-simple summand})$

Rmk • So don't have nasty summands ("solvable summands") in Lie algebra.

- Lie algebra theory can classify all simple Lie algebras!
- Consequence : can classify compact Lie groups!

— END OF COURSE —

FACT 1 If  $G$  simply-connected and simple, then  $G$  can be :

$\text{SU}(n)$  (Example  $\text{SU}(2) = \text{Spin}(3)$ )

$\text{Spin}(n) = \text{universal cover of } SO(n)$

$SO(n)$  = univ. cover of  $SO(3)$

$Sp(n)$  = symplectic group (Lecture 2)

$G_2, F_4, E_6, E_7, E_8$   $\leftarrow$  exceptional Lie groups.

Pf If omit assumption of simple, then  $G$  is a product of above groups

$\text{FACT 2 } G$  has a finite cover  $G' \xrightarrow{\pi} G$  (meaning  $\ker \pi$  finite)

with  $G' \cong$  torus  $\times$  finite product of groups from above list.

$\cong$  finite product of copies of  $S^1$  and groups from above list.

Rmk By Q1 of Q. sheet 5,  $G \cong G'/\Gamma$  where  $\Gamma \subseteq \text{Centre}(G')$  is a finite group (since discrete and compact).

The wikipedia page "List of simple Lie groups" has lots of info and examples about this classification.