

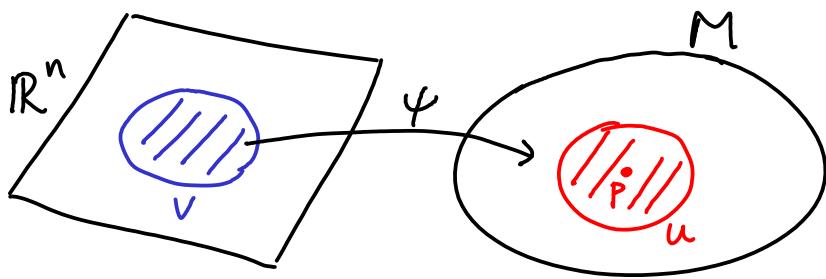
Let G be a group with operations

$$\begin{aligned}\mu: G \times G &\longrightarrow G & \mu(g, h) = gh \\ i: G &\longrightarrow G & i(g) = g^{-1}\end{aligned}$$

Def G is a Lie group if it is also a (smooth) manifold such that μ, i are smooth maps.

CRASH COURSE ON MANIFOLDS

A manifold $M = M^n$ of dimension n is a topological space which is locally parametrized by \mathbb{R}^n ,



A parametrization is a homeomorphism $\psi: V \rightarrow U$ (cts with cts inverse) from an open set $V \subset \mathbb{R}^n$ to an open set $U \subset M$.

Abbreviate this by

$$\psi: \mathbb{R}^n \dashrightarrow M$$

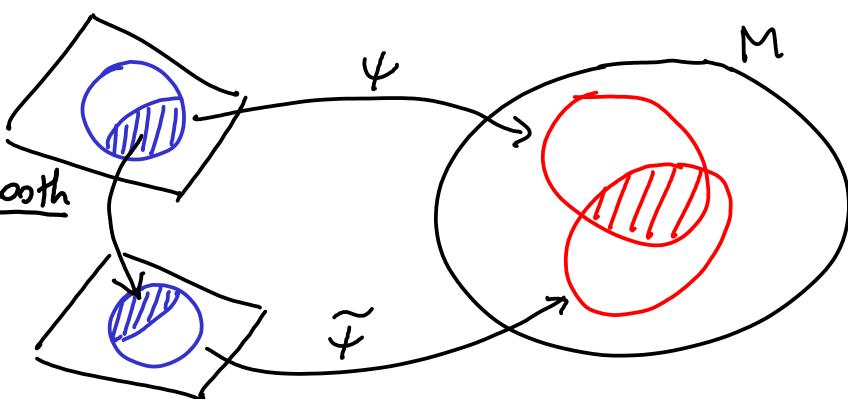
[Not standard notation - only this course]

such that on overlaps the parametrizations differ by smooth maps.

transition map

$$\tau = \tilde{\psi}^{-1} \circ \psi \text{ is smooth}$$

(all derivatives exist, as a map $\mathbb{R}^n \dashrightarrow \mathbb{R}^n$)

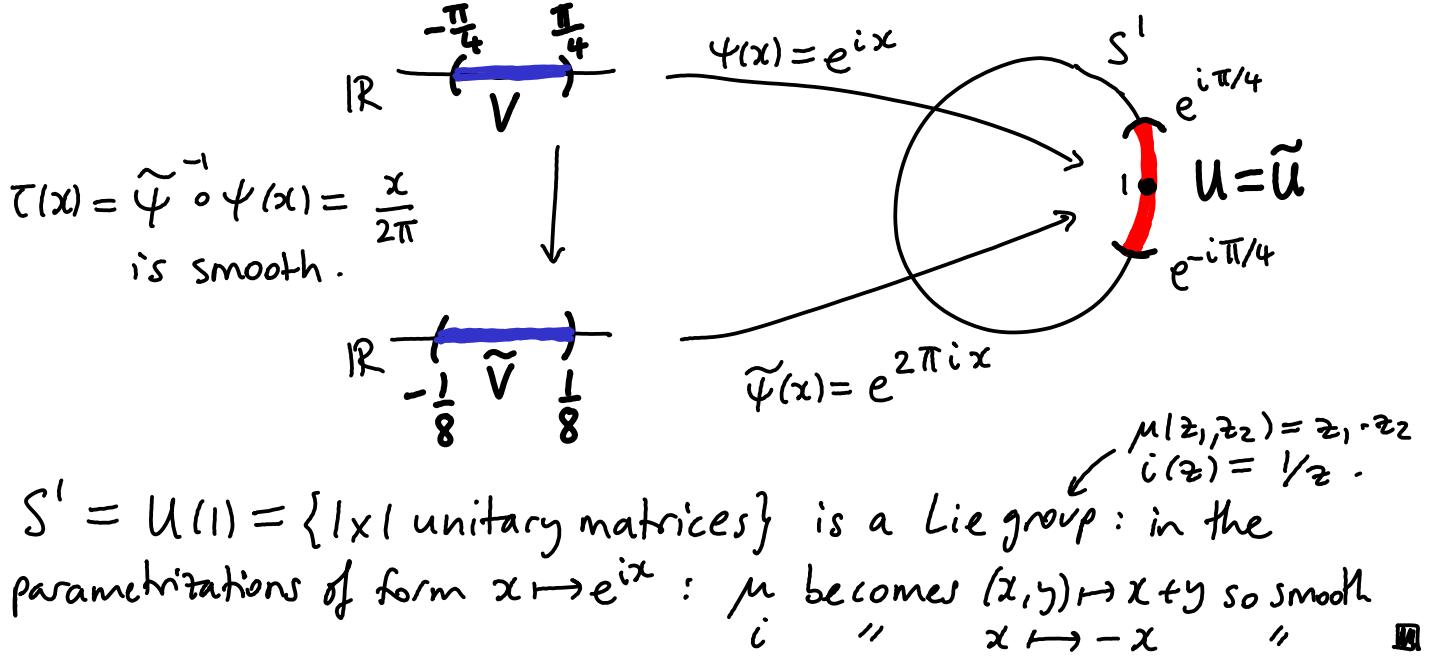


Rmks

- for a more precise definition, see the manifolds course.
- $\psi^{-1}: U \rightarrow V$ is called a chart

Def A parametrization $\psi: \mathbb{R}^n \dashrightarrow M$ defines local coordinates x_1, x_2, \dots, x_n on M
namely: $p \in U$ has coords $\psi^{-1}(p) = (x_1, \dots, x_n) \in \mathbb{R}^n$.

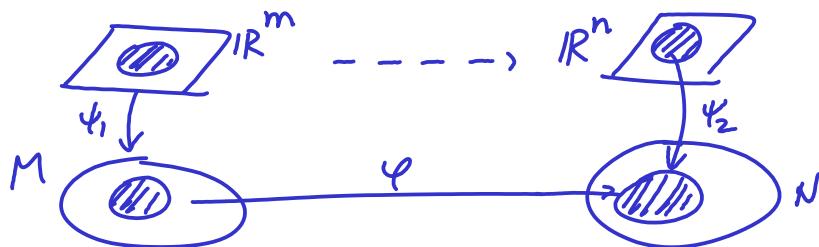
Example $M = S^1 = \text{circle} = \{z \in \mathbb{C} : |z|=1\}$



$S^1 = U(1) = \{1 \times 1 \text{ unitary matrices}\}$ is a Lie group: in the parametrizations of form $x \mapsto e^{ix}$: μ becomes $(x, y) \mapsto x + y$ so smooth
 i " " $x \mapsto -x$ " \blacksquare

SMOOTH MAPS

Def A continuous map $\varphi: M^m \rightarrow N^n$ of manifolds is smooth if locally in some (and hence all) parametrizations the map $\mathbb{R}^m \dashrightarrow \mathbb{R}^n$ is smooth.



locally: $\varphi(x) = (\varphi(x_1, \dots, x_n)) = (y_1(x), \dots, y_n(x))$
(so really mean) $\psi_2^{-1} \circ \varphi \circ \psi_1$ $x_i = \text{local coords near } p$ $y_i = \text{local coords near } \varphi(p)$

So: φ is smooth \Leftrightarrow the $y_i(x)$ are smooth functions of x .

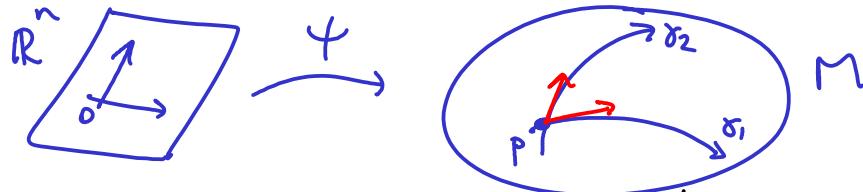
VECTORS

Def A (tangent) vector at $p \in M$ is an equivalence class $[\gamma]$ of smooth curves $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$, $\gamma(0) = p$ (some $\varepsilon > 0$)

equivalent $\gamma \sim \tilde{\gamma} \Leftrightarrow$ in some (hence all) parametrizations around p ,
 $\gamma'(0) = \tilde{\gamma}'(0) \in \mathbb{R}^n$.

[locally $\gamma(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n \Rightarrow$ derivative $\gamma'(0) = (x'_1(0), \dots, x'_n(0)) \in \mathbb{R}^n$]

Example



$\psi: \mathbb{R}^n \dashrightarrow M$, $\psi(0) = p$, determines n obvious vectors at p

$$[\gamma_1(t) = \psi(t, 0, \dots, 0)] \text{ called } \frac{\partial}{\partial x_1}$$

$$[\gamma_2(t) = \psi(0, t, 0, \dots, 0)] \quad " \quad \frac{\partial}{\partial x_2}$$

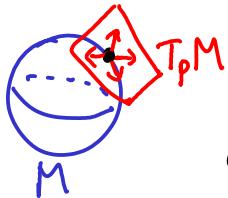
$$\dots \dots$$

$$[\gamma_n(t) = \psi(0, \dots, 0, t)] \quad " \quad \frac{\partial}{\partial x_n}$$

Rmks

- Locally in \mathbb{R}^n those curves correspond to the standard basis vectors of \mathbb{R}^n : $\gamma_1'(0) = \frac{\partial}{\partial t}|_0 (t, 0, \dots, 0) = (1, 0, \dots, 0) \in \mathbb{R}^n$
 $\gamma_j'(0) = \frac{\partial}{\partial t}|_0 (0, \dots, 0, t, 0, \dots, 0) = (0, \dots, 0, 1, 0, \dots, 0)$
 - don't really need $\psi(0) = p$: if $\psi(x) = p$ just translate: $[\gamma_i(t) = \psi(x + (t, 0, \dots, 0))]$
 - can add/scale vectors: if $\psi(0) = p$, then just add/scale the curves in \mathbb{R}^n
- EXAMPLE : $2 \frac{\partial}{\partial x_1} + 4 \frac{\partial}{\partial x_3} = 2[\gamma_1] + 4[\gamma_3] = [\gamma(t) = \psi(2t, 0, 4t, 0, \dots, 0)]$

The tangent space $T_p M$ at $p \in M$ is the vector space of vectors at p



$$T_p M \quad \cong$$

$$\mathbb{R}^n$$

$$a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n} = [\underbrace{\psi(x + (a_1 t, \dots, a_n t))}_{\gamma(t) = \text{curve in } \mathbb{R}^n}] \mapsto \gamma'(0) = (a_1, \dots, a_n)$$

Rmk

- The isomorphism depends on the choice of ψ with $\psi(x) = p$. We will see later that changing ψ to another parametrization $\tilde{\psi}$ corresponds to multiplying $\gamma'(0) \in \mathbb{R}^n$ by the derivative $D\tau$ of the transition map $\tau = \tilde{\psi}^{-1} \circ \psi$.

Vectors act on functions

For $v = [\text{curve } \gamma(t)] \in T_p M$ \Rightarrow $v \cdot f = \frac{\partial}{\partial t} \Big|_0 f(\gamma(t)) \in \mathbb{R}$
 $f: M \rightarrow \mathbb{R}$ defined near p

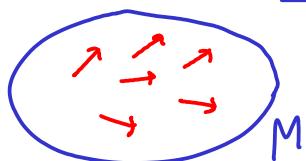
Locally it is just the obvious differentiation:

$$v = a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n} \quad v \cdot f = a_1 \frac{\partial f}{\partial x_1} + \dots + a_n \frac{\partial f}{\partial x_n} = \frac{\partial}{\partial t} \Big|_{t=0} f(a_1 t, \dots, a_n t)$$

where $\gamma(t) = (a_1 t, \dots, a_n t)$ is the local expression in parametrization ψ with $\psi(p)=0$

VECTOR FIELDS

Def A vector field is a map

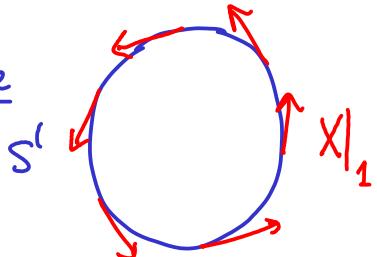


$$\begin{aligned} X: M &\longrightarrow TM = \bigsqcup_{p \in M} T_p M \\ p &\longmapsto X|_p \in T_p M \end{aligned}$$

such that locally $X|_x = a_1(x) \frac{\partial}{\partial x_1} + \dots + a_n(x) \frac{\partial}{\partial x_n}$ involves smooth functions $a_i(x) \in \mathbb{R}$.

[here $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i}|_x = [\psi(x + (0, \dots, 0, \underbrace{t}_\text{(position i)}, 0, \dots, 0))]$ is a vector at $x \in V$ so varies with x .]

Example



vector field $X = " \frac{\partial}{\partial \theta} "$ ($\theta = \text{angle} \in [0, 2\pi]$)

in a parametrization of type $\psi(x) = e^{ix}$
have $X|_x = \frac{\partial}{\partial x} = [\text{curve } t \mapsto e^{i(x+t)}]$

Vector fields act on functions: $(X \cdot f)|_p = X|_p \cdot f$ so $X \cdot f$ is a function

Locally it's just differentiation

$$X \cdot f = a_1(x) \frac{\partial f}{\partial x_1} + \dots + a_n(x) \frac{\partial f}{\partial x_n} \quad \leftarrow \text{gives a new function of } x$$

Rmk • X is determined locally by differentiating the coordinate functions x_i : $X \cdot x_i = a_i(x)$

\Rightarrow A vector field is uniquely determined by how it acts on functions!

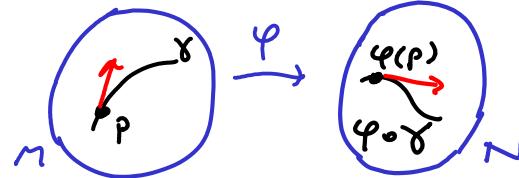
Rmk • When defining a concept locally, you always need to check that the choice of parametrization does not matter up to transition maps. So for $f: M \rightarrow \mathbb{R}$ we want $X \cdot f: M \rightarrow \mathbb{R}$ to be independent of the choices of ψ that locally define $X \cdot f$.
(see Appendix if you care)

DERIVATIVE MAP

Def The derivative (or differential) of $\varphi: M \rightarrow N$ is

$$D\varphi: TM \rightarrow TN \quad (\text{in particular } D_p \varphi: T_p M \rightarrow T_{\varphi(p)} N)$$

$$D_p \varphi \cdot [\gamma] = [\varphi \circ \gamma]$$



CLAIM the derivative of φ is a linear map which is locally the matrix of partial derivatives of φ .

Proof Locally $\varphi(x_1, \dots, x_m) = (y_1(x), \dots, y_n(x))$ as a map $\mathbb{R}^m \dashrightarrow \mathbb{R}^n$

(we abusively write φ although it really is $\psi_2^{-1} \circ \varphi \circ \psi_1$, for parametrizations ψ_1 on M near p and ψ_2 on N near $\varphi(p)$).

$$\text{By definition, } D\varphi \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = D\varphi \cdot \frac{\partial}{\partial x_i} = D\varphi \cdot [\gamma_i(t) = [t, 0, \dots, 0]] = [\varphi \circ \gamma_i(t)] \\ = [(y_1(t, 0, \dots, 0), \dots, y_n(t, 0, \dots, 0))]$$

Locally (that is using the above isomorphism $T_p M \cong \mathbb{R}^m$, $[\varphi \circ \gamma] \mapsto \gamma'(0)$) we just need to differentiate the curve in t at time $t=0$:

$$\frac{d}{dt}|_{t=0}(y_1(t, 0, \dots, 0), \dots, y_n(t, 0, \dots, 0)) = \left(\frac{\partial y_1}{\partial x_1}, \dots, \frac{\partial y_n}{\partial x_1} \right) \in \mathbb{R}^n$$

(which is now written in the basis $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ of $T_{\varphi(p)} N \cong \mathbb{R}^n$)

$$\left(\text{so explicitly: } D_p \varphi \cdot \frac{\partial}{\partial x_i} = \frac{\partial y_1}{\partial x_i} \frac{\partial}{\partial y_1} + \dots + \frac{\partial y_n}{\partial x_i} \frac{\partial}{\partial y_n} \in T_{\varphi(p)} N \right)$$

Similarly $D_p \varphi \cdot \frac{\partial}{\partial x_j} = \left(\frac{\partial y_1}{\partial x_j}, \dots, \frac{\partial y_n}{\partial x_j} \right)$ and in general:

$$D_p \varphi \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = D_p \varphi \cdot \left(a_1 \frac{\partial}{\partial x_1} + \dots + a_n \frac{\partial}{\partial x_n} \right) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \dots & \frac{\partial y_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$$

(recall the columns of a matrix are the images of the standard basis) ■

Claim

CHAIN RULE

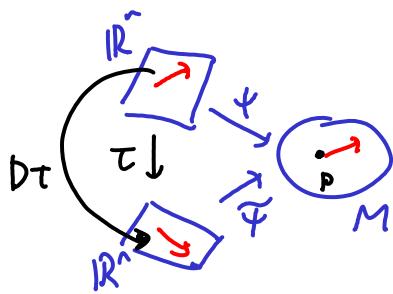
$$M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \Rightarrow D(\psi \circ \varphi) = D\psi \circ D\varphi$$

(locally it is
just the usual
chain rule)

Proof

$$D(\psi \circ \varphi) \cdot [\gamma] = [\psi \circ \varphi \circ \gamma] = D\psi \cdot [\varphi \circ \gamma] = D\psi \circ D\varphi \cdot [\gamma] ■$$

EXAMPLE The local expression of $\varphi = \text{identity} : M \rightarrow M$ if we use param. ψ near p on domain, and param. $\tilde{\psi}$ near $\varphi(p) = p$ on the image, is the transition map $\tau : \mathbb{R}^n \dashrightarrow \mathbb{R}^n$, $\tau = \tilde{\psi}^{-1} \circ \psi$.



Therefore $D\varphi = \text{identity}$ is locally $D\tau$.

Hence if $X^4, X^F \in \mathbb{R}^n$ are local expressions of the same vector X at $p \in M$ in parametrizations $\psi, \tilde{\psi}$ then

$$X^F = D\tau \cdot X^4$$

(see Appendix for more on this if you care)

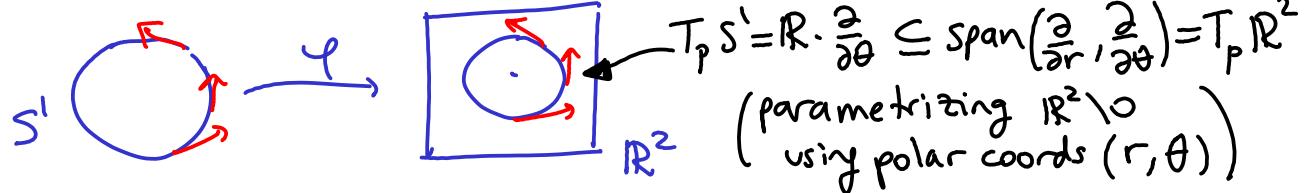
EXAMPLE $M = S^1$ vector field " $\frac{\partial}{\partial \theta}$ " is $X^4 = \frac{\partial}{\partial x}$ in param. $\psi(x) = e^{ix}$

However mathematicians often want to view S^1 as the quotient \mathbb{R}/\mathbb{Z} so they want to use $\tilde{\psi}(x) = e^{2\pi i x}$ (so that $\tilde{\psi}(\mathbb{Z}) = 1$). The local expression of " $\frac{\partial}{\partial \theta}$ " becomes: $X^F = D\tau \cdot X^4 = \frac{1}{2\pi} \cdot \frac{\partial}{\partial x}$ (since $\tau(x) = \frac{x}{2\pi}$, $D\tau = \frac{1}{2\pi} \cdot \text{Id}$)

Def $\varphi : M \rightarrow N$ is called an embedding if $\varphi : M \rightarrow \varphi(M)$ is a homeomorphism and $D_p \varphi : T_p M \rightarrow T_{\varphi(p)} N$ is injective $\forall p \in M$.

Rmks • Think of $\varphi(M) \subseteq N$ as an identical copy of M inside N
• $D_p \varphi$ injective $\Rightarrow D_p \varphi \cdot T_p M$ is a copy of $T_p M$ in $T_{\varphi(p)} N$ as a vector subspace

Example



EXAMPLE General linear group

$GL(n, \mathbb{R}) = \text{Lie group of invertible } n \times n \text{ real matrices} = \left\{ A : n \times n \text{ matrix with } \det A \neq 0 \right\}$
obvious parametrization near I :

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots \\ \vdots & & \\ a_{nn} & \dots & a_{nn} \end{pmatrix} \xleftarrow{\psi} (a_{11}, a_{12}, \dots, a_{21}, \dots, a_{nn}) \in \mathbb{R}^{n^2}$$

For any matrix $B = (b_{ij})$ have curve

$$\gamma(t) = A + tB = (a_{ij} + tb_{ij}) \xleftarrow{\text{still invertible for small } |t|} \text{so } A + tB \in GL(n, \mathbb{R})$$

$$\Rightarrow \text{vector } \frac{\partial}{\partial t} \Big|_0 \gamma(t) = B \in \text{Mat}_{n \times n}(\mathbb{R}) = \{n \times n \text{ real matrices}\}$$

$$\Rightarrow T_I GL(n, \mathbb{R}) = \text{Mat}_{n \times n}(\mathbb{R})$$

EXAMPLE orthogonal group

$O(n, \mathbb{R}) = \text{Lie group of orthogonal matrices} = \{A \in \text{Mat}_{n \times n} : A^T A = I\}$

Not easy to write down a parametrization near I

Note: the ψ for $GL(n, \mathbb{R})$ cannot work: ① the dimension is wrong and ② if you "wiggle" the a_{ij} then A may no longer be orthogonal!

(1) : $O(2, \mathbb{R}) = \{\text{rotations } \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\} \sqcup \{\text{reflections } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}\}$
 has dimension 1 obviously, since we give parametrizations in $\theta \in \mathbb{R}$
 Whereas $GL(2, \mathbb{R})$ has dimension $2^2 = 4 = \# \text{ entries.}$

Trick Consider the embedding $\psi: O(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}), A \mapsto A$

and find $T_I O(n, \mathbb{R})$ as vector subspace of $T_I GL = \text{Mat}_{n \times n}$

For a curve $A(t) \in O(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$ through $A(0) = I$,

$$0 = \frac{d}{dt} \Big|_0 \underbrace{A(t)^T A(t)}_1 = \underbrace{A'(0)^T A(0)}_{B^T} + \underbrace{A(0)^T A'(0)}_{I} = B^T + B$$

$$\Rightarrow T_I O(n, \mathbb{R}) \subseteq \text{vector subspace of skew symmetric matrices} \\ \uparrow \quad \{B \in \text{Mat}_{n \times n}(\mathbb{R}) : B^T + B = 0\}$$

in fact, equality. One way to prove it is to check that $\dim O(n, \mathbb{R}) = \dim \{\text{skew } B\}$ since you are comparing vector spaces.

Question sheet: why is $O(n, \mathbb{R})$ a manifold? Need :

Implicit function theorem

Assume: $\varphi: M^m \rightarrow N^n$ smooth, $D_p \varphi: T_p M \rightarrow T_q N$ surjective
 for all $p \in \varphi^{-1}(q) = \{p \in M : \varphi(p) = q\}$.

- $\varphi^{-1}(q) \subseteq M$ is a submanifold ($\varphi^{-1}(q)$ is a manifold and the inclusion into M is an embedding)
- $\dim \varphi^{-1}(q) = m - n \quad \leftarrow \text{idea: } \varphi = q \text{ imposes } n = \dim N \text{ conditions (independent equations)}$
- $T_p(\varphi^{-1}(q)) = \text{Ker } D_p \varphi \quad \leftarrow \text{idea: if } \varphi = q \text{ is constant then } \varphi(\text{curve}) = \text{constant so zero vector.}$

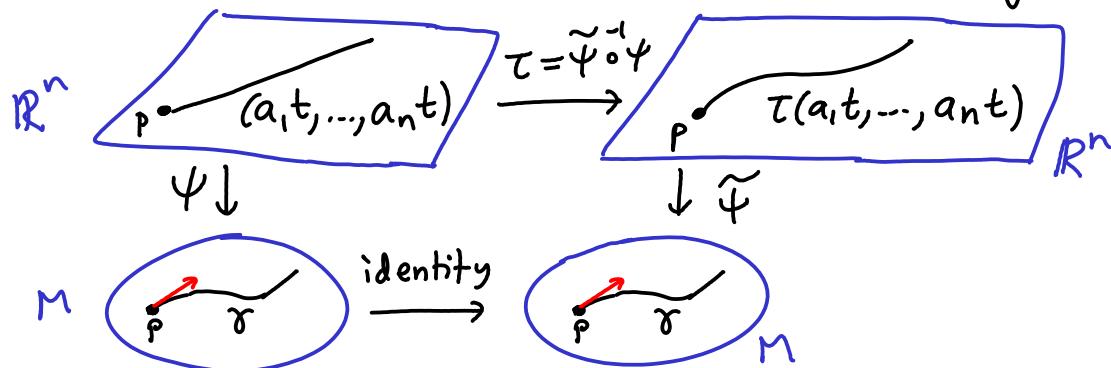
EXAMPLE: $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}, \varphi(x, y) = x^2 + y^2$. Take $q = 1$ then $\varphi^{-1}(1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

= circle S^1 . $D\varphi = \text{matrix } (2x \ 2y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is non-zero if $x^2 + y^2 = 1$ hence $D\varphi$ surjective.

Implicit fn thm $\Rightarrow S^1$ is mfd of $\dim = 2 - 1 = 1$ with $T_{(x,y)}S^1 = \text{Ker}((2x \ 2y) : \mathbb{R}^2 \rightarrow \mathbb{R})$
 as a vector subspace of $\mathbb{R}^2 \equiv T_{(x,y)}\mathbb{R}^2$. For example for $(x, y) = (1, 0)$ get $T_{(1,0)}S^1 = \text{span}(\begin{pmatrix} 2 \\ 0 \end{pmatrix})$. $S^1 \bigcirc \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Appendix (NON-EXAMINABLE, also NOT IMPORTANT)

Question: how do vectors transform locally if change parametrization?



$$T_p M \cong \mathbb{R}^n \longrightarrow \mathbb{R}^n \cong T_p M$$

$$\sum a_i \frac{\partial}{\partial x_i} \cong (a_i)_{i=1,\dots,n} = \frac{\partial}{\partial t} |(a_i, t) \longmapsto \frac{\partial}{\partial t} | \tau \cdot (a_i, t) = \left(\sum_j \frac{\partial \tau_i}{\partial x_j} \cdot a_j \right) \cong \sum_{j,i} \frac{\partial \tau_i}{\partial x_j} a_j \frac{\partial}{\partial \tilde{x}_i}$$

↑ chain rule ↑ $\tau(x) \in \mathbb{R}^n$ has coordinates $\tau_i(x) \in \mathbb{R}$

\Rightarrow vectors transform by left-multiplication by the derivative $D\tau$ of the transition map $\tau: \mathbb{R}^n \dashrightarrow \mathbb{R}^n$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \begin{pmatrix} \frac{\partial \tau_1}{\partial x_1} & \frac{\partial \tau_1}{\partial x_2} & \dots & \frac{\partial \tau_1}{\partial x_n} \\ \frac{\partial \tau_2}{\partial x_1} & \dots & \dots & \frac{\partial \tau_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial \tau_n}{\partial x_1} & \dots & \dots & \frac{\partial \tau_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

Question: is $X \cdot f = \sum a_i(x) \frac{\partial f}{\partial x_i} \in \mathbb{R}$ well-defined? (independent of choice of ψ)

Need more precise notation:

For $\psi: X^\psi = \sum a_i(x) \frac{\partial}{\partial x_i}, f^\psi(x) = f(\psi(x)), X^\psi \cdot f^\psi = \sum a_i(x) \frac{\partial f}{\partial x_i}$

For $\tilde{\psi}: X^{\tilde{\psi}} = D\tau \cdot X^\psi, f^{\tilde{\psi}}(y) = f(\tilde{\psi}(y)) = f^\psi(\tau^{-1}(y))$

$$X^{\tilde{\psi}} \cdot f^{\tilde{\psi}} = (D\tau \cdot X^\psi) \cdot (f^\psi \circ \tau^{-1}) \stackrel{\text{CHAIN RULE}}{=} D\tau \cdot X^\psi \cdot f^\psi \cdot D\tau^{-1} = X^\psi \cdot f^\psi$$

$\Rightarrow X^{\tilde{\psi}} \cdot f^{\tilde{\psi}} = X^\psi \cdot f^\psi$ agree at points of M . ■

check by writing it out with indices.

LECTURE 2

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C3.5 LIE GROUPS, HT2015, Oxford.

Examples of Lie groups

0) Finite groups (and discrete groups) : not so interesting since 0-dimensional manifolds (just a set of points)

lecture 1

$$\begin{cases} 1) \quad S^1 = U(1) \\ 2) \quad GL(n, \mathbb{R}) \end{cases}$$

3) $O(n, \mathbb{R}) = \{\text{isometries of } \mathbb{R}^n \text{ which fix origin}\}$ include translations

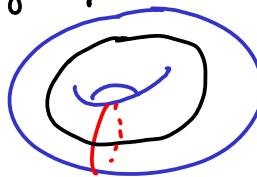
4) $\text{Isom}(\mathbb{R}^n) = \{\text{all Euclidean isometries of } \mathbb{R}^n\}$ $x \mapsto x + c$.

5) $\mathbb{R}^n \leftarrow \mu(x, y) = x + y \text{ and } i(x) = -x$.

6) G_1, G_2 Lie groups $\Rightarrow G_1 \times G_2$ Lie group

7) $T^n = S^1 \times \dots \times S^1$ n-dimensional torus

$$T^2 = S^1 \times S^1$$



The skew-field of quaternions

(non-commutative multiplication)

Def Quaternions \mathbb{H} : 3 equivalent definitions :

matrices	\mathbb{C} - v.s. dim=2	\mathbb{R} - v.s. dim=4
$\{h = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}\}$ $\mathbb{C} = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \right\} \subseteq \mathbb{H}$ $i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\{h = a + bj : a, b \in \mathbb{C}\}$ $\mathbb{C} = \{a + 0j\} \subseteq \mathbb{H}$ <u>how j multiplies</u> $j^2 = -1$ $jz = \bar{z}j \text{ for } z \in \mathbb{C}$	$\{h = x_1 + x_2i + x_3j + x_4k : x_i \in \mathbb{R}\}$ $\mathbb{C} = \{x_1 + x_2i\} \subseteq \mathbb{H}$ <u>quaternion relations</u> $i^2 = j^2 = k^2 = -1$ $ijk = -1$ (NOT COMMUTATIVE :) $ij = -ji = k$
$h^* := \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$ conjugate transpose $ h ^2 = \det = a ^2 + b ^2$	$h^* = \bar{a} - b j$ $ h ^2 = h^* h = a ^2 + b ^2$	$h^* = x_1 - x_2i - x_3j - x_4k$ $ h ^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$

scalar product $\langle h_1, h_2 \rangle = h_1^* h_2 \in \mathbb{H}$ induces the norm $|h| = \sqrt{\langle h, h \rangle}$

Rmk similar to \mathbb{C} : e.g. $h \neq 0 \Rightarrow h^{-1} = h^*/|h|^2$

but careful about non-commutativity: $(h_1 h_2)^* = \underline{\underline{h_2}} h_1^*$ ← what you expect when transpose matrices.

More Examples of Lie groups

- 8) $GL(1, \mathbb{H}) = \mathbb{H} \setminus \{0\}$ a "circle" in the world of quaternions
 9) $Sp(1) = \{ h \in \mathbb{H} : |h|=1 \} \subseteq \mathbb{H} \setminus \{0\}$ quaternion group
 10) Generalize 5, 2, 3:

\mathbb{R}^n	V vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H}$
$GL(n, \mathbb{R})$	$\text{Aut}(V) = \{\mathbb{F}\text{-linear ssos } V \rightarrow V\}$ automorphism group
$O(n, \mathbb{R})$	$G = \{A \in \text{Aut } V : \langle Av, Aw \rangle = \langle v, w \rangle \forall v, w \in V\}$ \star ↑ depending on a choice of scalar product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$
11) In \star : $V = \mathbb{F}^n$ with $\langle v, w \rangle = v^* w$	(* = conjugate transpose. Over \mathbb{R} just transpose)

\mathbb{F}	Lie group $G \approx \{A \in GL(n, \mathbb{F}) : v^* A^* A w = v^* w \text{ all } v, w \in \mathbb{F}^n\}$
\mathbb{R}	orthogonal group $O(n) = O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : A^T A = I\}$
\mathbb{C}	unitary group $U(n) = \{A \in GL(n, \mathbb{C}) : A^\dagger A = I\}$
\mathbb{H}	symplectic group $Sp(n) = \{A \in GL(n, \mathbb{H}) : A^* A = I\}$

Ranks

- implicit function theorem \Rightarrow these are Lie groups,
and $T_I G = \{B \in \text{Mat}_{n \times n}(\mathbb{F}) : B^* + B = 0\}$
- fact: $|Av| = |v| \quad \forall v \Leftrightarrow \langle Av, Aw \rangle = \langle v, w \rangle \quad \forall v, w \Leftrightarrow A^* A = I$
norm $|v| = \sqrt{\langle v, v \rangle}$

For $O(n)$ geometrically this says linear maps which preserve lengths must preserve angles since $\langle v, w \rangle = |v| \cdot |w| \cdot \cos(\text{angle between } v, w)$

- 12) special linear group $SL(n, \mathbb{F}) = \{A \in GL(n, \mathbb{F}) : \det A = 1\}$ = volume preserving automorphisms
 similarly: $SO(n) \subseteq O(n)$ impose $\det = 1$
 $SU(n) \subseteq U(n)$

(can't do it for $Sp(n)$ since \det doesn't make sense since \mathbb{H} not commutative)

Q.Sheet 2: $\det = 1$ imposes $\text{trace}(B) = 0$ for tangent vectors $B \in T_I G \subseteq \text{Mat}_{n \times n}$

Example $O(2) = SO(2) \sqcup \{\text{reflections}\}$
 $SO(2) = \begin{matrix} \text{rotations} \\ (\text{connected component of } I) \end{matrix}$
 $\{ \text{reflections} \} = \begin{matrix} \text{not a group} \\ \text{it is a coset} \end{matrix}$
 $O(2) \cong SO(2) \times \{\pm 1\}$ via $A \mapsto (\det A) \cdot A, \det A$

13) Def $G_0 = (\text{connected component of } 1 \text{ in } G)$ ↪ Lie group

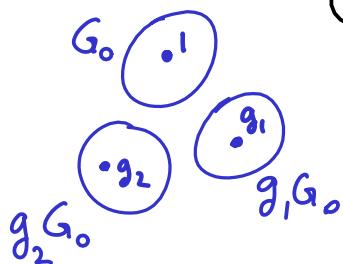
Lemma

① G_0 is a Lie group

② the cosets of G_0 in G are the connected components of G and they give an open cover $G = \bigsqcup gG_0$

③ G_0 is a normal subgroup of G $gG_0 \in G/G_0 = \{\text{cosets}\}$

④ The quotient group G/G_0 is a Lie group (discrete group so 0-manifold)



EXAMPLE $G = O(n) \Rightarrow G_0 = SO(n)$ and $G/G_0 \cong \{\pm 1\}$ (group with 2 elements)

Rmk's

- For subgroups $H \leq G$ the space of cosets G/H may not be group.

$H \leq G$ called normal subgroup if $hHh^{-1} \subseteq H \quad \forall h \in G$

This condition ensures that G/H is a group using $g_1H \cdot g_2H = g_1g_2H$

- In general $G \cong G_0 \times \frac{G}{G_0}$ is false! (see Appendix if you care)

Proof of Lemma

$\phi_g : G \rightarrow G, h \mapsto gh$ is a homeomorphism (inverse is $\phi_{g^{-1}}$)

(continuous - indeed smooth - since group multiplication is smooth)

$\Rightarrow \phi_g$ sends connected components to connected components

$\Rightarrow gG_0$ are connected components

Recall: any topological space = disjoint union of its connected components

Hence: $g \in G_0 \Rightarrow g \in G_0 \cap gG_0 \Rightarrow G_0 = gG_0 \quad (\text{since } G_0 \cap gG_0 \neq \emptyset)$
 $\Rightarrow g^{-1}G_0 = G_0$

Therefore can restrict multiplication and inversion to G_0 , proving ①

If C is a connected component and $g \in C$, then $g \in C \cap gG_0 \neq \emptyset$, so $C = gG_0$
 \Rightarrow ② follows (using general fact: connected components of a manifold are always open sets)

For ③ use homeomorphism $G \rightarrow G, h \mapsto ghg^{-1}$ (inverse $h \mapsto g^{-1}hg$)
 G_0 connected component $\Rightarrow gG_0g^{-1}$ connected component
but $1 \in G_0 \cap gG_0g^{-1} \neq \emptyset$ so $G_0 = gG_0g^{-1}$.

④ follows (not much content in ④ since silly manifold: $\dim = 0$) ■

TOPOLOGICAL PROPERTIES

COMPACTNESS

recall compact means open covers always have finite subcovers.

- useful trick: 1) first embed $G \subseteq \mathbb{R}^m$ (some large m)
 2) then use Heine-Borel theorem for \mathbb{R}^m :

$$\text{compact} \Leftrightarrow \text{closed \& bounded}$$

that is:

- check limits stay in G

- \mathbb{R}^m -norm on $G \subseteq \mathbb{R}^m$ is bounded

- EXAMPLE $S^1 \subseteq \mathbb{R}^2$
- if $z_n \rightarrow z$ with $|z_n|=1$ then $|z|=1$ so $z \in S^1$
 - $|z| \leq 1$ on S^1 (since $|z|=1 \forall z \in S^1$)

CONNECTEDNESS

- facts
- manifolds are metric spaces (since can always embed $M \subseteq \mathbb{R}^{\text{huge}}$)
 - for manifolds a subset is a connected component \Leftrightarrow open and closed
 - for manifolds: connected \Leftrightarrow path-connected

EXAMPLE

$$GL(1, \mathbb{R}) = \mathbb{R} \setminus \{0\} = \mathbb{R}^+ \sqcup \mathbb{R}^- \text{ not connected}$$

$$GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\} \text{ connected since path-connected.}$$

← DON'T NEED TO MEMORIZE THIS!

Some Topological facts (some are tricky to prove)

G	connected ($G=G_0$)	# Connected components if #1	compact
\mathbb{R}^n	✓		✗ (n ≥ 1) ✓
T^n			
$GL(n, \mathbb{R})$	✗		✗
$SL(n, \mathbb{R})$	✓		✗ (n ≥ 2)
$O(n)$	✗		✓
$SO(n)$	✓	$2 < \frac{\det}{\det} > 0$	✓
$GL(n, \mathbb{C})$			✗
$SL(n, \mathbb{C})$			✗ (n ≥ 2)
$U(n)$			✓
$SU(n)$	✓		✓
$GL(n, \mathbb{H})$			✗ (n ≥ 2)
$Sp(n)$	✓		✓

WHY ARE LIE GROUPS SUCH SPECIAL MANIFOLDS?

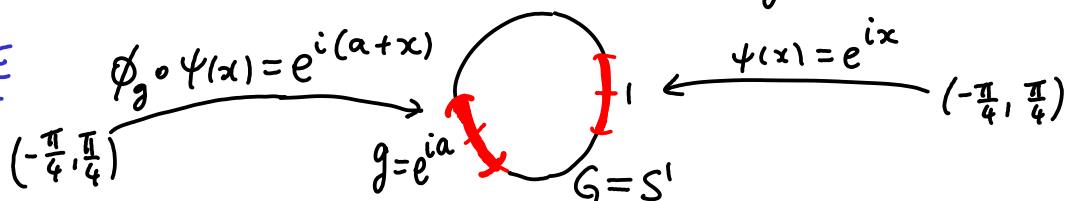
They have natural diffeomorphisms: (smooth with smooth inverse)
NOTE: $\phi_g^{-1} = \phi_{g^{-1}}$

$\phi_g: G \rightarrow G, h \mapsto gh$	left-translation by g
---	-------------------------

Amazing consequences:

- ① Once you pick a parametrization near 1, say $\psi: U \xrightarrow{\subset \mathbb{R}^n} V \subseteq G$ with $\psi(0) = 1$, get a parametrization near any $g \in G$: $\phi_g \circ \psi: U \rightarrow g \cdot V$ with $\phi_g \circ \psi(0) = g$

EXAMPLE



- ② In these parametrizations, ϕ_g is locally the identity map!

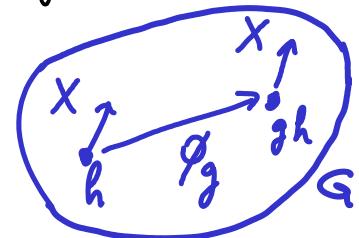
$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\text{Id=identity}} & \mathbb{R}^n \\ \psi \downarrow & & \downarrow \phi_g \circ \psi \\ G & \xrightarrow{\phi_g} & G \end{array}$$

Which vector fields X on G have exactly the same local expression near any g in parametrizations $\phi_g \circ \psi$?

$$\begin{array}{c} \sum a_i(x) \frac{\partial}{\partial x_i}: D\text{Id} = \text{Id} \xrightarrow{\quad} \sum a_i(x) \frac{\partial}{\partial x_i}: D(\phi_g \circ \psi) = D\phi_g \circ D\psi \\ D\psi \downarrow \qquad \qquad \qquad \text{chain rule} \\ X|_{\psi(x)} \xrightarrow{D\phi_g} X|_{g \cdot \psi(x)} \end{array} \left. \begin{array}{l} \text{hence need } X|_{g \cdot \psi(x)} = D\phi_g \cdot X|_{\psi(x)} \end{array} \right\}$$

Def A vector field X on G is left-invariant if

$$D_h \phi_g \cdot X|_h = X|_{gh} \quad (\forall g, h \in G)$$



Def

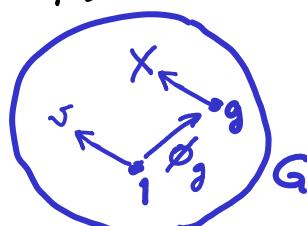
$$\boxed{\text{Lie } G = \{ \text{left-invariant vector fields on } G \}}$$

Rmk this is a vector space: adding/scaling vector fields preserves left-invariance because $D\phi_g$ is linear.

Theorem There is a natural isomorphism of vector spaces:

$$\begin{array}{ccc} \text{Lie } G & \longrightarrow & T_1 G \\ X & \longmapsto & X|_1 \\ \left(\begin{array}{c} \text{vector field } X \\ X|_g = D_1 \phi_g \cdot v \end{array} \right) & \longleftarrow & v \end{array}$$

In particular $\dim \text{Lie } G = \dim T_1 G = (\dim G \text{ as a manifold})$.



basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$

THE LIE BRACKET

For any vector fields X, Y on a manifold M there is a bracket operation $[X, Y] =$ a new vector field, defined locally by

$$\left[\sum a_i(x) \frac{\partial}{\partial x_i}, \sum b_i(x) \frac{\partial}{\partial x_i} \right] = \sum_j \sum_i \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

EXAMPLE in \mathbb{R}^2 :

$$[x^2 y^3 \frac{\partial}{\partial x}, x^4 y^5 \frac{\partial}{\partial y}] = x^2 y^3 \frac{\partial}{\partial x} (x^4 y^5) \frac{\partial}{\partial y} - x^4 y^5 \frac{\partial}{\partial y} (x^2 y^3) \frac{\partial}{\partial x} = -3x^6 y^7 \frac{\partial}{\partial x} + 4x^5 y^8 \frac{\partial}{\partial y}$$

Recall: vector fields act on functions, and this locally determines the vector field since $X \cdot x_i = a_i(x)$ if $X = \sum a_i(x) \frac{\partial}{\partial x_i}$ locally. For brackets:

$$[X, Y] \cdot f = X \cdot \underbrace{(Y \cdot f)}_{\text{new function}} - Y \cdot (X \cdot f)$$

↑ new function, then X differentiates it

IDEA: $[X, Y]$ measures how badly X, Y fail to commute as differential operators

e.g. $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ says partial derivatives commute on smooth functions:

$$\frac{\partial^2}{\partial x_i \partial x_j} (f) = \frac{\partial^2}{\partial x_j \partial x_i} (f)$$

PROPERTIES

- i) $[\cdot, \cdot]$ is bilinear : \mathbb{R} -linear in each entry.
- ii) antisymmetric : $[X, Y] = -[Y, X]$, so $[X, X] = 0$
- iii) Jacobi's identity :

exercise ↑

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

LIE ALGEBRAS

Def A Lie algebra is a vector space V together with a bilinear antisymmetric map $[\cdot, \cdot] : V \times V \rightarrow V$ satisfying the Jacobi identity.

EXAMPLES

- $V = \text{Mat}_{n \times n}(\mathbb{R})$, $[B, C] = B \cdot C - C \cdot B$ (matrix multiplication)
- $V = \mathbb{R}^3$, $[v, w] = v \times w$ cross-product
- Abelian Lie algebras : any vector space V with $[\cdot, \cdot] = 0$

Theorem Lie G is a Lie algebra of dimension $\dim G$

Pf Need show : X, Y left-invt $\Rightarrow [X, Y]$ left-invt.

In parametrizations $\phi_g \circ \psi$ the v.f. X has the same local expression near any $g \in G$, so does Y , hence so does $[X, Y]$, so $[X, Y]$ is left-invt.

EXAMPLE $gl(n, \mathbb{R}) = \text{Lie } GL(n, \mathbb{R})$

$$gl(n, \mathbb{R}) \cong \text{Mat}_{n \times n}(\mathbb{R}) = T_I GL(n, \mathbb{R})$$

$$X = \sum_{i,j} X_{ij}(x) \frac{\partial}{\partial x_{ij}} \longleftrightarrow B = X|_I = \begin{pmatrix} X_{ij}(I) \\ i=1, \dots, n \\ j=1, \dots, n \end{pmatrix}$$

left-invt.

$$X|_x = D_x \phi_x \cdot X|_I = x \cdot B$$

$x \in GL(n, \mathbb{R})$

$x = (x_{ij}) \in \mathbb{R}^{n^2}$ entries

are local coords near I

$= \varphi(x) + t\varphi(v)$ since φ linear

(For linear maps φ , " $D\varphi = \varphi$ " since $D_x \varphi \cdot v = \lim_{t \rightarrow 0} \frac{\varphi(x+tv) - \varphi(x)}{t} = \varphi(v)$).

$$[X, Y] = \sum_{i,j,k} \left(X_{ij} \frac{\partial Y_{ek}}{\partial x_{ij}} - Y_{ij} \frac{\partial X_{ek}}{\partial x_{ij}} \right) \frac{\partial}{\partial x_{ek}} \longleftrightarrow \sum (B_{ij} C_{jk} - C_{ij} B_{jk}) \frac{\partial}{\partial x_{ik}}$$

$$X|_x = x \cdot B \quad Y|_x = x \cdot C \quad (\text{so } Y_{ek} = \sum_j x_{ej} C_{jk} \text{ and } \frac{\partial Y_{ek}}{\partial x_{ij}} = C_{jk} \text{ for } l=i \text{ and zero otherwise})$$

Corollary $gl(n, \mathbb{R}) \cong \text{Lie algebra Mat}_{n \times n}(\mathbb{R})$ with bracket $[B, C] = BC - CB$

iso of Lie algebras
(= iso of vector spaces preserving bracket) \longleftrightarrow precise definition: $\varphi: V \rightarrow W$ iso of v.s.
 $\varphi([v_1, v_2]) = [\underbrace{\varphi v_1}_{\text{in } V}, \underbrace{\varphi v_2}_{\text{in } W}]$ all $v_1, v_2 \in V$

Same calculation shows:

$$\text{being in } O(n) \text{ puts some constraints on the } X_{ij}(x) \text{ and on } x, \text{ but calculation still holds viewing } O(n) \subseteq GL(n)$$

$$O(n) \subseteq GL(n)$$

APPENDIX (NON-EXAMINABLE - can ignore it)

Remarks about G_0

• Not true in general that $G \cong G_0 \times G/G_0$ as groups (not even $G_0 \times G/G_0$) because there is no reason homomorphisms $G \rightarrow G_0$ and $G/G_0 \rightarrow G$ should exist. It is true that $G \cong G_0 \times G/G_0$ as manifolds since you just pick some representatives g_i : then $G_0 \times \{g_i G\} \rightarrow G$ but can't make choices consistently with the group structure.
Later in course we prove $G \cong G_0 \times G/G_0$ works for abelian G . $(g, g_i G) \mapsto g \cdot g_i$

• There may be bigger subgroups than G_0 : for $G = S^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2 : G_0 = S^1 \times \{1\} \times \{1\} \subseteq S^1 \times \mathbb{Z}_2 \times \{1\} \subseteq G$. But of course G_0 is the largest connected subgroup of G .

ABOUT $[X, Y]$ Why is $[X, Y]$ a well-defined global vector field?

Answer: If for any function f defined near $p \in M$ you define a new function $Z \cdot f$ defined near $p \in M$, and you show the Leibniz rule holds $Z \cdot (f_1 f_2) = (Z \cdot f_1) f_2 + f_1 (Z \cdot f_2)$ then Z is a vector field on M (try proving this using the Appendix of lecture 1).

We defined $[X, Y] \cdot f$ and we defined $[X, Y]$ in local coordinates: the two definitions agree when compute $[X, Y] \cdot f$ locally, and the local definition clearly satisfies Leibniz. ■

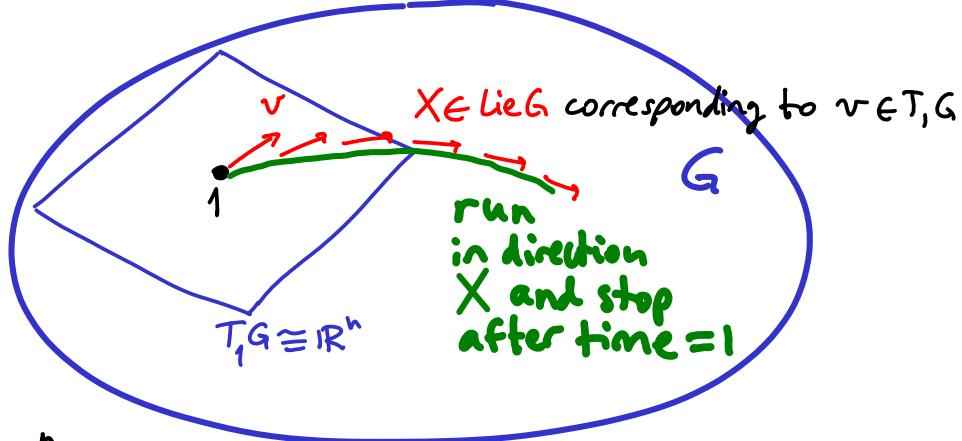
LECTURE 3

Dr Alexander F. Ritter, ritter@maths.ox.ac.uk
C3.5 LIE GROUPS, HT2015, Oxford.

LAST TIME: Parametrization γ near $1 \in G \rightsquigarrow$ parametrizations $\phi_g \circ \gamma$ near any g

AIM: Find the best parametrization γ near 1

IDEA:



$$\psi: \mathbb{R}^n \stackrel{\text{choice of basis}}{\cong} T_1 G \stackrel{\text{left invariant}}{\cong} \text{Lie } G \longrightarrow G$$

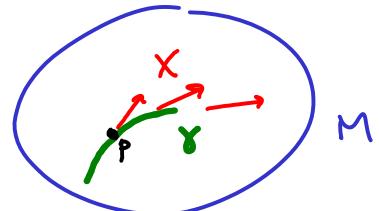
$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$

FLows

Def For a vector field X on a manifold M , a flowline of X through p is a curve

$$\gamma: (-\varepsilon, \varepsilon) \longrightarrow M \quad (\varepsilon > 0)$$

$$\gamma(0) = p \quad \gamma'(t) = X|_{\gamma(t)}$$

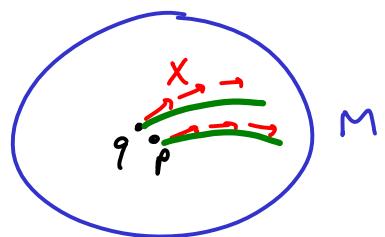


If let p vary: flow of X near p is

$$F: (-\varepsilon, \varepsilon) \times U \xrightarrow{\text{small open set around } p} M$$

$$F(0, q) = q \quad \frac{\partial F}{\partial t}|_{(t, q)} = X|_{F(t, q)}$$

$\Rightarrow F(\cdot, q)$ is flowline through q .



Rmks • $\gamma'(t)$ is an abbreviation for $D_t \gamma \cdot \frac{\partial}{\partial t}$. Locally it's really $\gamma'(t)$.

• The equation is locally a 1st order diff. eqn. on \mathbb{R}^n :

$$x'(t) = f(x(t)) \quad x(0) = p$$

$$f: \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^n$$

ODE theory \Rightarrow • x exists for small $\varepsilon > 0$

• x is unique given p

• x depends smoothly on the initial condition p

So for small ε and U , γ and F exist, unique, smooth.

Example $M = (-1, 1) \subseteq \mathbb{R}$, $X = \frac{\partial}{\partial t}$ then $F(q, t) = q + t$ only defined if $q + t \in (-1, 1)$.

Abbreviate $F_t(g) = F(t, g)$

Lemma 1 $F_s(F_t(g)) = F_{s+t}(g)$ (where defined)

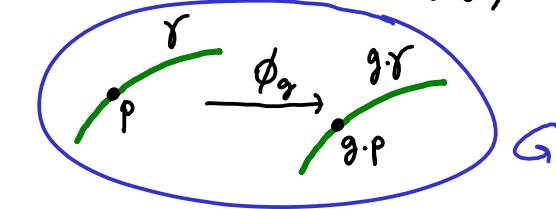
Pf for $s=0$: LHS = RHS = $F_t(g)$

$$\frac{\partial}{\partial s} \Big|_{s=0} (\text{LHS}) = X|_{F_s(F_t(g))}, \quad \frac{\partial}{\partial s} \Big|_{s=0} (\text{RHS}) \stackrel{\text{chain rule}}{=} X|_{F_{s+t}(g)}$$

Now suppose $M=G$ is a Lie group
 $X \in \text{Lie } G$ left-invariant

Lemma 2 γ flowline $\Rightarrow g \cdot \gamma$ flowline

$$\text{If } \frac{\partial}{\partial t} \Big|_t (g \cdot \gamma(t)) = \frac{\partial}{\partial t} (\phi_g \circ \gamma(t)) = D\phi_g \cdot \gamma'(t) = D\phi_g \cdot X|_{\gamma(t)} = X|_{g \cdot \gamma(t)} \blacksquare$$



Start being sloppy:
doing the derivative wherever it
is relevant, here $\gamma'(t)$ is a
vector at $\gamma(t)$ so we take $D_{\gamma(t)} \phi_g$

Cor 1 flowlines of $X \in \text{Lie } G$ are defined for all time, and flow defined everywhere

Pf $\gamma: (-\varepsilon, \varepsilon) \rightarrow G \Rightarrow$ can extend on $(0, \varepsilon + \frac{\varepsilon}{2})$ using $\gamma(\frac{\varepsilon}{2}) \gamma(0)^{-1} \gamma(t - \frac{\varepsilon}{2})$

Explanation: $t \mapsto \gamma(t - \text{constant})$ is a flowline: by chain rule $\frac{\partial}{\partial t} \gamma(t - c) = \gamma'(t - c) = X|_{\gamma(t - c)}$ then apply Lemma 1 to flowline $\gamma(t - \frac{\varepsilon}{2})$ with $g = \gamma(\frac{\varepsilon}{2}) \gamma(0)^{-1}$. So $\delta(t) = \gamma(\frac{\varepsilon}{2}) \gamma(0)^{-1} \gamma(t - \frac{\varepsilon}{2})$ is a flowline for X . We constructed it so that $\delta(\frac{\varepsilon}{2}) = \gamma(\frac{\varepsilon}{2}) \gamma(0)^{-1} \gamma(0) = \gamma(\frac{\varepsilon}{2})$. But then δ, γ are both flowlines for X and equal at $t = \frac{\varepsilon}{2}$ so they equal on overlap by uniqueness. So δ extends γ beyond ε . Similarly can extend beyond $-\varepsilon$. Finally argue by contradiction: if $(-\varepsilon, \varepsilon)$ was the largest interval where γ is defined, then we just showed that cannot be true since we can extend.

Key trick: let $\gamma: \mathbb{R} \rightarrow G$ be the (unique) flowline of X with $\gamma(0) = 1$ then by Lemma 1 the flow of X is: $F_t(g) = g \cdot \gamma(t)$ ■

Theorem Let γ be the flowline of $X \in \text{Lie } G$ with $\gamma(0) = 1$. Then

$$\gamma(s) \gamma(t) = \gamma(s+t) \quad \forall s, t \in \mathbb{R} \quad (\text{in particular: } \gamma(s) \gamma(t) = \gamma(t) \gamma(s))$$

So $\gamma: \mathbb{R} \rightarrow G$ is a Lie group homomorphism. ← (hom of gps &)

Conversely, all Lie gp homs $\gamma: \mathbb{R} \rightarrow G$ arise in this way for some $X \in \text{Lie } G$.

Pf. By Lemma 1 + Cor 1: $F_s(F_t(1)) = F_{s+t}(1) = 1 \cdot \gamma(s+t)$
 $\Downarrow F_s(1 \cdot \gamma(t)) = 1 \cdot \gamma(t) \cdot \gamma(s)$

Conversely, if γ homom $\mathbb{R} \rightarrow G$ then $\gamma(0) = 1$, and we claim $F(t, g) = g \cdot \gamma(t)$ is flow for $\frac{\partial}{\partial t} |_{t=0} F(t, g) = D\phi_g \cdot \gamma'(0) = X|_g$ left-inv v.f. determined by $\gamma'(0) \in T_g G$.

proof: $F_{s+t}(g) = g \cdot \gamma(s+t) = g \cdot \gamma(t+s) = g \cdot \gamma(t) \cdot \gamma(s) = F_s(g \cdot \gamma(t)) = F_s F_t(g)$

and in general $F_0 = \text{id}$, $F_s F_t = F_{s+t}$ ensures you are the flow for the v.f. $X|_p = \frac{\partial}{\partial t} |_{t=0} F_t(p)$ since:

(for manifold M)
and smooth map
 $F: \mathbb{R} \times M \rightarrow M$.
Call $F_t = F(t, \cdot)$

$$\frac{\partial}{\partial t} |_t F_t(p) = \frac{\partial}{\partial s} |_0 F_{t+s}(p) \stackrel{\text{chain rule}}{=} \frac{\partial}{\partial s} |_0 F_s(F_t(p)) = X|_{F_t(p)} \quad \checkmark \quad \blacksquare$$

Def The Lie group homomorphisms $\mathbb{R} \xrightarrow{\gamma} G$ are called 1-parameter subgroups of G

Cor

$$\begin{array}{ccc} \text{Lie } G & \xrightarrow{\text{v.s.}} & T_1 G \\ X & \longleftrightarrow & v = X|_1 = \gamma'_v(0) \end{array} \xrightarrow{\text{bijection}} \{ \text{1-parameter subgroups of } G \}$$

Def Exponential map

$$\exp : \text{Lie } G \cong T_1 G \longrightarrow G$$

$$v \longmapsto \gamma_v(1)$$

Next time:

$\psi : \mathbb{R}^n \cong T_1 G \xrightarrow{\exp} G$ is a parametrization since \exp is smooth, and invertible near $0 \in T_1 G$.

- ① determined by conditions, $\gamma_v(s+t) = \gamma_v(s)\gamma_v(t) \forall s, t$, $\gamma_v'(0) = v$
- ② also determined by equation: $\gamma_v(0) = 1$, $\gamma_v'(t) = X|_{\gamma_v(t)}$.

(flow for time 1 in direction X where X is the left-invt vector field corresponding to $v \in T_1 G$ (so $X|_g = D_g \psi \cdot v$))

EXAMPLE 1

Torus $G = \mathbb{R}^n / \mathbb{Z}^n$: obvious 1-param. subgrps $\gamma_v(t) = t v \bmod \mathbb{Z}^n$

Check condition ①:

$$\gamma_v(s+t) = (s+t)v = sv + tv \text{ hence } \gamma_v : \mathbb{R} \rightarrow G \text{ homomorphism.}$$

This classifies all 1-param. subgrps since $\gamma_v'(0) = v \in \mathbb{R}^n \cong T_1 G$ is general.

$$\Rightarrow \exp(v) = v \bmod \mathbb{Z}^n$$

$\Rightarrow \exp : \mathbb{R}^n \rightarrow G$ is the map Π of questionsheet 1
 $v \longmapsto v \bmod \mathbb{Z}^n$

Alternative approach: check condition ② for $\gamma_v(t) = tv \bmod \mathbb{Z}^n$

$$\gamma_v(0) = 0, \quad \gamma'_v(0) = v, \quad \gamma'_v(t) = v$$

For the last two equalities we used the parametrization $\Pi : \mathbb{R}^n \rightarrow G$
 $\Pi(v) = v \bmod \mathbb{Z}^n$

Recall from Question sheet 1 that the left-invt v.f. on T^n in the parametr. Π are the constant vectors $v \in \mathbb{R}^n$.

Case n=1 $G = \mathbb{R}/\mathbb{Z} = S^1$ get $\exp(x) = e^{2\pi i x} \in S^1$

Remark

here we identify $T_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}$ via $[\begin{matrix} \text{curve} \\ t \mapsto tv \bmod \mathbb{Z} \end{matrix}] \leftrightarrow v$

so we must get $\exp(\mathbb{Z}) = 1$. This corresponds to parametrizing the circle with " $x \in [0, 1]$ " rather than with " $\theta \in [0, 2\pi]$ ".

If you instead parametrize with $\theta \mapsto e^{i\theta}$ then $\exp(\theta) = e^{i\theta} \in S^1$.

EXAMPLE 2

$$G = GL(n, \mathbb{R})$$

$$\gamma_B(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} B^n$$

$$\exp(B) = \sum \frac{1}{n!} B^n$$

$$B \in \text{Mat}_{n \times n}(\mathbb{R}) \\ = T_1 GL(n, \mathbb{R})$$

CLAIM makes sense (converges) & can differentiate term by term

proof below

$$\Rightarrow \gamma_B(t) = I + tB + \frac{1}{2!} t^2 B^2 + \frac{1}{3!} t^3 B^3 + \dots$$

$$\gamma'_B(t) = B + tB^2 + \frac{1}{2!} t^2 B^3 + \dots$$

$$\Rightarrow \gamma_B(0) = I, \quad \gamma'_B(0) = B, \quad \gamma'_B(t) = \left. \gamma_B(t) \cdot B \right|_{\gamma_B(t)} = X \quad \text{for } X = \text{left-inv.}$$

$$\Rightarrow 1\text{-param subgp for } B \quad \blacksquare \quad \text{v.f. for } B: X|_x = x \cdot B$$

$$\text{compare } \frac{\partial}{\partial t} e^{tx} = e^{tx} \cdot x$$

finite dim'l, so all norms are equivalent

proof of claim The vector space $\text{Mat}_{n \times n}(\mathbb{R})$ is a normed space using the norm

$$\|B\| = n \cdot \max_{i,j} |B_{ij}|$$

Nice properties:

- $\|BC\| \leq \|B\| \cdot \|C\|$
- complete, i.e. Cauchy sequences converge.
(since finite dim'l)
- $\text{Mat}_{n \times n}(\mathbb{R})$ is an algebra

$$\begin{aligned} \|BC\| &= n \cdot \max_{i,k} |\sum_j B_{ij} C_{jk}| \\ &\leq n \cdot \max_{i,k} \underbrace{\sum_j |B_{ij}|}_{\leq \|B\|} \cdot \underbrace{|C_{jk}|}_{\leq \|C\|} \leq \frac{\|B\|}{n} \|C\| \\ &\leq \cancel{n} \cdot \cancel{n} \frac{\|B\|}{\cancel{n}} \frac{\|C\|}{\cancel{n}} \end{aligned}$$

(v.s. V with bilinear $V \times V \rightarrow \mathbb{R}$)
defining multiplication

Def Complete normed algebras satisfying \star are called Banach algebras

For Banach algebras can reprove all the usual results about series, absolute convergence, radius of convergence, etc. \square (SAME PROOFS!)

If $I \in \text{Banach}$ $\Rightarrow \exp(tx) = I + tx + \frac{t^2}{2!} x^2 + \dots$ converges absolutely and radius of convergence $= \infty$

\Rightarrow can differentiate in t term by term \blacksquare

LECTURE 4

LAST TIME :

$$\begin{array}{c} \text{Lie } G \xrightleftharpoons[\text{v.s.}]{\cong} T_1 G \xrightleftharpoons[\text{bijection}]{\cong} \{\text{l-parametr-subgroups}\} \\ X \longleftrightarrow v \longleftrightarrow (\gamma_X = \gamma_v : \mathbb{R} \rightarrow G) \end{array}$$

recall : $\begin{cases} \gamma_v(0) = 1 \\ \gamma'_v(0) = v = X|_1, \\ \gamma'_v(t) = X|_{\gamma_v(t)} \end{cases}$ also recall that
 IF $\begin{cases} \gamma \text{ smooth} \\ \gamma : \mathbb{R} \rightarrow G \text{ hom} \\ (\gamma(s+t) = \gamma(s)\gamma(t)) \end{cases}$ THEN $\gamma = \gamma_v$ where $v = \gamma'(0)$.

THE EXPONENTIAL MAP

Def $\exp : \text{Lie } G \cong T_1 G \rightarrow G$, $\exp(v) = \gamma_v(1)$.

Lemma $\exp(sv) = \gamma_v(s)$

If $t \mapsto \gamma_{sv}(t)$, $t \mapsto \gamma_v(st)$ are homs $\mathbb{R} \rightarrow G$, so just compare $\frac{\partial}{\partial t}|_{t=0}$:

$\frac{\partial}{\partial t}|_0 \gamma_{sv}(t) = sv$, $\frac{\partial}{\partial t}|_0 \gamma_v(st) = s \cdot \gamma'_v(s \cdot 0) = sv$. So by uniqueness $\gamma_{sv}(t) = \gamma_v(st)$
 chainrule \Rightarrow claim, taking $t=1$ ■

Theorem $\exp : \text{Lie } G \rightarrow G$ is smooth

side remark: vector spaces V are manifolds: just pick a basis to get a (global!) parametrization $\mathbb{R}^n \cong V$. Their tangent spaces are:
 $T_v V \cong V$, $[curve \gamma(t) = v + tw] \mapsto w = \gamma'(0)$
 Hence Lie G is a manifold.

If \exp is the composite of 3 smooth maps:

$$\begin{array}{ccccccc} \text{Lie } G & \longrightarrow & \mathbb{R} \times (G \times \text{Lie } G) & \longrightarrow & G \times \text{Lie } G & \longrightarrow & G \\ X & \longmapsto & (1, 1, X) & \longmapsto & (t, g, X) & \longmapsto & (g, X) \longmapsto g \end{array}$$

flow of γ on $G \times \text{Lie } G$ where $\gamma|_{(g,X)} = (X|_g, 0)$ ■
 hence smooth

Lemma $D_0 \exp = \text{Id}$

$$\text{Pf } D_0(\exp) \cdot w = \frac{\partial}{\partial s}|_{s=0} \exp(0 + sw) = \frac{\partial}{\partial s}|_{s=0} \gamma_{sw}(1) \stackrel{\text{above Lemma}}{=} \frac{\partial}{\partial s}|_0 \gamma_w(s) = \gamma'_w(0) = w \blacksquare$$

Cor $\exp : \text{Lie } G \rightarrow G$ is a local diffeomorphism near 0.

$(\exists \text{ open sets } \overset{0}{U} \subseteq \text{Lie } G, \overset{0}{V} \subseteq G \text{ s.t. } \exp : U \rightarrow V \text{ is a diffeomorph.})$

Pf

FACT Inverse Function Theorem $\varphi : M \rightarrow N$ smooth map of mfds

$D_m \varphi : T_m M \rightarrow T_{\varphi(m)} N$ invertible $\Rightarrow \varphi$ local diffeo near m ■

Cor $\psi : \mathbb{R}^n \cong T_1 G \cong \text{Lie } G \xrightarrow{\exp} G$ is a parametrization near $1 \in G$
 choice of basis ($GL(n, \mathbb{R})$ choices)
 (defined on a nbhd of $0 \in \mathbb{R}^n$)
 (Hence get nice parametrizations $\phi_g \circ \psi \dots$)

EXAMPLES

1) $\exp : \mathbb{R} \rightarrow S^1$, $\exp(x) = e^{2\pi i x}$ local diffeo near 0, but not global (not injective!).
 a local inverse near $e^{2\pi i \cdot 0} = 1$ is $\frac{1}{2\pi i} \log(y) \leftarrow$ pick a branch of complex log.
 ($2\pi i \mathbb{Z}$ choices)

2) $\exp : \text{Mat}_{n \times n}(\mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R})$, $\exp(B) = \sum \frac{1}{n!} B^n$, then a local inverse near I is
 $A \mapsto \log(A) = \log(I + (A - I))$ (for $\|A - I\| < 1$)

EXPLANATION (NON-EXAMINABLE):

For a Banach algebra with 1 define

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n \text{ for } \|x\| < 1$$

radius of convergence

Remark we know $\exp(\log(1+tx)) = 1+tx$ for $t, x \in \mathbb{R}$ is an absolutely convergent series in tx (where defined). Hence same must be true for Banach algebras with 1. The reason is: take $x = 1 \in$ Banach algebra and $t \in \mathbb{R}$. Then the coefficients of those series in $t \cdot 1$ must agree with the coefficients of the series you got when working with \mathbb{R} ! (consider the coefficients inductively letting $t \rightarrow 0$ allows you to ignore higher order terms).

Def $\varphi : G \rightarrow H$ Lie group homomorphism means

- 1) φ group homomorphism
- 2) φ smooth

Theorem (Naturality of \exp)

If $\varphi : G \rightarrow H$ Lie group hom then:

$$\begin{array}{ccc} T_1 G & \xrightarrow{D_1 \varphi} & T_1 H \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\varphi} & H \end{array} \quad \text{commutes}$$

(i.e. composing $\xrightarrow{\quad}$
 equals composing \downarrow)

Pf $v = [\gamma_v(t)] \longmapsto D_1 \varphi \cdot v = [\varphi \circ \gamma_v(t)]$
 $(\text{since } \gamma'_v(0) = v)$ $\gamma_v(1) \longmapsto \varphi \circ \gamma_v(1)$

Note $\varphi \circ \gamma_v(t)$ is a 1-parameter group since φ is a group hom. Hence \exp is evaluation at $t=1$. ■

EXAMPLES ① $\mathbb{R} \cong TS^1 \longrightarrow \mathbb{C} = \text{Mat}_{1 \times 1}(\mathbb{C})$

$$\mathbb{R}/\mathbb{Z} = S^1 \longrightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\} = \text{GL}(1, \mathbb{C})$$

(Note: $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ is $\exp(z) = e^z$ since $= \sum \frac{1}{n!} z^n$ as 1×1 matrix)

$$x \longmapsto z = 2\pi i x$$

$$x \bmod \mathbb{Z} \longmapsto e^z = e^{2\pi i x}$$

② $\text{SkewSym}_{n \times n}(\mathbb{R}) \rightarrow \text{Mat}_{n \times n}(\mathbb{R})$

$$\downarrow \text{inclusion} \quad \downarrow$$

$$\text{O}(n) \xrightarrow{\text{inclusion}} \text{GL}(n, \mathbb{R})$$

implies that $\exp(B) = \sum \frac{1}{n!} B^n$ also for $\text{O}(n)$

(indeed for any Lie subgroup of $\text{GL}(n, \mathbb{R})$, not just $\text{O}(n)$)

LECTURE 5

HOMOMORPHISMS

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Def Lie group homomorphism $\varphi: G \rightarrow H$ means 1) φ hom of gp's
2) φ smooth map

- When $H = GL(n, \mathbb{R})$, φ is called a representation of G
- Lie group isomorphism if φ bijective and φ^{-1} Lie gp hom $\begin{pmatrix} \text{so:} \\ 1) \varphi \text{ iso of gp's} \\ 2) \varphi \text{ diffeo} \end{pmatrix}$

Warning Let $\mathbb{R}_{\text{disc}} = \mathbb{R}$ with discrete topology (each point is an open set) is Lie gp using + identity: $\mathbb{R}_{\text{disc}} \rightarrow \mathbb{R}$ is a bijective Lie gp hom, inverse not continuous!
(Exp/naturality tells you nothing since $T_0 \mathbb{R}_{\text{disc}} = \{0\}$) $\begin{pmatrix} \uparrow \\ \text{o-dim mfd, but} \\ \text{not 2nd countable} \end{pmatrix}$

Rmk Local diffeo + bijective = diffeo

Lemma 1 $\varphi: G \rightarrow H$ bijective Lie gp hom and φ^{-1} continuous near 1 $\Rightarrow \varphi$ Lie gp iso

Pf By naturality of exp: $(\text{nbhd of } 0 \in T_1 G) \xrightarrow{D_1 \varphi} (\text{nbhd of } 0 \in T_1 H)$

$\begin{array}{l} \text{May not be diffeo} \\ \text{May need to pick smaller} \\ (\text{nbhd of } 1 \in G) = V \subseteq \varphi^{-1}(U) \\ \Rightarrow \text{replace } U \text{ by } \varphi(V) \\ \varphi(V) = (\varphi^{-1})^{-1}(V) \text{ open} \\ \text{since } \varphi^{-1} \text{ cts near 1.} \\ \text{This fails in Warning above} \end{array} \xrightarrow{\downarrow \text{exp}} \varphi \xrightarrow{\text{exp } \downarrow \text{diffeo}}$

$$\varphi^{-1}(U) = (\text{nbhd of } 1 \in G) \xrightarrow{\varphi} U = (\text{nbhd of } 1 \in H)$$

φ bijective $\Rightarrow D_1 \varphi$ bijective near 0 hence iso (since linear map)
inverse function theorem $\Rightarrow \varphi$ local diffeo near 1 $\in G$

$$\varphi \text{ hom} \Rightarrow (\phi_{\varphi(g)} \circ \varphi \circ \phi_{g^{-1}})(h) = \varphi(g) \varphi(g^{-1}h) = \varphi(h)$$

$\Rightarrow \varphi$ local diffeo near g (since ϕ maps are diffeos and)

Alternative proof: $D_1 \phi_{\varphi(g)} \circ D_1 \varphi \circ D_g \phi_{g^{-1}} : T_g G \rightarrow T_1 G \rightarrow T_1 H \rightarrow T_{\varphi(g)} H$

The proof also shows:

$= D_g \varphi$ is iso, then apply inverse function thm \blacksquare

(for
 $\varphi: G \rightarrow H$
Lie gp hom)

Lemma 2

$D_1 \varphi$ iso $\Rightarrow D_g \varphi$ iso for any g \Rightarrow

Lemma 3 φ locally homeo near 1 $\Rightarrow \varphi$ local diffeo near any g .

Exercise φ bijective and $\dim G = \dim H \Rightarrow \varphi$ Lie gp iso

(Hint: use injectivity of $\varphi: V \rightarrow U$ above to get $D_1 \varphi$ injective, then use $\dim G = \dim H$)

Harder exercise Manifolds are usually required to be second countable (\exists countable basis for the topology). If we require Lie gps to be 2nd countable, is it true that

Bijective Lie gp hom \Rightarrow Lie gp iso?

Idea: above if $D_1 \varphi(T_1 G) \neq T_1 H$ then it is a strictly lower dimensional vector subspace, so $\varphi(V) \subseteq U$ is a strictly lower dimensional submanifold in H , so you need uncountably many left translates $\phi_h(\varphi(V))$ to get a disjoint cover of U (by non-open sets). Then $\varphi^{-1}(\phi_h(\varphi(V)))$ give uncountably many disjoint open sets covering V contradicting G is 2nd cble $\Rightarrow \varphi^{-1}(\phi_{c(g)}(V)) = \varphi^{-1}\varphi(\phi_g V) = \phi_g(V)$ where $c(g) = h$.

Def Lie algebra homomorphism $\psi: (V, [\cdot, \cdot]_V) \rightarrow (W, [\cdot, \cdot]_W)$ means

1) ψ linear map (homomorphism of vector spaces)

2) $[\psi x_1, \psi x_2]_W = \psi [x_1, x_2]_V \quad \text{all } x_1, x_2 \in V$

• When $W = \text{Mat}_{n \times n}(\mathbb{R})$, $[B, C]_W = BC - CB$, ψ is called a representation of V

• Lie algebra isomorphism if ψ also bijective (hence ψ iso of v-s.)

MORE ABSTRACTLY FOR REPRESENTATIONS CAN REPLACE :

For Lie gps:	$GL(n, \mathbb{R})$	$\text{Aut}(\mathbb{R})$
For Lie algs:	$\text{Mat}_{n \times n}(\mathbb{R})$	$\text{End}(\mathbb{R}) = \text{Hom}(\mathbb{R}, \mathbb{R})$

where \mathbb{R} is a vector space

$$\varphi(g)(r) \\ + (g)(r)$$

Often call \mathbb{R} the representation and write $\begin{matrix} g \cdot r \\ x \cdot r \end{matrix}$ instead of

EXAMPLES

1) $\gamma: \mathbb{R} \rightarrow G$ 1-param. subgroups are Lie gp homs

example: $\mathbb{R} \rightarrow S^1$, $x \mapsto e^{2\pi i x}$ (or $x \bmod \mathbb{Z}$ if view $S^1 = \mathbb{R}/\mathbb{Z}$)

2) $SU(2) \rightarrow SO(3)$ on Q-sheet 2

3) $A_g: G \rightarrow G$

$$A_g(h) = g h g^{-1}$$

Lie group isomorphism

(the inverse is $A_g^{-1} = A_{g^{-1}}$)

4)

$$\text{Ad}: G \rightarrow \text{Aut}(T_e G) \cong \text{Aut}(\text{Lie } G)$$

$$\text{Ad}(g) = D_{g^{-1}} A_g$$

ADJOINT
REPRESENTATION
OF G

$\nwarrow D_{g^{-1}} A_g$ is an automorph since has inverse $D_{g^{-1}} A_g^{-1}$ (chain rule)

5)

$$\text{ad} = D_{e^{-1}} \text{Ad}: \text{Lie } G \cong T_e G \rightarrow \text{End}(T_e G)$$

Why is $D_{e^{-1}} \text{Ad}$ a Lie alg hom?

ADJOINT
REPRESENTATION
OF LIE(G)
(in Lecture 6 will prove that)
 $(\text{ad } X)(Y) = [X, Y]$

Theorem For $\varphi: G \rightarrow H$ Lie group hom

$$\varphi: \text{Lie } G \cong T_e G \xrightarrow{D_e \varphi} T_e H \cong \text{Lie } H \text{ a Lie algebra hom}$$

Rmk: not obvious since φ usually not a diffeo, so can't push-forward a v.f.

FROM NOW ON ABBREVIATE $\mathfrak{g} = \text{LIE}(G)$

Consequences By naturality of \exp :

Cor

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{Ad}(g)} & \mathfrak{g} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{dg} & G \end{array}$$

$$\begin{array}{ccc} g & \xrightarrow{\text{ad}} & \text{End}(g) \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\text{Ad}} & \text{Aut}(\mathfrak{g}) \end{array}$$

$$g \cdot \exp(X) \cdot g^{-1} = \exp(\text{Ad}(g) \cdot X)$$

$$\text{Ad}(\exp X) = \exp(\text{ad } X)$$

Example For $G = GL(n, \mathbb{R})$ $\mathfrak{g} = \text{Mat}_{n \times n}(\mathbb{R})$

$$\text{Ad}_A(x) = Ax A^{-1} \text{ and } \text{Ad}(A) \cdot B = ABA^{-1}$$

1st box says: $A e^B A^{-1} = e^{ABA^{-1}}$ (holds because $(ABA^{-1})^n = A B^n A^{-1}$)

Exercise What does 2nd box say, using result from lecture 6 that $\text{ad } X = [X, \cdot]$

this \exp we know
 $\exp(B) = I + B + \frac{B^2}{2!} + \dots$

PROOF OF THEOREM

$$Z \in \text{Lie } G \Rightarrow Z|_g = D_g \phi^G \cdot Z|_1$$

ϕ^G_g means
left-translation in G
 $\phi^G_g(\tilde{g}) = g\tilde{g}$

$$\text{call } \tilde{Z} = \varphi(Z), \text{ so: } \tilde{Z}|_h = D_h \phi^H \cdot (D_g \varphi \cdot Z|_1) \stackrel{\substack{\text{chain} \\ \text{rule}}}{=} D_g \varphi \cdot D_{\tilde{g}} \phi^G \cdot Z|_1 = (D\varphi \cdot Z)|_g$$

$$\Rightarrow \tilde{Z}|_{\varphi(g)} = D_{\tilde{g}} \underbrace{(\phi^H_{\varphi(g)} \circ \varphi)}_{\substack{\text{because } \varphi \text{ hom:} \\ \varphi(g)\varphi(\bullet) = \varphi(g \cdot \bullet)}} \cdot Z|_1 \stackrel{\substack{\text{chain} \\ \text{rule}}}{=} D_g \varphi \cdot D_{\tilde{g}} \phi^G \cdot Z|_1 \stackrel{\substack{\text{since left-inv} \\ Z|_g}}{=} (D\varphi \cdot Z)|_g$$

$$\Rightarrow \boxed{\tilde{Z}|_{\varphi(\cdot)} = D\varphi \cdot Z}$$

In general such vector fields Z, \tilde{Z}
are called φ -related

Rmk If φ was a diffeo, this would say that \tilde{Z} is the pushforward
of Z . If φ not diffeo, then $D\varphi \cdot Z$ need not be a vector field

Proposition
(proof later)

If X, \tilde{X} and Y, \tilde{Y} are φ -related then $[X, Y], [\tilde{X}, \tilde{Y}]$ are φ -related

continue proof:

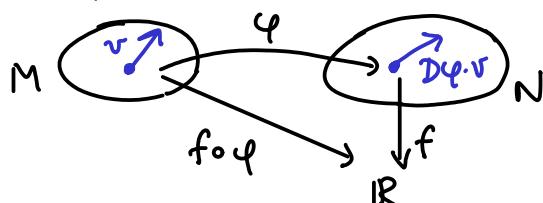
$$[\tilde{X}, \tilde{Y}]|_{\varphi(\cdot)} = D\varphi \cdot [X, Y] = \overbrace{[X, Y]}^{\substack{\text{above equation for } Z = [X, Y]}}|_{\varphi(\cdot)}$$

$$\begin{aligned} &\Rightarrow [\varphi(X), \varphi(Y)]|_{\varphi(g)} = (\varphi[X, Y])|_{\varphi(g)} \quad \forall g \in G \\ &\xrightarrow{g=1} \Rightarrow [f(X), f(Y)]|_1 = (\varphi[X, Y])|_1 \\ &\Rightarrow [f(X), f(Y)] = \varphi[X, Y] \quad \begin{array}{l} \text{(since left-invariant vector fields)} \\ \text{are determined uniquely by value at 1)} \end{array} \end{aligned}$$

PROOF OF PROPOSITION

Given: $\varphi: M \rightarrow N$ some v.f. X, Y on M want: $[\tilde{X}, \tilde{Y}]|_{\varphi(\cdot)} = D\varphi \cdot [X, Y]$
smooth map of manifolds $\tilde{X}|_{\varphi(\cdot)} = D\varphi \cdot X$ " $\tilde{Y}|_{\varphi(\cdot)} = D\varphi \cdot Y$ " \tilde{X}, \tilde{Y} " N

Need TRICK:



PROOF OF TRICK Locally:

$$v = \sum a_j \frac{\partial}{\partial x_j} \text{ and } \varphi = (\varphi_1, \dots, \varphi_n): \mathbb{R}^m \dashrightarrow \mathbb{R}^n$$

$$v \cdot (f \circ \varphi) = \sum a_j \frac{\partial}{\partial x_j} (f \circ \varphi)$$

$$\stackrel{\text{chain rule}}{=} \sum a_j \frac{\partial f}{\partial x_i} \cdot \frac{\partial \varphi_i}{\partial y_j} = (D\varphi \cdot v) \cdot f$$

matrix for $D\varphi$

For $v \in T_m M$ and $f: N \rightarrow \mathbb{R}$
 $v \cdot (f \circ \varphi) = (D_m \varphi \cdot v) \cdot f \in \mathbb{R}$
for vector field X on M :
 $X \cdot (f \circ \varphi) = (D\varphi \cdot X) \cdot f$
as functions $M \rightarrow \mathbb{R}$

means:

$$\begin{cases} ((D\varphi \cdot X) \cdot f)(m) = \\ = (D_m \varphi \cdot X|_m) \cdot f \end{cases}$$

$$\begin{aligned}
 (D\varphi \cdot [x, y]) \cdot f &= [x, y] \cdot (f \circ \varphi) \\
 \stackrel{\text{TRICK}}{=} X \cdot (Y \cdot (f \circ \varphi)) - Y \cdot (X \cdot (f \circ \varphi)) \\
 &= X \cdot ((D\varphi \cdot Y) \cdot f) - \text{switch } X, Y \\
 \stackrel{\text{TRICK}}{=} X \cdot ((\tilde{y} \cdot f) \circ \varphi) &- " \\
 &= (D\varphi \cdot X) \cdot (\tilde{y} \cdot f) - " \\
 \stackrel{\text{TRICK}}{=} (\tilde{X} \cdot (\tilde{y} \cdot f)) \Big|_{\varphi(1)} &- " \\
 &= [\tilde{X}, \tilde{y}] \Big|_{\varphi(1)} \quad \blacksquare
 \end{aligned}$$

Example question sheet 2: $\varphi: SU(2) \rightarrow SO(3)$ double cover (in particular a diffeo near $1 \in SU(2)$)
 $\Rightarrow \psi = D_1 \varphi: su(2) \rightarrow so(3)$ Lie algebra isomorphism!
 $\{2 \times 2 \text{ complex}\}$ $\stackrel{\text{skew Hermitian}}{=} \{3 \times 3 \text{ real skew-symmetric}\}$
 $\text{(connected component of } \mathfrak{leg})$

Lemma A neighbourhood $V \subseteq G_0$ of 1 generates G_0 as a group

 Pf Can assume V is open (interior(V) is smaller than V , $1 \in \text{Int}(V)$)
 $\langle V \rangle = \text{subgroup generated by } V \text{ in } G_0$
 $\Rightarrow \langle V \rangle \subseteq G_0$ open subset (since $v_1^{\pm 1} \dots v_k^{\pm 1}$ has nbhd $v_1^{\pm 1} \dots v_{k-1}^{\pm 1} \cdot V \subseteq G_0$)
 $\Rightarrow \text{cosets } g \cdot \langle V \rangle \text{ are open (since } \not\exists g \text{ diffeo)}$
 $\Rightarrow \langle V \rangle \text{ closed subset (since complement of open set } \bigcup g \cdot \langle V \rangle \text{ is homeom. to open disjoint union of cosets of } \langle V \rangle \text{.)}$
 $\Rightarrow \langle V \rangle \text{ connected component (since open \& closed)}$
 $\Rightarrow \langle V \rangle = G_0 \quad \blacksquare$

Theorem Let G be connected

A Lie gp hom $\varphi: G \rightarrow H$ is uniquely determined by $D_1 \varphi: T_1 G \rightarrow T_1 H$
(meaning: if $\varphi, \tilde{\varphi}: G \rightarrow H$ Lie gp homs with $D_1 \varphi = D_1 \tilde{\varphi}$ then $\varphi = \tilde{\varphi}$)

Pf Naturality of exp: (small nbhd $0 \in T_1 G$) $\xrightarrow{D_1 \varphi} T_1 H$
 $\exp \downarrow \text{Diffeo} \qquad \qquad \qquad \downarrow \exp$
 $V = (\text{small nbhd of } 1 \in G) \xrightarrow{\varphi} H$

$\Rightarrow \varphi$ determined by $D_1 \varphi$ on V

$\varphi \text{ hom} \rightarrow \Rightarrow \varphi$ determined by $D_1 \varphi$ on $\langle V \rangle = G_0 = G \quad \blacksquare$

Warning ("not everything is determined at the identity")

$su(2) \cong so(3)$ but $SU(2) \not\cong SO(3)$: different topologically:

$$\overset{\text{"}}{S^3} \qquad \overset{\text{"}}{\mathbb{R}P^3}$$

FACT (non-examinable) S^3 is simply connected (all loops are contractible)
 $\mathbb{R}P^3$ is not simply connected.

LECTURE 6

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ADJOINT REPRESENTATION

LAST TIME

$\text{Ag} : G \rightarrow G$ conjugation by g $\text{Ag}(h) = ghg^{-1}$

$\text{Ad} : G \rightarrow \text{Aut}(g)$, $\text{Ad}(g) = D_g$, Ag

$\text{ad} = D_g, \text{Ad} : g \rightarrow \text{End}(g)$

We will need 2 TRICKS from LECTURE 1: ① $D\varphi \cdot v = \frac{\partial}{\partial s} \Big|_{s=0} \varphi(\gamma(s))$
(where $\varphi : M \rightarrow N$, $f : M \rightarrow \mathbb{R}$, $v = [\text{curve } \gamma(s)]$) ② $v \cdot f = \frac{\partial}{\partial s} \Big|_{s=0} f(\gamma(s))$

Theorem

$$\underbrace{\text{ad}(X) \cdot Y}_{\in \text{End}(g)} = [X, Y] \quad X, Y \in g$$

pf. $\text{ad}(X) \cdot Y \stackrel{\text{def}}{=} \underbrace{(D_{\gamma_X(s)}, (\text{Ad}) \cdot X) \cdot Y}_{\in \text{End}(g)}$

$$\stackrel{\text{①}}{=} \frac{\partial}{\partial s} \Big|_0 \text{Ad}(\gamma_X(s)) \cdot Y \quad \leftarrow \text{recall } \gamma'_X(s) = X \text{ for 1-param. subgp } \gamma_X(s)$$

$$\stackrel{\text{def}}{=} \frac{\partial}{\partial s} \Big|_0 D_{\gamma_X(s)} A_{\gamma_X(s)} \cdot Y$$

$$\stackrel{\text{①}}{=} \frac{\partial}{\partial s} \Big|_0 \frac{\partial}{\partial t} \Big|_0 A_{\gamma_X(s)} (\gamma_Y(t))$$

$$\stackrel{\text{def}}{=} \frac{\partial^2}{\partial s \partial t} \Big|_{(0,0)} \gamma_X(s) \gamma_Y(t) \underbrace{\gamma_X(s)^{-1}}_{\gamma_X(-s)} \leftarrow \text{Question sheet 2}$$

$$= \frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} \gamma_X(s) \gamma_Y(t) \gamma_X(0) - \frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} \gamma_X(0) \gamma_Y(t) \gamma_X(s)$$

$$= \frac{\partial}{\partial s} \Big|_0 \frac{\partial}{\partial t} \Big|_0 F^Y(t, \gamma_X(s)) - \frac{\partial}{\partial t} \Big|_0 \frac{\partial}{\partial s} \Big|_0 F^X(s, \gamma_Y(t))$$

partial derivatives commute
and
chain rule for composition
 $\mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} \longrightarrow G$
 $s \mapsto (s, -s)$

$$(x_1, x_2) \mapsto \gamma_X(x_1) \gamma_Y(t) \gamma_X(x_2)$$

$$\Rightarrow \frac{\partial}{\partial s} = \sum \frac{\partial x_i}{\partial s} \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}$$

partial derivs
commute
and recall
 $\gamma_X(0) = 1$ and
 $F^Y(t, g) = g \cdot \gamma_Y(t)$
is flow of Y
(LECTURE 3)

$$= \frac{\partial}{\partial s} \Big|_0 Y \Big|_{\gamma_X(s)} - \frac{\partial}{\partial t} \Big|_0 X \Big|_{\gamma_Y(t)}$$

$$= [X, Y] \Big|_0 \text{ by next Lemma } \blacksquare$$

Lemma $X, Y \in \mathfrak{g} \Rightarrow [X, Y] \Big|_0 = \frac{\partial}{\partial s} \Big|_0 Y \Big|_{\gamma_X(s)} - \frac{\partial}{\partial s} \Big|_0 X \Big|_{\gamma_Y(s)}$

Pf locally $y = \sum b_i(x) \frac{\partial}{\partial x_i}$ so $y \Big|_{\gamma_X(s)} = \sum b_i(\gamma_X(s)) \frac{\partial}{\partial x_i}$
 $\frac{\partial}{\partial s} \Big|_0 y \Big|_{\gamma_X(s)} = \sum \frac{\partial}{\partial s} \Big|_0 (b_i \circ \gamma_X) \frac{\partial}{\partial x_i} \stackrel{(2)}{=} \sum (X \cdot b_i) \frac{\partial}{\partial x_i}$ ■

Proof of Thm also showed:

Corollary $X, Y \in \mathfrak{g} \Rightarrow \frac{\partial^2}{\partial s \partial t} \Big|_0 \gamma_X(s) \gamma_Y(t) \gamma_X(-s)$

Def A matrix group is a closed subgroup of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$
(or linear group) \uparrow (or $\text{Aut}(R)$ for v.s. R)

Rmk By LECTURE 7, this condition ensures they are Lie groups.

Examples $O(n), SO(n), U(n), SU(n), SL(n, \mathbb{R}), \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}, \dots$

EXAMPLE 1 $[X, Y] = XY - YX$ for matrix groups ($X, Y \in \text{Mat}_{n \times n}$)

$$\gamma_X(s) = I + sX + o(s) \quad \left(\begin{array}{l} f \text{ is "little-o-hes" if } \lim_{s \rightarrow 0} \frac{f(s)}{s} = 0 \\ \text{hence } \frac{\partial}{\partial s} \Big|_0 f(s) = 0 \end{array} \right)$$

↑ ↑
since $\gamma_X(0) = I$ since $\gamma_X'(0) = X$

$$\Rightarrow [X, Y] = \frac{\partial^2}{\partial s \partial t} \Big|_0 \gamma_X(s) \gamma_Y(t) \gamma_X(-s) = \frac{\partial^2}{\partial s \partial t} \Big|_0 ((I + sX + o(s))(I + tY + o(t))(I - sX + o(s)))$$

$\left(\begin{array}{l} \text{The } s^2, t^2 \text{ terms vanish} \\ \text{if do } \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \end{array} \right) = \frac{\partial^2}{\partial s \partial t} \Big|_0 (I + tY + st(XY - YX) + o(st)) = XY - YX$ ■

EXAMPLE 2 G abelian ($S^1 = U(1), T^n, \mathbb{R}^n, \dots$)

$$\Rightarrow \text{Ad}_g(h) = hgh^{-1} = hh^{-1}g = g$$

$$\Rightarrow \text{Ad}_g = \text{Id}$$

$$\Rightarrow \text{Ad}(g) = D, \text{Ad}_g = \text{Id} \quad ("D \text{ of linear map is the linear map}")$$

$$\Rightarrow \text{ad} = D, \text{Ad} = 0 \quad ("D \text{ of constant map } g \mapsto \text{Id} \text{ is } 0")$$

$$\Rightarrow [., .] \equiv 0$$

$$\Rightarrow \mathfrak{g} \text{ abelian Lie algebra} \quad (\text{so } \cong (\mathbb{R}^n, [., .] \equiv 0))$$

USING EXP TO DETERMINE \mathfrak{g} AND G

Lemma If $H \xrightarrow{\text{Lie group}} G$ embedding then the 1-param. subgps of H are precisely those $\gamma_X(s) \subseteq G$ which lie in H .

$$\begin{array}{ccc} \text{Proof Naturality} & \begin{array}{c} \mathfrak{g} \xrightarrow{\exp} \mathfrak{g} \\ \downarrow \quad \downarrow \exp \\ H \xrightarrow{\quad} G \end{array} & \begin{array}{c} sY \mapsto sX \\ \downarrow \quad \downarrow \\ \gamma_y^H(s) \mapsto \gamma_x^G(s) \end{array} \end{array}$$

since embedding can view
\$H \subseteq G\$ and \$\mathfrak{g} \subseteq \mathfrak{g}\$
So \$Y \equiv X\$, \$\gamma_y^H(s) = \gamma_x^G(s) \subseteq H\$

converse: if \$\gamma_x^G(s) \subseteq H\$ then \$\gamma_x^G: \mathbb{R} \rightarrow H\$ is a Lie gp hom hence 1-param. subgp. in \$H\$ (smooth in \$H\$ because smooth in \$G\$)

Consequences

- Can identify \$\mathfrak{g} = \text{Lie}(H)\$ with a vector subspace of \$\mathfrak{g}\$:
 $\mathfrak{g} \cong \{X \in \mathfrak{g} : \gamma_X(s) \subseteq H \text{ for small (hence all) } s \in \mathbb{R}\}$
 ↪ Lie alg. iso. (respect bracket by above Corollary)
- \$\exp\$ for \$H\$ agrees with \$\exp\$ for \$G\$: \$\exp(X) = \gamma_X(1) \in H\$ if \$X \in \mathfrak{g} \subseteq \mathfrak{g}\$

EXAMPLE 3 \$\sigma(n) = \{X \in \text{Mat}_{n \times n}(\mathbb{R}) : X^T + X = 0\}

$$\begin{array}{ll} \text{Proof} & \gamma_X(s) \subseteq O(n) \Rightarrow I = \gamma_X(s)^T \gamma_X(s) \\ & \qquad \qquad \qquad \uparrow \\ & \qquad \qquad \qquad \text{in } GL(n) \\ & = (I + sX)^T (I + sX) + \sigma(s) \\ & = I + s(X^T + X) + \sigma(s) \text{ hence } X^T + X = 0. \end{array}$$

$\left(\begin{array}{l} \sigma(n) = \text{Lie } O(n) \\ \cong SO(n) \\ \uparrow \\ \text{since } O(n), SO(n) \\ \text{locally diffeo near } I \end{array} \right)$

converse:

$$\begin{array}{l} X^T + X = 0 \Rightarrow \gamma_X(s)^T \gamma_X(s) = \exp(sX)^T \exp(sX) = \exp(sX^T) \exp(sX) = \exp(-sX) \exp(sX) = I \\ \Rightarrow \gamma_X(s) \subseteq O(n) \end{array}$$

$\begin{array}{l} \text{exp series} \\ \text{for } gl(n). \\ \uparrow \\ \text{inverses} \\ (\text{Q. sheet 2}) \end{array}$

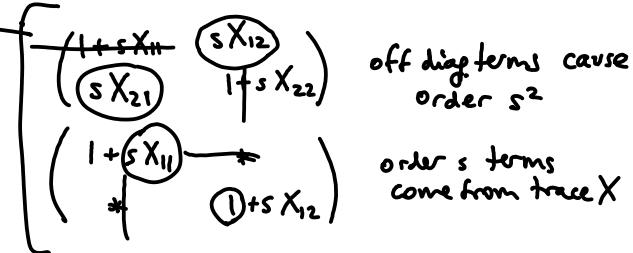
SAME PROOF SHOWS \$u(n) = \{X \in \text{Mat}_{n \times n}(\mathbb{C}) : X^* + X = 0\}

EXAMPLE 4 \$sl(n) = \{X \in \text{Mat}_{n \times n} : \text{Trace}(X) = 0\}\$ (\$sl(n) = \text{Lie } SL(n)\$, work over \$\mathbb{R}\$ or \$\mathbb{C}\$)

$$\text{Pf } \det \gamma_X(s) = \det(I + sX) + \sigma(s) = I + s \cdot \text{Tr}(X) + \sigma(s)$$

$$\Rightarrow \frac{\partial}{\partial s} \Big|_0 \det \gamma_X(s) = \text{Tr}(X)$$

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial t} \Big|_t \det \gamma_X(t) &= \frac{\partial}{\partial s} \Big|_0 \det \underbrace{\gamma_X(t+s)}_{\gamma_X(t)\gamma_X(s)} \\ &\stackrel{\text{chain rule}}{=} \det \gamma_X(t) \cdot \frac{\partial}{\partial s} \Big|_0 \det \gamma_X(s) \\ &= \det \gamma_X(t) \cdot \text{Tr}(X) \end{aligned}$$



order \$s^2\$ terms come from trace \$X\$

$$\text{Now: } \gamma_X(s) \in SL(n) \Rightarrow \det \gamma_X(s) = 1 \Rightarrow 1 + s \cdot \text{Tr}(X) + \sigma(s) = 0 \Rightarrow \text{Tr}(X) = 0.$$

converse: \$\text{Tr } X = 0 \Rightarrow \det \gamma_X(t)\$ constant in \$t \Rightarrow \det \gamma_X(t) = \det \gamma_X(0) = 1 \Rightarrow \gamma_X(t) \in SL(n)\$

SAME PROOF SHOWS: \$su(n) = \{X \in u(n) : \text{Tr}(X) = 0\}

Theorem Let \$G\$ be connected.

1) \$\exp : \mathfrak{g} \rightarrow G\$ is a group hom \$\iff G\$ abelian

2) \$G\$ abelian \$\iff G \cong \text{torus} \times \text{vector space}\$

Cor 1 \$G\$ abelian \$\Rightarrow \exp : \mathfrak{g} \rightarrow G_0\$ surjective hom. onto \$G_0 = \text{comm. comp. of } \text{leg.}\$

Cor 2 \$G\$ compact connected abelian \$\Rightarrow G \cong T^n\$

Cultural Rmk In non-connected case in 2) get torus \$\times\$ vector space \$\times\$ discrete abelian group

PROOFS Thm 2 \$\Rightarrow\$ Cor 2: because a vectorspace \$\cong \mathbb{R}^k\$ is non-compact (\$k \neq 0\$)

Pf 1: \Rightarrow : $(\mathfrak{g}, +)$ is an abelian group $\Rightarrow \exp(\mathfrak{g})$ abelian

But $\exp(\mathfrak{g})$ generates $G_0 = G$ (LECTURE 5: $\exp \mathfrak{g} \supseteq$ nbhd V of $1 \in G$) ■

Thm 1 \Rightarrow Cor 1 $\langle \exp(g) \rangle = G_0$ and, since image of a hom is a subgp, get $\exp(\mathfrak{g}) = \langle \exp(g) \rangle$ ■

Pf 1: \Leftarrow : G abelian \Rightarrow multiplication $\mu: G \times G \rightarrow G$ is a Lie gp hom

\Rightarrow naturality of \exp :

$$\begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} & \xrightarrow{D, \mu} & \mathfrak{g} \\ \downarrow \exp & & \downarrow \exp \\ G \times G & \xrightarrow{\mu} & G \end{array} \quad \begin{array}{c} (X, Y) \longmapsto D, \mu \cdot (X, Y) \\ \downarrow \quad \downarrow \\ (\exp X, \exp Y) \longmapsto \exp(X) \exp(Y) \end{array} \quad \begin{array}{l} \left(\mu((g_1, h_1) \cdot (g_2, h_2)) = \mu(g_1 g_2, h_1 h_2) = \right. \\ \left. = g_1 g_2 h_1 h_2 \stackrel{G \text{ abelian}}{=} g_1 h_1 g_2 h_2 = \mu(g_1, h_1) \cdot \mu(g_2, h_2) \right) \end{array}$$

(Rmk general fact $\text{Lie}(G_1 \times G_2) = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \xrightarrow{\exp} G_1 \times G_2$ is just \exp in each entry.
Indeed $\gamma_{X_1+X_2}(t) = (\gamma_{X_1}(t), \gamma_{X_2}(t)) \in G_1 \times G_2$ since solve flow equation in each entry)

$$D, \mu \cdot (X, Y) \stackrel{\text{naturality}}{=} \frac{\partial}{\partial t} \Big|_0 \mu(\gamma_X(t), \gamma_Y(t)) \stackrel{\substack{\text{chain rule} \\ \text{rule}}}{=} \frac{\partial}{\partial x_1} \mu(\gamma_X(x_1), 1) + \frac{\partial}{\partial x_2} \mu(1, \gamma_Y(x_2)) = X + Y$$

$\Rightarrow \exp(X+Y) = \exp(X) \exp(Y)$ so \exp is hom

Pf Thm 2: \Rightarrow Idea is to use the 1st isomorphism theorem for groups

We already know $\text{Im}(\exp: \mathfrak{g} \rightarrow G) = G_0 = G$ (by "Thm 1 \Rightarrow Cor 1"). Need find $\text{Ker}(\exp)$

Claim $K := \text{Ker}(\mathfrak{g} \xrightarrow{\exp} G)$ is a discrete subgroup of the vector space \mathfrak{g} .

(any point is an open set)

Proof $\exp: U \rightarrow V$ diffeo, and for $X \in K$: $\exp(X+U) = \exp(X) \cdot \exp(U)$

$\Rightarrow (X+U) \cap K = \{X\}$ (Note: $X+U$ is an open set around X in K for the subspace topology for $K \subseteq \mathfrak{g}$)

only get $\exp(u)=1$
if $u=1 \in U$

Def Discrete subgps of a vector space are called lattices.

FACT: discrete subgroups of a vector space are generated (as group, so over \mathbb{Z} not \mathbb{R})
by a finite collection of linearly independent vectors.

(This is proved by induction on dim of the vector space. We take it as a fact for this course)

$$\Rightarrow K = \text{span}_{\mathbb{Z}}(X_1, \dots, X_k) \cong \mathbb{Z}^k \subseteq \mathbb{R}^k = \text{span}_{\mathbb{R}}(X_1, \dots, X_k)$$

$$\begin{array}{ccc} \text{complete} & \mathfrak{g} = \text{span}_{\mathbb{R}}(X_1, \dots, X_n) \cong \mathbb{R}^k \times \mathbb{R}^{n-k} \\ \text{to a basis} & \xrightarrow{\text{U1}} & \xrightarrow{\text{U1}} \\ K & \xrightarrow{\quad} & \mathbb{Z}^k \times 0 \end{array}$$

$$\begin{array}{ccc} \xrightarrow{\text{1st isomorph}} & \mathfrak{g}/K \cong \mathbb{T}^k \times \mathbb{R}^{n-k} & \xrightarrow{\exp} \text{Image}(\exp) = G \quad \text{by 1st iso. theorem} \\ \text{theorem} & \cong & \blacksquare \end{array}$$

Above proof showed in general: Lemma $D_{(1,1)} \mu \cdot (X, Y) = X + Y$ for $\mu: G \times G \xrightarrow{\text{multiply}} G$

LIE SUBGROUPS

Lie subgroup $H \subseteq G$ means

- $H \subseteq G$ subgroup
- H is a Lie group
- inclusion $H \hookrightarrow G$ is smooth

Equivalently: a Lie subgp is an injective Lie group hom $H \xrightarrow{j} G$
 (identify $H \equiv j(H) \subseteq G$)

EXAMPLES

- $H = \mathbb{R} \cdot (1, 3) \subseteq T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ Lie subgp $\cong S^1$, namely $S^1 \xrightarrow{j} S^1 \times S^1$, $j(e^{2\pi i t}) = (e^{2\pi i t}, e^{3 \cdot 2\pi i t})$
 In this case, H is also a submanifold: a circle wrapping around first S^1 factor once and wrapping 3 times around second S^1 factor.
- $H = \mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ Lie subgp $\cong \mathbb{R}$, but not a submanifold (Q.sheet 3)
- $H = \text{Image } (\gamma_X) \subseteq G$ Lie subgp ($\cong \mathbb{R}$ or S^1 depending on whether γ_X is injective or not)
- $H = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_j \in S^1 \right\} \subseteq U(3)$ (where $X \neq 0 \in \text{Lie } G$)
- $H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \subseteq SL(3, \mathbb{R})$
- $G_0 \subseteq G$ for example $SO(2) \subseteq O(2)$

NON-EXAMPLE: Let $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ be a linear iso of vector spaces over \mathbb{Q} such that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ map to $\sqrt{2}, \sqrt{5}, \sqrt{3}$ respectively. Then φ is a group hom (using addition on \mathbb{R}) but φ is not smooth since not continuous (continuous bijections $\mathbb{R} \rightarrow \mathbb{R}$ are either strictly increasing or strictly decreasing)

GENERAL EXAMPLES

- $H \xrightarrow{\varphi} G$ Lie gp hom $\Rightarrow \text{Ker } \varphi \subseteq H$ is closed normal subgp (Q.sheet 4)
- FACT 1 $N \subseteq H$ closed normal subgp $\Rightarrow H/N$ is a Lie gp in natural way
 (see Lecture 8)
- $H \xrightarrow{\varphi} G$ Lie gp hom $\xrightarrow{\text{1st iso thm + Fact 1}} H/\text{Ker } \varphi \cong \text{Im } \varphi$ Lie subgp of G
- example $\gamma_X: \mathbb{R} \rightarrow G, X \neq 0 \Rightarrow \mathbb{R}/\text{Ker } \gamma_X \cong \text{Im } \gamma_X \subseteq G$ Lie subgp (indeed $\cong \mathbb{R}$ or S^1)
- COR All closed normal Lie subgps arise as Kernels: $N = \text{Ker } (H \xrightarrow{\text{quotient map}} H/N)$

WARNING : LIE SUBGROUPS MAY NOT BE SUBMANIFOLDS

$j: H \rightarrow G$
may not be
an embedding

Examples $\mathbb{R} \rightarrow T^2 = \mathbb{R}^2 / \mathbb{Z}^2$, $x \mapsto (x, \lambda x)$, for λ irrational, is injective and has dense image, not a homeomorphism onto image so not a submfd (the subspace topology for the image is not the usual topology on \mathbb{R})

Lemma 1 $H \xrightarrow{j} G$ Lie subgroup $\Rightarrow D_h \varphi: T_h H \rightarrow T_{\varphi(h)} G$ injective for all $h \in H$

Pf D_{ij} is injective by naturality of \exp since j is injective nbhd $o \in \mathfrak{g}$ \rightarrow nbhd $o \in g$
(linear map is injective if it is injective near 0) $\xrightarrow{\text{nbhd } i \in H \xrightarrow{\text{injective}} \text{nbhd } \varphi(i) \in G}$

To show $D_h j$ injective:

$$D_h \varphi \cdot X|_h = D_h \varphi \cdot D \phi_h^H \cdot X|_1 \xlongequal{\quad} D_1 \phi_{\varphi(h)}^G \cdot D_1 \varphi \cdot X|_1$$

\uparrow \uparrow \uparrow
 $\varphi \circ \phi_h^H = \varphi(h \cdot)$ $\varphi(h) \varphi(\cdot)$ injective
 $= \phi_{\varphi(h)}^G \circ \varphi$

$\left(\begin{array}{l} \text{span } T_h H \\ \text{as vary } X \in \mathfrak{h} \end{array} \right)$



Rmk Maps $N \xrightarrow{\varphi} M$ with $D\varphi$ injective are called immersions.

φ immersion $\Leftrightarrow \varphi$ is a local embedding (i.e. $\forall p \in N \exists$ nbhd $U \subseteq N$ s.t. $\varphi: U \rightarrow M$ embedding)
 \uparrow
 (Non-examinable: implicit function theorem argument)

Cor Lie subgps are locally embedded.

Example locally $\mathbb{R} \rightarrow T^2$, $x \mapsto (x, \lambda x)$ looks like /

LIE SUBALGEBRAS

Lie subalgebra $W \subseteq (V, [\cdot, \cdot])$ means

- vector subspace $W \subseteq V$
- $[w_1, w_2] \in W$ for all $w_1, w_2 \in W$

Equivalently: a Lie subalgebra is an injective Lie algebra hom $i: W \rightarrow V$
 \uparrow (identify $W \cong i(W) \subseteq V$)

Lemma $H \xrightarrow{j} G$ Lie subgp $\Rightarrow \mathfrak{h} \xrightarrow{D_{ij}} \mathfrak{g}$ Lie subalg

Pf By Lecture 5, D_{ij} is Lie alg hom. By Lemma 1, D_{ij} injective. ■

EXAMPLES

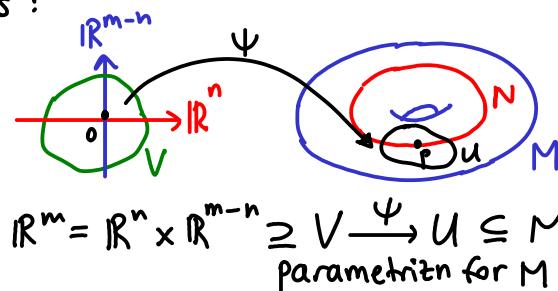
- $H = \gamma_x(\mathbb{R}) \subseteq G$ gives $\mathfrak{h} = \text{span}(X) = \mathbb{R} \cdot X \subseteq \mathfrak{g}$ (abelian Lie subalg since $[X, X] = 0$)
- Q. sheet 4 : $\text{Lie}(\text{Ker}(\varphi: H \rightarrow G)) = \text{Ker } D_\lambda \varphi \subseteq \mathfrak{h}$ Lie subalg
- $\text{Lie} \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_i \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} ix & 0 & 0 \\ 0 & iy & 0 \\ 0 & 0 & iz \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \subseteq u(3)$
- $\text{Lie} \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \subseteq sl(3, \mathbb{R})$

SUBMANIFOLDS

We defined submanifolds $N \subseteq M$ by saying that the inclusion is an embedding, but that is not very practical. Better:

FACT A subset $N \subseteq M$ is a submfd $\iff \forall p \in N \exists$ product parametrization near p

Means :



such that

- $\psi(p) = p$
- $\psi((\mathbb{R}^n \times \{0\}) \cap V) = N \cap U$

Notice: $\psi|_{\mathbb{R}^n \times \{0\}}$ gives a parametrization for N

FACT / COR Local coordinates x_1, \dots, x_n for N can be extended to local coordinates $x_1, \dots, x_n, \dots, x_m$ for M , so locally N is the subset defined by the equations $\begin{cases} x_{n+1} = 0 \\ \dots \\ x_m = 0 \end{cases}$

WHEN ARE LIE SUBGROUPS ALSO SUBMANIFOLDS?

Theorem Let G be a Lie group.

A subgroup $H \subseteq G$ is a submfd $\iff H \subseteq G$ closed subset

\uparrow (NOT assuming that H) \uparrow (and hence an
is a Lie group embedded Lie subgroup)

Example Any matrix group (= closed subgp of $GL(n)$) is a Lie group!

LECTURE 8

Dr Alexander F. Ritter, ritter@maths.ox.ac.uk
C3.5 LIE GROUPS, HT2015, Oxford.

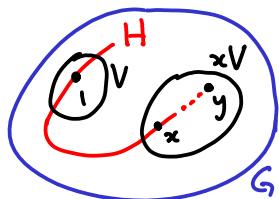
WHEN ARE LIE SUBGROUPS ALSO SUBMANIFOLDS?

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 ↑
 (Not assuming that H)
 ↓
 (and hence an embedded Lie subgroup)

Example Any matrix group (= closed subgp of $GL(n)$) is a Lie group!

Proof of Thm. \Rightarrow : H submfld $\Rightarrow H$ locally closed in G $\Rightarrow \exists$ nbhd V of 1 with $V \cap H$ closed



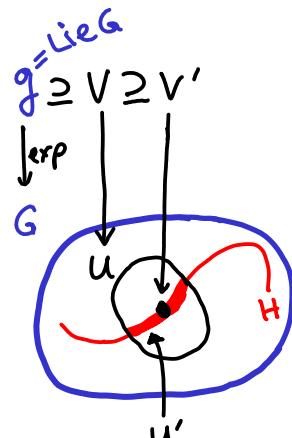
Let $y \in \overline{H}$, pick $x \in H$ close to y : $x \in yV^{-1}$

$\Rightarrow y \in xV \cap \overline{H}$ (note: V contains open nbhd of $1 \in G$, so V^{-1} does also since inversion is a diffeo, so yV^{-1} contains open nbhd of y , so $yV^{-1} \cap H \neq \emptyset$)

$$\Rightarrow x^{-1}y \in V \cap \overline{H} = V \cap H \Rightarrow y \in xH \subseteq H \Rightarrow \overline{H} \subseteq H \text{ hence equal. } \checkmark$$

$x^{-1}\overline{H} = \overline{x^{-1}H} = \overline{H}$

$$\begin{aligned} \text{pf} \Leftarrow: g = \text{Lie } G &\ni V^{\circ} \ni V' = \log(U') \quad (\text{we will show this is a nbhd of } 0 \in \text{Lie}(H)) \\ \exp \downarrow \text{diffeo} \quad \uparrow \log &= \text{inverse of } \exp: V \rightarrow U \\ G \ni U_{y_1} &\ni U' = H \cap U = \text{nbhd of } 1 \in H \end{aligned}$$



NOTE V' is just a subset of g . We want to build a vector subspace W from V' which is a likely candidate for $W = \text{Lie } H \subseteq g$.

STRATEGY: Pick an inner product on g (identify $g \cong \mathbb{R}^m$ and use usual i.p.)

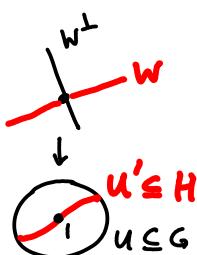
Define a subset W which is likely to be the set of tangent vectors to H at 1.

Show that W is a vector subspace of g .

Then show that $g = W \oplus W^\perp$ $\xrightarrow{\exp(1) \exp(-1)} G$ is a product parametrization near 1

$$\mathbb{R}^n \times \mathbb{R}^{m-n}$$

Finally, $\phi_{\mathbf{h}}^G \circ \psi$ is a product parametrization near $h \in H$.



DEFINE $W = \{tX : X = \lim_{n \rightarrow \infty} \frac{v_n'}{|v_n'|} \text{ some } v_n' \in V', v_n' \rightarrow 0, t \in \mathbb{R}\}$

MOTIVATION: you are trying to understand which are the likely tangent directions to H at 1 if it really were a submanifold.

- So you consider sequences of vectors $v_n' = \log(h_n)$ for $h_n \in H$ close to 1: this explains the conditions $v_n' \in V'$ and $v_n' \rightarrow 0$.

- You are not interested in the zero vector, so you might as well normalize: $\frac{v_n}{\|v_n\|}$
- The condition that $\|v_n'\|/\|v_n\|$ converges is not restrictive at all, since $\forall v_n \in g$, $v_n/\|v_n\|$ lie in the compact unit sphere of g , so passing to a subsequence you can always assume that it converges in g .
- Morally it should be enough to take $W = \mathbb{R} \cdot (V' \cap \text{small sphere } S \text{ in } g)$ since this would be the \mathbb{R} -span of a small sphere in $\text{Lie } H$ and hence is all of $\text{Lie } H$. The problem you would encounter is that it's not clear that $V' \cap (\text{ball boundary } S \text{ in } g) \subseteq W$

SANITY CHECK

If we knew H was an embedded Lie subgp and we knew $V' = n\text{bhd of } 0 \in \text{Lie}(H)$ then $W = \text{Lie}(H)$. Proof: $X \in \text{Lie}(H) \Rightarrow v_n' = \frac{1}{n}X \in V'$ for large n , $v_n' \rightarrow 0$, $\frac{v_n'}{\|v_n'\|} = \frac{X}{\|X\|} \rightarrow \frac{X}{\|X\|} \in W$ hence $t \cdot \frac{X}{\|X\|} \in W$ all $t \in \mathbb{R}$. Take $t = \|X\|$ to get $X \in W$ ■

Claim 1 $\exp(W) \subseteq H$

Pf We need to show $\exp(tX) \in H$ (where $X = \lim \frac{v_n'}{\|v_n'\|}$, $v_n' \in V'$, $v_n' \rightarrow 0$)

$$\text{Note } \|v_n'\| \rightarrow 0 \text{ and } \frac{t}{\|v_n'\|} \cdot v_n' \rightarrow tX$$

Pick $m_n \in \mathbb{Z}$ with $m_n \|v_n'\| \rightarrow t \leftarrow (\text{idea: approximating } \frac{t}{\|v_n'\|} \text{ by integers } m_n\right)$

$$\Rightarrow \exp(m_n v_n') = \exp(\underbrace{m_n \cdot \|v_n'\|}_{t} \cdot \underbrace{\frac{v_n'}{\|v_n'\|}}_X) \rightarrow \exp(tX) \quad \left. \begin{array}{l} \text{q.sheet 3 //} \\ \exp(v_n')^{m_n} \in H^{m_n} \subseteq H \end{array} \right\} \Rightarrow \exp(tX) \in \overline{H} = H. \quad \blacksquare$$

Claim 2 $W \subseteq g$ is a vector subspace

$\in H$ by claim 1, so $\gamma/t \in V'$ (small t)

Pf scaling ✓ adding: $X, Y \in W \Rightarrow \text{let } \gamma(t) = \log(\underbrace{\exp(tX) \exp(tY)}_{\in H \text{ by claim 1}}) \text{ so } \gamma(0) = 0$

$$\Rightarrow \gamma'(t) = \lim_{t \rightarrow 0} \frac{\gamma(t)}{t} = X + Y \quad \leftarrow \begin{array}{l} \text{(lecture 6:} \\ D_{(1,1)} \mu \cdot (X, Y) = X + Y \end{array}$$

$$\Rightarrow \lim_{t \rightarrow 0} \frac{\gamma(t)}{\|\gamma(t)\|} = \lim_{t \rightarrow 0} \frac{\gamma(t)}{t} \cdot \frac{t}{\|\gamma(t)\|} = \frac{X + Y}{\|X + Y\|} \in W \quad \begin{array}{l} \mu: G \times G \rightarrow G \\ \text{taking } \gamma(\frac{1}{n}) \in V' \text{ as the} \\ "v_n'" \text{ in the definition of } W \end{array}$$

$$\Rightarrow \text{rescale } X + Y \in W \quad \blacksquare$$

Claim 3 Define $\mathbb{R}^n \times \mathbb{R}^m \cong W \oplus W^\perp = \text{Lie } G \xrightarrow{\psi} G$, $\psi(w, \bar{w}) = \exp(w) \cdot \exp(\bar{w})$ Then ψ is a product parametrization for $H \subseteq G$ near $1 \in G$.

Pf $D_0 \psi \cdot (X, Y) = X + Y \leftarrow \text{q.sheet 3 (compare } D_0 \mu \text{ from lecture 6)}$

$\Rightarrow D_0 \psi$ iso, so by inverse function theorem ψ is diffco near 0 hence parametrization near $1 \in G$.

Remains to show $\psi(W)$ is a neighbourhood of $1 \in H$ (Note by claim 1, $\psi(W) \subseteq H$).

Suppose not, by contradiction. Since ψ is surjective near $1 \in G$, this implies:

$\exists \psi(w_n, \bar{w}_n) = \exp(w_n) \exp(\bar{w}_n) \in H \setminus \psi(W)$ arbitrarily close to 1, so: $\bar{w}_n \neq 0$, and $(w_n, \bar{w}_n) \rightarrow 0 \in V$

STRATEGY: build a non-zero $\bar{w} \in W^\perp \cap W = \{0\}$ (\Rightarrow contradiction!)

$\frac{\bar{w}_n}{\|\bar{w}_n\|} \in$ compact unit sphere in W^\perp , so passing to a subsequence can assume $\frac{\bar{w}_n}{\|\bar{w}_n\|} \rightarrow \bar{w} \in \underline{W^\perp}$, $\|\bar{w}\|=1$.
 \downarrow so $\bar{w} \neq 0$!

But $\exp(w_n) \in H$ (Claim 1) so $\exp(\bar{w}_n) = \underbrace{\exp(w_n)^{-1}}_{\in H} \underbrace{\exp(w_n)\exp(\bar{w}_n)}_{\in H} \in H$ since subgp.
 $\Rightarrow \bar{w}_n = \log \exp(\bar{w}_n) \in V'$, $\bar{w}_n \rightarrow 0$, $\bar{w} = \lim \frac{\bar{w}_n}{\|\bar{w}_n\|}$ so by definition of W get $\bar{w} \in \underline{W}$. ■

NON-EXAMINABLE REMARK: QUOTIENT LIE GROUPS

For N closed normal subgrp of $G \Rightarrow G/N$ Lie group

- $G \xrightarrow{\pi} G/N$ smooth
- $\text{Lie}(G/N) = \mathfrak{g}/\mathfrak{n}$ where $\mathfrak{n} = \text{Lie}(N)$
- $[X+n, Y+n] = [X, Y] + n$ (well-defined since n ideal)
(Q. sheet 4) $= \underline{U \cdot N}$

- Topology on G/N is quotient topology ($U \subseteq G/N$ open $\Leftrightarrow \pi^{-1}(U) \subseteq G$ open).
- Local parametrization near $1 \cdot N \in G/N$ is
 $\psi: (\text{nbhd of } 0 \in W^\perp) \xrightarrow{\subseteq} W^\perp \xrightarrow{\subseteq} W \oplus W^\perp = \mathfrak{g} \xrightarrow{\exp} G \xrightarrow{\pi} G/N$ (using notation of above proof for $H=N$)
- Near $gN \in G/N$ the local param. is $\phi_g \circ \psi$.

CONTINUOUS LIE GP HOMS ARE SMOOTH

Theorem $\varphi: H \rightarrow G$ continuous group homomorph $\Rightarrow \varphi$ smooth
 H, G Lie gps (hence Lie gp hom) !

EXAMPLE Q.sheet 2: $SU(2) \rightarrow SO(3)$ obviously cts gp hom \Rightarrow smooth

Pf graph of $\varphi: \Gamma_\varphi = \{(h, \varphi(h)): h \in H\} \subseteq H \times G$ is a closed subgp since φ cts
hence Lie subgp, so submfld, so $\Gamma_\varphi \xrightarrow{i} H \times G$ smooth.

- i) $H \times G \xrightarrow{\pi} H$ projection (smooth of course)
- $\Rightarrow \tilde{\pi} = \pi \circ i: \Gamma_\varphi \rightarrow H$ smooth
- ii) $\tilde{\pi}$ is homeomorphism (inverse $h \mapsto (h, \varphi(h))$)
- iii) φ gp hom $\Rightarrow \tilde{\pi}$ gp hom

Lemma 3
Lecture 5

$\tilde{\pi}$ homeomorph + local diffeo \Rightarrow diffeo (see Lecture 5)

$\Rightarrow \varphi: H \xrightarrow{\tilde{\pi}^{-1}} \Gamma_\varphi \xrightarrow{i} H \times G \xrightarrow{\text{project}} G$ smooth ■

$(G, \xrightarrow{\varphi} G_2$ Lie gp hom
 φ locally homeo near 1
 $\Rightarrow \varphi$ local diffeo)

LECTURE 9

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C3.5 LIE GROUPS, HT2015, Oxford.

THE SUBGROUP–SUBALGEBRA CORRESPONDENCE

G Lie group, $\mathfrak{g} = \text{Lie } G$.

Chevalley's Theorem There is a 1-to-1 correspondence

$$\begin{array}{c} \{\text{Lie subalgebras } \mathfrak{h} \subseteq \mathfrak{g}\} \longleftrightarrow \{\text{connected Lie subgroups } H \subseteq G\} \\ \mathfrak{h} = \text{Lie}(H) \longleftrightarrow H \\ \mathfrak{h} \longmapsto H = \langle \exp \mathfrak{h} \rangle = \text{subgp generated by } \exp \mathfrak{h} \end{array}$$

EXAMPLE

dim	$\mathfrak{h} \subseteq \mathfrak{g}$	$H \subseteq G$
0	$\{0\}$	$\{1\}$
1	$\mathbb{R} \cdot X$ (Lie subalgebra) (since $[X, X] = 0$)	$\gamma_X(\mathbb{R})$ (image of the (1-parameter subgp))
$n = \dim G$	\mathfrak{g}	$G_0 \subseteq G$

EXAMPLE $G = T^n = \mathbb{R}^n / \mathbb{Z}^n \Rightarrow [\cdot, \cdot] = 0$

\Rightarrow any vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra

Recall $\exp : \mathbb{R}^n \rightarrow T^n$ is the homomorphism $\pi(v) = v \bmod \mathbb{Z}^n$

$\Rightarrow \langle \exp \mathfrak{h} \rangle = \exp(\mathfrak{h}) = \mathfrak{h} \bmod \mathbb{Z}^n (\cong \mathfrak{h} / \mathfrak{h} \cap \mathbb{Z}^n)$

\Rightarrow correspondence is: $\begin{pmatrix} \text{vector subspaces} \\ \mathfrak{h} \subseteq \mathbb{R}^n = \text{Lie}(T^n) \end{pmatrix} \xleftrightarrow{1:1} \begin{pmatrix} \text{abelian subgroup} \\ \mathfrak{h} \bmod \mathbb{Z}^n \subseteq T^n \end{pmatrix}$

MORE EXAMPLES Q. sheet 4: $SO(3)$, $SL(2, \mathbb{Z})$.

have the form
subtorus \times vectorspace
e.g. $\mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2$ is $\cong \mathbb{R}$
 $\mathbb{R} \cdot (1, 3) \subseteq T^2$ is $\cong S^1$

Remarks • It's because we want the above correspondence that we do not require Lie subgrps to be submfds (Lecture 7, Q.sheet 3)

- The correspondence is difficult to prove because $H = \langle \exp \mathfrak{h} \rangle$ need not be a submfd. In general we need to define a new topology on H , which may not be the subspace topology, in order to prove that H is a Lie group!

(Example from Lecture 7, $\mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2$: the subspace topology has less open sets than the usual topology on \mathbb{R})



Connected is necessary: $O(3), SO(3) \subseteq GL(3, \mathbb{R})$ have Lie algs $\mathfrak{o}(3) = \mathfrak{so}(3) \subseteq \mathfrak{gl}(3, \mathbb{R})$.

Proof of uniqueness: H connected $\Rightarrow H$ generated by nbhd of $1 \in H$

$\Rightarrow \exp(\mathfrak{h})$ generates H (since contains nbhd of 1)
 $\Rightarrow H = \langle \exp(\mathfrak{h}) \rangle$ is the only possible choice if you want $\text{Lie } H = \mathfrak{h}$.

Proof of existence:

① Consider $D = \text{span}(\mathfrak{h}) \subseteq TG$. Pick basis X_1, \dots, X_d of \mathfrak{h} .

Notice: at each $g \in G$, $D_g = \text{span}(X_1|_g, \dots, X_d|_g) \subseteq T_g G$ is a d -dim'l v.s. and locally near $g \in G$ there are vector fields Y_1, \dots, Y_d with $D = \text{span}(Y_1, \dots, Y_d)$ e.g. take $Y_j = X_j$. Such D are called a d -dim'l distribution on the manifold G .

② Say that a vector field X on G is in D , written $X \in D$, if $X|_g \in D_g$ all $g \in G$.

Claim 1 D is integrable (or involutive), meaning: $[X, Y] \in D \quad \forall X, Y \in D$

Proof all $X \in D$ are pointwise in span of X_1, \dots, X_d hence

$$X = \sum a_j(x) X_j \Rightarrow [X, Y] = \sum_{i,j} a_i b_j \underbrace{[X_i, X_j]}_{\in \mathfrak{h} \text{ since Lie subalg}} + a_i (X_i \cdot b_j) X_j - b_j (X_j \cdot a_i) X_i \in D$$

③

Local Frobenius Theorem

d -dim'l integrable distributions are locally of the form:

$$\exists x_1, \dots, x_m \text{ local coords for the manifold with } D_x = \text{span}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$$

Proof idea (NON-EXAMINABLE)

Locally $D = \text{span}(Y_1, \dots, Y_d)$. By integrating Y_1 get local coordinate $x_1 = \text{time coord. of flow of } Y_1$. Using the flow of Y_1 , one can build local coordinates y_1, y_2, \dots, y_m with $y_1 = \frac{\partial}{\partial y_1}$ and $y_i = x_i$. Then modify other v.f. $Z_j = Y_j - (y_j \cdot x_1) Y_1$ ($j \geq 2$) - notice $Z_j \cdot x_1 = 0$.

The slice $\{x_1 = 0\}$ is locally a submfld S and $D' = \text{span}(Z_2, \dots, Z_d)$ is a $(d-1)$ -dim'l integrable distribution on S (since $Z_j \cdot x_1 = 0$ have $Z_j \in TS$)

Then use an induction on dim of distribution to get local coords x_2, \dots, x_m on S .

Extend x_2, \dots, x_m to local coords near S by projecting to S (in y -coord system)

By construction $Y_1 = \frac{\partial}{\partial x_1}$ but need to check $Z_j \cdot x = 0$ for coords $x = x_{d+1}, \dots, x_m$

We know this on S (by induction) so we need to show it holds also near S .

$$\text{Observe: } \frac{\partial}{\partial x_1} Z_j \cdot x = Y_1 \cdot (Z_j \cdot x) = \underbrace{[Y_1, Z_j]}_{\in D} \cdot x \text{ since } Y_1 \cdot x = 0 \text{ (for } x = x_{d+1}, \dots, x_m\text{)}$$

$$\text{use } Y_1 \cdot x = 0 \Rightarrow \frac{\partial}{\partial x_1} Z_j \cdot x = \sum f_{jk} \cdot Z_k \cdot x \text{ some functions } f_{jk}$$

Now fix values of x_2, \dots, x_m , then $Z_j \cdot x = y_j(x_1)$ and get system of ODE's.

$$y'_j(x_1) = \sum f_{jk} y_k \text{ hence unique solution given initial condition}$$

Initial condition is $y_j(x_1) = Z_j \cdot x = 0$ on $S = \{x_1 = 0\}$ (by induction) so $y_j \equiv 0$ unique solution

$\Rightarrow D = \text{span}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$ (since $D \cdot x = 0$ all $x = x_{d+1}, \dots, x_m$) (Note: we are not claiming that $Z_j = \frac{\partial}{\partial x_j}$)

④ locally can integrate D meaning \exists submfld $S \subseteq G$ with $T_S S = D_S$

$$\text{namely: } S = \left\{ x = (x_1, \dots, x_d, \underbrace{x_{d+1}, \dots, x_m}_{\text{constants}}) \right\} \leftarrow \text{called slice}$$

By ③ these S are the only connected integral manifolds of D (meaning $T_S S = D_S \forall s \in S$)
 $\Rightarrow H$ near $l \in G$ is the unique slice of D through l .

⑤ piece together slices of D starting with this one) to build the manifold H .

Rmk 1: slices are embedded, so we simply define the topology and manifold structure as the subspace topology and submanifold structure of $S \subseteq G$.

Definition A leaf L is a connected integral manifold meaning:

a manifold L together with an injective immersion $L \xrightarrow{\varphi} G$ such that $D\varphi \cdot TL = D$.

Examples: • a slice is an embedded leaf. $\curvearrowleft (\text{D}\varphi \text{ injective})$

- $\mathbb{R} \cdot (1, \sqrt{2})$ is a leaf in T^2

Recall (Lecture 7) φ immersion $\Leftrightarrow \varphi$ local embedding

Claim 2 A leaf is a union of slices, and each slice is an open subset of the leaf.

Pf Using φ cts, for small connected $U \subseteq L$ have $\varphi(U) \subseteq$ local model ④ so $\varphi(U) =$ some slice.

Since φ immersion, $\varphi: U \rightarrow \varphi(U) \subseteq G$ local embedding (for small U) so $U \cong \varphi(U)$ diffeo
 so the topology & manifold structure on U is the same as for slice (Rmk 1) ■

⑥ Rmk 1 + Claim 2 \Rightarrow topology and manifold structure on leaves is determined by G

\Rightarrow finer topology on G called leaf topology given by taking leaves as basis of open sets.

Example • $\mathbb{R} \cdot (1, \sqrt{2}) \subseteq T^2$  \curvearrowleft all these segments are now open sets!
 (leaves)

\Rightarrow Let $H =$ the connected component of l in the leaf topology on G
 = maximal connected leaf in G through l

$\Rightarrow H$ is a connected manifold (with the leaf topology) and $H \rightarrow G$ is an immersion.

Non-examinable FACT A map $S \xrightarrow{\alpha} H$ is smooth \Leftrightarrow composition $S \xrightarrow{\alpha} H \rightarrow G$ smooth
 $(\Leftarrow \text{requires some work to show } \alpha \text{ cts})$

⑦ Claim H is a subgp

connected components
 for the leaf topology

Pf $(\phi_g^G)_* X = X$ for all $X \in g$ $\Rightarrow (\phi_g^G)_* D = D \Rightarrow \phi_g^G$ permutes the maximal leaves
 (Q.Sheet 1) \curvearrowleft (Indeed: $L \xrightarrow{\varphi} G$ leaf $\Rightarrow L \xrightarrow{\varphi} G \xrightarrow{\phi_g^G} G$ leaf since injective, $D\phi_g^G \circ D\varphi$ injective,
 and $D\phi_g^G \circ D\varphi \cdot TL = D\phi_g^G \cdot D = D$)

For $h \in H$: $\phi_{h^{-1}}^G \cdot H$ leaf containing $l \Rightarrow \phi_{h^{-1}}^G \cdot H \subseteq H \Rightarrow h^{-1} \in H$
 $\Rightarrow \phi_{(h^{-1})^{-1}}^G \cdot H \subseteq H$ so $h \cdot h' \in H$ all $h' \in H$ ■

⑧ Claim group operations in H are smooth

Pf Want inversion $H \xrightarrow{i} H$ smooth. By above FACT need show composite $S = H \xrightarrow{i} H \xrightarrow{\varphi} G$ is smooth. This composite equals the composition $H \xrightarrow[\text{smooth}]{} G \xrightarrow[\text{smooth}]{} G$ hence smooth ✓

Want multiplication $H \times H \xrightarrow{m} H$ smooth. By FACT need show $S = H \times H \xrightarrow{m} H \xrightarrow{\varphi} G$ smooth.

This equals the composition $H \times H \xrightarrow[\text{smooth}]{} G \times G \xrightarrow[\text{smooth}]{} G$ hence smooth ✓ ■

COVERING MAPS

A smooth surjective map of manifolds $\pi: M \rightarrow N$ is a covering map if there is an open set U around any point $p \in N$ with:

- $\pi^{-1}(U) = \bigsqcup \tilde{U}_i$ disjoint union of open sets (the sheets over U)
- $\tilde{U}_i \xrightarrow{\pi} U$

The fibre over p is $\pi^{-1}(p) = \bigsqcup \{\tilde{p}_i\}$ (discrete set)

EXAMPLES

- $\mathbb{R} \xrightarrow{e^{2\pi i x}} S^1$, pictorially:  pictorially: $\exp: \mathbb{R} \rightarrow S^1$, $\exp^{-1}(1) = \mathbb{Z}$
- Q. sheet 1: $\exp = \pi: \mathbb{R}^n \rightarrow T^n = \mathbb{R}^n / \mathbb{Z}^n$ cover, fibre $\cong \mathbb{Z}^n$
- Q. sheet 2: $SU(2) \rightarrow SO(3)$ double cover, fibre $\cong \{\pm I\}$
- NON-EXAMPLE: $(0, 3) \xrightarrow{e^{2\pi i x}} S^1$ local diffeo but not covering

Rmks

- π local diffeo
- If N is connected, the fibres are all homeomorphic
(pick a path γ from p to p' in $N \Rightarrow \pi^{-1}(\gamma)$ are paths from \tilde{p}_i to \tilde{p}'_i)
- FACT If M, N compact mfds of same dimension, N connected then
 $M \xrightarrow{\pi} N$ covering $\Leftrightarrow \pi$ local diffeo $\Leftrightarrow D\pi$ surjective

Lemma For $\pi: H \rightarrow G$ Lie group hom and covering, then

- | | | |
|------------------|------|--|
| $\cdot 1$ | V | i) $\text{Ker } \pi = \pi^{-1}(1)$ is a discrete closed normal subgp of H |
| $\cdot h$ | hV | ii) fibres are homeomorphic to $\text{Ker } \pi$ |
| $\downarrow \pi$ | | iii) for small enough nbhd V of $1 \in H$, |
| $\cdot 1$ | U | $\bigsqcup_{k \in \text{Ker } \pi} k \cdot V \rightarrow U = \pi(V)$ are the sheets over U |
| | | iv) $H/\text{Ker } \pi \cong G$ (by 1st iso theorem) |

Pf for (ii): $\pi^{-1}(g) = h \cdot \text{Ker } \pi$ if $\pi(h) = g$

for (iii): pick $U \ni 1$ as in definition of covering, $V = \tilde{U}$ sheet containing 1.
 \tilde{U}' another sheet $\Rightarrow \tilde{U}' \cap \text{Ker } \pi = \{k\}$ some $k \Rightarrow \tilde{U}' = kV$ ■

Theorem 1 $H \xrightarrow{\pi} G$ Lie gp hom, G connected, then

π covering $\Leftrightarrow D\pi: \mathfrak{h} \rightarrow \mathfrak{g}$ isomorphism

Pf of " \Rightarrow ": π covering $\Rightarrow \pi$ local diffeo near 1 $\Rightarrow D\pi: T_1 H \rightarrow T_1 G$ iso ■

Pf of " \Leftarrow ": $D\pi$ iso $\Rightarrow D_h \pi$ iso, all $h \in H$
(Lecture 5)

$\Rightarrow \pi$ local diffeo $\Rightarrow \pi^{-1}(1) = \text{Ker } \pi$ discrete ①

inverse function thm $\Rightarrow \text{image}(\pi)$ is subgrp of G containing nbhd of 1 (G connected) $\Rightarrow \pi$ surjective ②

Trick $H \times H \rightarrow H$
 $(h, l) \mapsto h^{-1}l$ smooth $\Rightarrow \exists$ nbhd V of $1 \in H$ with $(V^{-1} \cdot V) \cap \text{Ker } \pi = \{1\}$

Claim $\bigsqcup_{k \in \text{Ker } \pi} h \cdot k \cdot V \xrightarrow{\pi} g \cdot \pi(V) = \text{nbhd of } 1 \in G$ whenever $\pi(h) = g$
(use ②)

$h \cdot k \cdot V \xrightarrow{\pi} g \cdot \pi(V)$ are the sheets over $g \cdot \pi(V)$.

Pf similar to pf of Lemma, in particular:

$h \cdot k \cdot V \xrightarrow{\pi} g \cdot \pi(V)$ local diffeo ✓

surjective ✓

injective since: $\pi(v_1) = \pi(v_2) \Rightarrow \pi(\underbrace{v_1^{-1} v_2}_{} \cdot 1) = 1 \Rightarrow v_1^{-1} v_2 \in V^{-1} \cdot V$ ✓

hence diffeo \blacksquare

SIMPLY-CONNECTED GROUPS

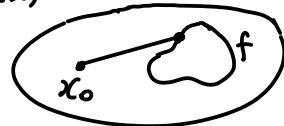
A path-connected manifold is simply-connected if continuous maps $S' \xrightarrow{f} M$ are contractible
(connected mfd)
see Lecture 2) meaning: \exists continuous $F: S' \times [0,1] \rightarrow M$

$$F(\cdot, 0) = f, \quad F(\cdot, 1) = \text{constant}$$

EXAMPLES

- $\mathbb{R}^n: F(x, t) = (1-t)f(x)$

- Convex subsets of \mathbb{R}^n



$$F(x, t) = t x_0 + (1-t)f$$

- $\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \xrightarrow{\text{diffeo}} \mathbb{R}^3$

- $S^n \subseteq \mathbb{R}^{n+1}$ spheres ($n \geq 2$) \leftarrow FACT it's simply connected

- $SU(2) \cong S^3$

- $SU(n), SL(n, \mathbb{C})$ \leftarrow FACT simply connected.

e.g. $GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$

NON-EXAMPLES $S^1, T^n, SO(n), U(n), SL(n, \mathbb{R})$ ($n \geq 2$), $GL(n, \mathbb{C})$

A covering $M \rightarrow N$ is called universal cover if M is simply connected.
 \nwarrow (path connected)

FACT Universal covers exist and are unique up to diffeomorphism.

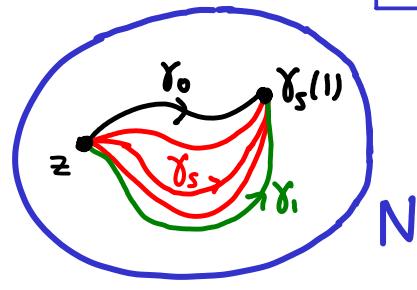
Sketch of existence (Non-examinable)

Fix $z \in N$ $M = \{ [\gamma] : \gamma: [0,1] \rightarrow N \text{ continuous path}, \gamma(0) = z \}$

equivalence class: identify paths γ_0, γ_1 if can continuously deform γ_0 to γ_1 keeping endpoints fixed

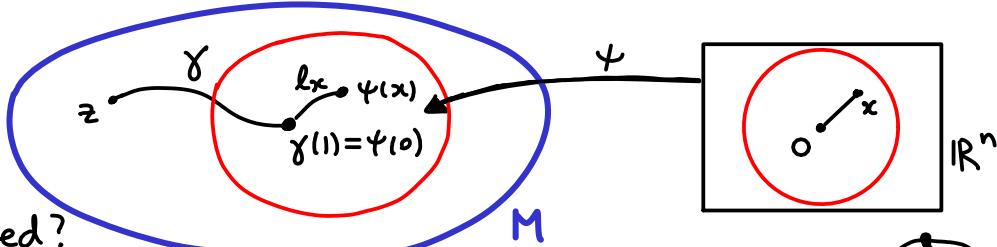
$(F: [0,1] \times [0,1] \xrightarrow{\text{cts}} N, F(\cdot, 0) = \gamma_0, F(\cdot, 1) = \gamma_1, F(0, \cdot) = z, F(1, \cdot) = \gamma_0(1) = \gamma_1(1))$

(think of this as a continuous family of paths $\gamma_s = F(\cdot, s)$)



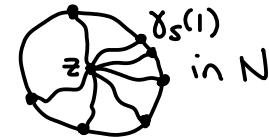
$$M \xrightarrow{\pi} N, \pi([\gamma]) = \gamma(1) = \text{end point of } \gamma$$

Rmk A local parametrization $\mathbb{R}^n \xrightarrow{\psi} N$ near $\psi(0) = \gamma(1)$ also parametrizes M near $[\gamma]$ via $[\gamma \# l_x]$ attach straight line segment $\gamma(tx)_{0 \leq t \leq 1}$ to γ so $\pi[\gamma \# l_x] = \psi(x)$.



Why M simply-connected?

Idea: a loop $S^1 \rightarrow M, s \mapsto [\gamma_s]$ corresponds to a picture of form in N so just contract it down to \bar{z} by $F(s, r) = [\text{path } t \mapsto \gamma_s((1-r) \cdot t)]$.



Lemma For G Lie group, the universal cover \tilde{G} is a Lie group in a natural way so that $\tilde{G} \xrightarrow{\pi} G$ is surj. Lie group hom, hence $\text{Lie}(\tilde{G}) \xrightarrow{D_1 \pi} \mathfrak{g}$ Lie algebra iso

Proof In the construction of \tilde{G} above pick $\underline{z=1}$. also $\tilde{G}/\ker \pi \cong G$.

unit: $\tilde{1} = [\text{constant path at 1}]$

multiplication: $[\gamma_1] \cdot [\gamma_2] = [\text{path } t \mapsto \gamma_1(t) \cdot \gamma_2(t)]$ notice via π these are the operations in N : $\gamma_1(1) \cdot \gamma_2(1)$ and $\gamma(1)^{-1}$
inversion: $[\gamma]^{-1} = [\text{path } t \mapsto \gamma(t)^{-1}]$ \Rightarrow locally, in above parametrizations, the operations are smooth since smooth in N .

EXAMPLES OF $\tilde{G} \rightarrow G$

- $\mathbb{R} \xrightarrow{\exp} S^1$ and $\mathbb{R}^n \xrightarrow{\exp} T^n$
- $SU(2) \rightarrow SO(3)$ Q.sheet 2
- $\text{Spin}(n) = \widetilde{SO(n)}$ definition of spin group for $n > 3$ (example: $\text{Spin}(3) \cong SU(2)$)
FACT $\text{Spin}(n) \rightarrow SO(n)$ is a double cover

FACT "Can't make universal covers any larger":

$\pi: \tilde{M} \rightarrow M$ covering, M simply conn. $\Rightarrow \pi$ diffeo
 \tilde{M} connected

Non-examinable proof idea: suffices to show fibre $\pi^{-1}(m)$ is a point.

By contradiction, if $\tilde{m}_1, \tilde{m}_2 \in \pi^{-1}(m)$, a curve connecting \tilde{m}_1, \tilde{m}_2 would give a contractible loop in M via π . Lifting this contraction to \tilde{M} then shows $\tilde{m}_1 = \tilde{m}_2$.

Cor $\tilde{G} \xrightarrow{\pi} G$ Lie gp hom + covering $\Rightarrow \pi$ Lie group iso
connected \uparrow simply connected (and connected) using that fibres are discrete

CORRESPONDENCE BETWEEN LIE ALG. & LIE GP. HOMOMORPHISMS

Theorem If H is simply-connected (and connected),

$$\left\{ \begin{array}{l} \text{Lie algebra homs} \\ g \xrightarrow{\psi} \mathfrak{g} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{Lie group homs} \\ H \xrightarrow{\varphi} G \end{array} \right\}$$

$$\psi = D_1 \varphi \quad \varphi$$

Proof Recall Lecture 5 : a Lie gp hom $H \xrightarrow{\varphi} G$ is uniquely determined by D, φ .
 Remains to show existence.

$$\psi: \mathfrak{h} \rightarrow \mathfrak{g} \Rightarrow \text{graph } \Gamma = \{(x, \psi(x)) : x \in \mathfrak{h}\} \subseteq \mathfrak{h} \times \mathfrak{g}$$

is a Lie subalgebra since ψ Lie alg hom.

(Chevalley) \Rightarrow corresponds to a connected Lie subgroup $S \subseteq H \times G$, $\text{Lie}(S) = \Gamma$.

Consider the projection $H \times G \rightarrow H$ Lie grp hom $\Rightarrow \mathfrak{h} \oplus \mathfrak{g} \rightarrow \mathfrak{h}$ Lie alg hom

Observe : $\pi : S \subseteq H \times G \rightarrow H$ has $D, \pi : \Gamma \subseteq \mathfrak{h} \oplus \mathfrak{g} \rightarrow \mathfrak{h}$ isomorphism

(Theorem 1)

$\Rightarrow \pi : S \rightarrow H$ covering

(Corollary) " diffeo (using H simply connected)

$\Rightarrow H \xrightarrow{\pi^{-1}} S \subseteq H \times G \xrightarrow{\text{project}} G$ Lie grp hom inducing ψ since :

$$\mathfrak{h} \longrightarrow \Gamma \subseteq \mathfrak{h} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$x \longmapsto (x, \psi(x)) \longrightarrow \psi(x)$$

■

Cor H, G simply-connected Lie gps with $\mathfrak{h} \cong \mathfrak{g} \Rightarrow H \cong G$ iso Lie gps

Pf $\mathfrak{h} \xrightarrow{\varphi} \mathfrak{g}$ gives a unique $H \xrightarrow{\varphi} G$

$\mathfrak{g} \xrightarrow{\varphi'} \mathfrak{h}$ " " $G \xrightarrow{\varphi'} H$

$\mathfrak{h} \xrightarrow{\varphi} \mathfrak{g} \xrightarrow{\varphi'} \mathfrak{h}$ " " $H \longrightarrow H$ which must be both $\varphi' \circ \varphi$ and identity $\} \Rightarrow \varphi' \circ \varphi = \text{id}$ $\} \Rightarrow \varphi$ iso ■

FACT Ado's theorem For any Lie algebra V , there is an injective Lie algebra hom $V \longrightarrow \text{gl}(m, \mathbb{R})$, some m .

Lie's third theorem

There is a 1-to-1 correspondence

$$\left\{ \begin{array}{l} \text{Lie algebras} \\ \text{isos} \end{array} \right\} / \text{Lie alg isos} \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{l} \text{simply-connected} \\ \text{Lie groups} \end{array} \right\} / \text{Lie gp isos}$$

Pf By Cor, get uniqueness of G (up to iso) for given \mathfrak{g} (up to iso)

Remains to show existence of G given Lie algebra V .

Ado's thm $\Rightarrow V \subseteq \text{gl}(m, \mathbb{R})$ Lie subalg

(Chevalley) \exists connected Lie subgp $H \subseteq \text{GL}(m, \mathbb{R})$ with $\mathfrak{h} = V$

\Rightarrow take $G = \widetilde{H}$ universal cover : simply connected and $\mathfrak{g} \cong \mathfrak{h} = V$ ■

REPRESENTATION THEORYLecture 5: Representation V of Lie group G means:Continuous hom $G \rightarrow \text{Aut}(V)$ where V vector space

- Rmk 1)
- we work over field $\mathbb{F} = \mathbb{R}$ or \mathbb{C}
 - always assume V finite-dimensional v.s. / \mathbb{F} , $d = \dim_{\mathbb{F}}(V)$
- 2) $\text{Aut}(V) = \{\text{linear bijections } V \rightarrow V\}$. If pick basis of V , $\text{Aut}(V) \cong \text{GL}(d, \mathbb{F})$
- 3) Recall continuous hom \Rightarrow smooth hom

Lemma Equivalent definition of rep:Group action $G \times V \rightarrow V$ continuous in G , linear in V

Explicitly:

- $1 \cdot v = v$ all $v \in V$
- $(g_1 g_2) \cdot v = g_1 \cdot (g_2 \cdot v)$ all $g_1, g_2 \in G, v \in V$

For this reason, we often call V a G -module or G -mod

Pf Action determines $G \xrightarrow{\varphi} \text{Bijections}(V, V)$, $g \mapsto (v \mapsto g \cdot v)$
continuous in G and linear in V , hence $G \xrightarrow{\varphi} \text{Aut}(V)$ rep.

Conversely, for rep $G \xrightarrow{\varphi} \text{Aut}(V)$ define action $g \cdot v = \varphi(g)(v)$ ■Lecture 5: Representation V of Lie algebra \mathfrak{g} means:Lie algebra hom $\mathfrak{g} \rightarrow \text{End}(V)$ where V vector spaceusing the bracket $[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha$ on $\text{End}(V)$ Rmk $\text{End}(V) = \{\text{linear maps } V \rightarrow V\}$. If pick basis of V , $\text{End}(V) \cong \text{Mat}_{d \times d}(\mathbb{F})$ Lecture 10 \Rightarrow For G simply-connected (and connected) Lie gp:

$$\{\text{Lie gp reps } G \rightarrow \text{Aut}(V)\} \xleftrightarrow{1:1} \{\text{Lie alg reps } \mathfrak{g} \rightarrow \text{End}(V)\}$$

Q.5 Question sheet 4 (case $SU(2) \rightarrow SO(3)$) generalizes to universal covers $\widetilde{G} \xrightarrow{\pi} G$:
For G connected Lie gp:

$$\{\text{Lie gp reps } \widetilde{G} \xrightarrow{\tilde{\varphi}} \text{Aut}(V)\} \xleftrightarrow{1:1} \{\text{Lie alg reps } \mathfrak{g} \xrightarrow{\psi} \text{End}(V)\}$$

For rep $G \xrightarrow{\varphi} \text{Aut}(V) \Rightarrow$ get rep $\tilde{\varphi} = \varphi \circ \pi: \widetilde{G} \xrightarrow{\pi} G \xrightarrow{\varphi} \text{Aut}(V)$ conversely given
 \Rightarrow forces $\ker \pi \subseteq \ker \tilde{\varphi}$ (i.e. $\ker \pi \subseteq \widetilde{G}$ acts by Id on V) $\tilde{\varphi}: \widetilde{G} \rightarrow \text{Aut}(V)$ with $\ker \pi \subseteq \ker \tilde{\varphi}$ then get
 $\psi: \mathfrak{g} \cong \widetilde{G}/\ker \pi \xrightarrow{\tilde{\varphi}} \text{Aut}(V)$

$$\Rightarrow \{\text{Lie gp reps } G \rightarrow \text{Aut}(V)\} \xleftrightarrow{1:1} \{\text{Lie gp reps } \widetilde{G} \xrightarrow{\tilde{\varphi}} \text{Aut}(V) \text{ with } \ker \pi \subseteq \ker \tilde{\varphi}\}$$

EXAMPLES

- Trivial representation $G \rightarrow \text{Aut}(V)$, $g \mapsto \text{Id}$
- Adjoint representation $\text{Ad}: G \rightarrow \text{Aut}(g)$

Def faithful rep means injective rep.

In practice it means you can identify G or g with a subset of matrices.

EXAMPLES Standard representations

$$\left. \begin{array}{l} O(n) \rightarrow \text{Aut}(\mathbb{R}^n) \\ U(n) \rightarrow \text{Aut}(\mathbb{C}^n) \\ \dots \end{array} \right\} \text{given by left-multiplication: } A \longmapsto \left(\begin{array}{c} \mathbb{F}^n & \xrightarrow{\quad} & \mathbb{F}^n \\ v & \longmapsto & A \cdot v \end{array} \right)$$

Def V, W G -mods, a G -linear map (or G -mod homomorphism) means

- \mathbb{F} -linear map $f: V \rightarrow W$
- f commutes with G -action: $f(g \cdot v) = g \cdot f(v)$

$$\text{Hom}_G(V, W) = \{G\text{-linear maps } f: V \rightarrow W\}$$

Def V, W equivalent reps if \exists G -isomorphism $f: V \rightarrow W$

Write $\underline{V \cong W}$.

\nwarrow (bijective G -linear map)

Explicitly:

$$\rho_i: G \rightarrow \text{Aut}(\mathbb{R}^n) = GL(n, \mathbb{R}) \text{ reps}$$

$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ iso \Rightarrow given by invertible matrix F

$$\rho_i \text{ equivalent} \Leftrightarrow f(\rho_i(g)v) = \rho_2(g)f(v) \Leftrightarrow \rho_i(g) = F^{-1} \cdot \rho_2(g) \cdot F$$

$\Leftrightarrow \rho_i(g)$ are conjugate via an iso F , for all g

INVARIANT INNER-PRODUCTS

Def For a rep V , an inner product (Hermitian i.p. if $\mathbb{F} = \mathbb{C}$) $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$ is G -invariant if

$$\langle g v, g w \rangle = \langle v, w \rangle \quad \text{all } g \in G, v, w \in V$$

and call V an orthogonal rep ($\mathbb{F} = \mathbb{R}$) or unitary rep ($\mathbb{F} = \mathbb{C}$).

Explicitly If pick o.n./unitary basis e_i for V

$$\Rightarrow \langle \sum a_i e_i, \sum b_j e_j \rangle = a^* b \quad (a, b \in \mathbb{F}^n \text{ and } * = \text{conjugate transpose})$$

$$\Rightarrow v^* \rho(g)^* \rho(g) w = \langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle = v^* w \quad \text{for all } v, w$$

$$\Rightarrow \rho(g)^* \rho(g) = \text{Id} \quad \text{so } \rho(g) \text{ is an orthogonal/unitary matrix}$$

$$\Rightarrow \mathbb{F} = \mathbb{R}: \rho: G \rightarrow O(n) \subseteq GL(n, \mathbb{R}) \cong \text{Aut}(V)$$

$$\mathbb{F} = \mathbb{C}: \rho: G \rightarrow U(n) \subseteq GL(n, \mathbb{C}) \cong \text{Aut}(V)$$

NEW FROM OLD

Let V, W be G -mods : we want to build new G -mods from V, W

- 1) direct sum $V \oplus W$ $g(v, w) = (gv, gw)$
- 2) tensor product $V \otimes_{\mathbb{F}} W$ $g(v \otimes w) = gv \otimes gw$

Recall : $V \otimes W$ is a vector space of dimension $\dim V \cdot \dim W$.

We use the symbols $v_i \otimes w_j$ to denote a basis of $V \otimes W$, whenever $\{v_i\}$ is a basis of V and $\{w_j\}$ is a basis of W .

EXAMPLE $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{n \cdot m}$

It is convenient to extend the symbol \otimes to any vectors by declaring that: \leftarrow So $v \otimes 0 = 0$ and $0 \otimes w = 0$

$$(\sum \lambda_i v_i) \otimes (\sum \mu_j w_j) = \sum \lambda_i \mu_j (v_i \otimes w_j) \quad (\text{often call } v \otimes w \text{ generators})$$

Warning: not all vectors in $V \otimes W$ arise as $v \otimes w$: you need to allow sums $\sum_i v_i \otimes w_i$

For example in $\mathbb{R}^2 \otimes \mathbb{R}^2$, $e_1 \otimes e_2 + e_2 \otimes e_1 \neq v \otimes w$ for any $v, w \in \mathbb{R}^2$.

A linear map $\varphi: V \otimes W \rightarrow U$ is determined by its values on generators, $\varphi(v \otimes w)$, since that determines φ on the basis $v_i \otimes w_j$. Conversely, if you define $\varphi(v \otimes w)$ in a way that is linear in v and in w , then φ extends to a well-defined linear map $V \otimes W \rightarrow U$.

- 3) $f \in \text{Hom}_G(V, W) \Rightarrow$ get G -mods : $\text{Ker } f$ $f(v) = 0 \Rightarrow f(g \cdot v) = g \cdot f(v) = 0$
 $\text{Im } f$ $g \cdot f(v) = f(g \cdot v) \in \text{Im } f$
 $\text{Coker } f = W / \text{Im } f$ $g \cdot (w + \text{Im } f) = gw + \text{Im } f$

- 4) conjugate space \bar{V} : as sets $\bar{V} = V$, use same G -action, but change the \mathbb{C} -action:
 $(\text{for } \mathbb{F} = \mathbb{C})$ $\lambda \cdot v = \bar{\lambda} v$ for $\lambda \in \mathbb{C}$

- 5) dual space $V^* = \{\text{IF-linear } V \xrightarrow{\psi} \mathbb{F}\}$ $(g \cdot \psi)(v) = \psi(g^{-1} \cdot v)$

Rmk : Need inverse g^{-1} to make it a left-action (check axiom (iii))

Another reason is that you want the following diagram to commute :

$$\begin{array}{ccc} V & \xrightarrow{\psi} & \mathbb{F} \\ g \downarrow & & \downarrow \text{id} \\ V & \xrightarrow{g \cdot \psi} & \mathbb{F} \end{array}$$

- 6) hom space $\text{Hom}_{\mathbb{F}}(V, W) = \{\text{IF-linear } V \xrightarrow{\psi} W\}$ $(g \cdot \psi)(v) = g \cdot (\psi(g^{-1} \cdot v))$

This ensures that the following diagram commutes :

$$\begin{array}{ccc} V & \xrightarrow{\psi} & W \\ g \downarrow & & \downarrow g \\ V & \xrightarrow{g \cdot \psi} & W \end{array}$$

Lemma $V^* \otimes_{\mathbb{F}} W \simeq \text{Hom}_{\mathbb{F}}(V, W)$ G -iso

Pf $\psi \otimes w \mapsto \left(\begin{array}{c} \psi: V \rightarrow W \\ \psi(v) = \underbrace{\psi(v)w}_{\in \mathbb{F}} \end{array} \right)$ and extend bilinearly in ψ and w .

- Linear by construction. ✓
- Bijective : pick basis v_i of V , get dual basis v_i^* so $v_i^*(v_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$. Pick basis w_i of W . Then $v_i^* \otimes w_j$ is a basis for $V^* \otimes W$. The corresponding ψ maps are : $\varphi_{ij}: V \rightarrow W$ with $\varphi_{ij}(v_k) = v_i^*(v_k)w_j = \begin{cases} w_j & \text{if } i=k \\ 0 & \text{if } i \neq k \end{cases}$ so φ_{ij} is the matrix with $\begin{cases} 1 & \text{in position } (j, i) \\ 0 & \text{in all other positions} \end{cases}$. These matrices are a basis for $\text{Hom}_{\mathbb{F}}(V, W)$. ✓

- preserves G -action:

$$(g \cdot (\psi \otimes w))v = ((g\psi) \otimes (gw))v = (g\psi)(v) \cdot gw = \underbrace{\psi(g^{-1}v)}_{\in \mathbb{F}} gw = g(\psi(g^{-1}v)w) = (g \cdot \psi)(v). \quad \blacksquare$$

REDUCIBILITY

Def A G-submodule or subrepresentation of V is a G -invariant vector subspace $W \subseteq V$.
 Call V reducible if \exists subrep $W \neq 0, V$ $\nwarrow G \cdot W \subseteq W$ (meaning: $g \cdot w \in W$ for all $g \in G, w \in W$)
irreducible if $W = 0, V$ are the only subreps. \leftarrow call V irrep

EXAMPLES

- $G = S^1$ acts on \mathbb{C}^2 by $e^{i\theta}: (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$. This is reducible: $\mathbb{C} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \subseteq \mathbb{C}^2$ is invariant.
- $G = \{(a b) : a, b \in \mathbb{R}, a \cdot c \neq 0\}$ acts naturally on \mathbb{R}^2 ($v \mapsto Av$ for $v \in \mathbb{C}^2, A \in G$)
 But \mathbb{R}^2 is reducible: $W = \mathbb{R} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a G -submod since $G \cdot W \subseteq W$.
- $G = U(2)$ acting on \mathbb{C}^2 is irreducible: if $v \neq 0 \in \mathbb{C}^2$, extend $u_i = \frac{v}{|v|}$ to a unitary basis u_1, u_2 . Then $A = (u_1, u_2) \in U(2)$, and $A \cdot e_i = u_i = \frac{v}{|v|}$. So $A^{-1} \cdot v = |v|e_i$. Similarly for $B = (u_2, u_1)$, $B^{-1} \cdot v = |v|e_2$. So if $W \subseteq \mathbb{C}^2$ is a subrep and $v \neq 0 \in W$, then $G \cdot W \subseteq W$, in particular $A^{-1}v, B^{-1}v \in W$, so $e_1, e_2 \in W$, so $W = \mathbb{C}^2$ (since it is a vector subspace, $\text{span}(e_1, e_2) \subseteq W$)

Schur's Lemma $f \in \text{Hom}_G(V, W)$ for V, W irreps $\Rightarrow f$ iso or $f = 0$

Pf $\text{Ker } f = 0$ or V since V irrep, $\text{Im } f = W$ or 0 since W irrep. \blacksquare

Schur's Lemma over \mathbb{C} $f \in \text{Hom}_{\mathbb{C}}(V, V)$ for V irrep $\Rightarrow f = \lambda \cdot \text{Id}$ some $\lambda \in \mathbb{C}$

Pf $\text{IF } \mathbb{F} = \mathbb{C} \Rightarrow \exists \lambda$ eigenvalue of f
 $\Rightarrow f - \lambda \cdot \text{Id} \in \text{Hom}_{\mathbb{C}}(V, W)$ with non-zero kernel
 $\xrightarrow{\text{Schur}} f - \lambda \cdot \text{Id} = 0 \quad \blacksquare$

Cor V, W irreps $\Rightarrow \begin{cases} \text{Hom}_G(V, W) = 0 & \text{if } V, W \text{ not equivalent} \\ \dim \text{Hom}_G(V, W) \geq 1 & \text{if } V, W \text{ equivalent} \\ & (= 1 \text{ when } \mathbb{F} = \mathbb{C}) \end{cases}$

Abbreviate $nV = \underbrace{V \oplus \dots \oplus V}_{n \text{ copies}}$ $n \in \mathbb{N}$

Theorem V_i non-equivalent irreps, then
 $\bigoplus m_i V_i \simeq \bigoplus n_i V_i \iff m_i = n_i$
 (we assume only finitely many m_i, n_i are non-zero)

\leftarrow IDEA: compare prime factorizations over \mathbb{N} .

$$\text{Pf} \Rightarrow: \text{Hom}_G(V_k, \bigoplus m_i V_i) \simeq \text{Hom}_G(V_k, \bigoplus n_i V_i)$$

$$\bigoplus m_i \text{Hom}_G(V_k, V_i) \stackrel{12}{\simeq} \bigoplus n_i \text{Hom}_G(V_k, V_i)$$

Schur: $\text{Hom}_G(V_j, V_i) = 0$ unless $i=j$

$$m_k \text{Hom}_G(V_k, V_k) \stackrel{12}{\simeq} n_k \text{Hom}_G(V_k, V_k)$$

Take dimensions $\Rightarrow m_k = n_k \quad \blacksquare$

COMPLETE REDUCIBILITY

Def A rep V is completely reducible if $V = \bigoplus n_i V_i$ is a sum of irreps.

Question Is every rep a sum of irreps?

Answer No: $G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C} \right\}$ (abelian!)

$V = \mathbb{C}^2$ is reducible, and subreps are:

$$0, W = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{C} \right\}, V$$

but $V \not\cong W \oplus$ irrep

More details:
If $\begin{pmatrix} x \\ y \end{pmatrix} \in$ subrep W
then:
 $\begin{pmatrix} 1 & -\frac{x}{y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} \in W$
so $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} \in W$
but W is v.-subspace, so
 $\begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} y \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ y \end{pmatrix} \in W$
span \mathbb{C}^2 so $W = \mathbb{C}^2$

Lemma For $\mathbb{F} = \mathbb{C}$, G abelian \Rightarrow Irreps are 1-dimensional

Pf For V irrep, G abelian, the multiplication $\phi_g: V \rightarrow V, \phi_g(v) = gv$ is G -linear:

$$\phi_g(g'v) = gg'v = g'gv = g' \phi_g(v)$$

$$\xrightarrow{\text{Schur}} \phi_g = \lambda_g \cdot \text{Id} \quad \text{some } \lambda_g \in \mathbb{C}.$$

$$\Rightarrow \phi_g(\mathbb{C}v) = \lambda_g \cdot \mathbb{C}v = \mathbb{C}v \quad \text{for all } g \in G$$

$$\Rightarrow \mathbb{C}v \text{ subrep of } V, \text{ so } V = \mathbb{C}v \text{ (since irrep)} \blacksquare$$

AIM: prove that for compact G , reps are completely reducible.

The proof for finite groups G is as follows:

① Given an inner product $(\cdot, \cdot): V \times V \rightarrow \mathbb{F}$ (e.g. pick basis, so $V \cong \mathbb{F}^d$, then use standard inner product on \mathbb{F}^d)

can produce a G -invariant inner product $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{F}$

by averaging $\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} (g \cdot v, g \cdot w).$

② If V irrep, done ✓

If V reducible, let $W \subseteq V$ be a subrep $\neq 0, V$. Then W^\perp is a subrep

proof: $\langle gv, w \rangle = \langle gv, gg^{-1}w \rangle = \underbrace{\langle v, g^{-1}w \rangle}_{\substack{\text{G-inverse} \\ \in W}} = 0 \quad \text{for all } w \in W$
 $\Rightarrow gv \in W^\perp \blacksquare$

③ Induction on $\dim V \Rightarrow$ can completely reduce $W, W^\perp \Rightarrow$ can completely reduce $V = W \oplus W^\perp$. ■

Theorem 1 If V admits a G -invt inner product, then V is completely reducible

Proof By steps ② & ③ above. ■

↙ PROOF NON-EXAMINABLE - SEE NON-EXAMINABLE HAND-OUT (integrals
are limits of
Riemann sums)

Theorem G compact Lie group

$\Rightarrow \exists$ unique normalized left-invariant integral over G , meaning:

To any continuous function $f: G \rightarrow \mathbb{R}$ it associates a value $\int_G f = \int_{g \in G} f(g) \in \mathbb{R}$
such that • $\int_G 1 = 1$

- If $f > 0$ then $\int_G f > 0$ (positivity)
- $\int_G (f_1 + \lambda f_2) = \int_G f_1 + \lambda \int_G f_2$ for $\lambda \in \mathbb{R}$ (linearity)
- $\int_G f \circ \phi_h = \int_{g \in G} f(hg) = \int_G f$ (left-invariance)

Rmk 1) linearity + positivity \Rightarrow monotonicity:

- if $f > g$ then $\int_G f > \int_G g$ (proof: consider $f-g > 0$)

(using normalization, this also holds with " \geq ". Proof: if $f > 0$ then $f > -\varepsilon$ so $\int_G f > -\varepsilon \int_G 1 = -\varepsilon$)

2) In fact the integral is bi-invariant, i.e. left and right invariant. Indeed it satisfies

- $\int_{g \in G} f(gh) = \int_G f$ (right-invariance)
- $\int_{g \in G} f(g^{-1}) = \int_G f$ (inversion-invariance)
- $\int_{G_1} f \circ \varphi = \int_{G_2} f$ for cts $f: G_2 \rightarrow \mathbb{R}$ and Lie gp iso $\varphi: G_1 \rightarrow G_2$ (isomorphism-invariance)

3) Can integrate continuous maps $f: G \rightarrow \mathbb{F}^d$ by integrating in each entry so $\int f \in \mathbb{F}^d$
 $\Rightarrow \dots$ " " " " $f: G \rightarrow V$ (pick basis, so $V \cong \mathbb{F}^d$) so $\int_G f \in V$

Exercise: $\frac{\text{LINEARITY}}{(V \text{ any v.s.)}} \quad \frac{\varphi: V \rightarrow V \text{ linear}}{f: G \rightarrow V} \Rightarrow \int \varphi \circ f = \varphi \left(\int f \right)$

Corollary G compact \Rightarrow for any rep V there is a G -invt inner product
 \Rightarrow all reps are completely reducible

Pf Pick any inner product (\cdot, \cdot) on V (e.g. standard i.p. on $\mathbb{F}^d \cong V$)

Define

$$\langle v, w \rangle = \int_{g \in G} (gv, gw)$$

Linear in each entry since G -action linear, (\cdot, \cdot) bilinear, and \int_G is linear.

Positive definite since $\langle v, v \rangle = \underbrace{\int (gv, gv)}_{>0 \text{ (for } v \neq 0\text{)}} > 0$ using positivity of \int_G .

G -invariant: for $h \in G$,

$$\langle hv, hw \rangle = \int_{g \in G} (ghv, ghw) = \int_{g \in G} f(gh) = \int_G f = \int_{g \in G} (gv, gw) = \langle v, w \rangle$$

Define $f: G \rightarrow \mathbb{R}$

$$f(g) = (gv, gw) \quad (\text{fixed } v, w)$$

right-invariance

CHARACTERS

Def The character $\chi_v = \chi_{\rho} : G \rightarrow \mathbb{F}$ of a rep $\rho : G \rightarrow \text{Aut } V$ is

$$\boxed{\chi_v(g) = \text{Tr}(\rho(g))}$$

$\text{Tr} = \text{Trace.}$

Q.sheet 5: • χ_v smooth

• $\chi_v(1) = \dim V$

• $\chi_v(h^{-1}gh) = \chi_v(g)$

• $V \cong W \Rightarrow \chi_v = \chi_w$

• $\chi_{V \oplus W}(g) = \chi_v(g) + \chi_w(g)$

• $\chi_{V \otimes W}(g) = \chi_v(g) \cdot \chi_w(g)$

• $\chi_{V^*}(g) = \chi_v(g^{-1})$

• $\chi_{\overline{V}}(g) = \overline{\chi_v(g)}$

Lemma 1 ($\mathbb{F} = \mathbb{C}$) G compact $\Rightarrow \chi_v(g^{-1}) = \overline{\chi_v(g)}$

Pf $\chi_v(g^{-1}) = \chi_{V^*}(g) = \overline{\chi_V(g)} = \overline{\chi_v(g)}$.

$V^* \cong \overline{V}$ Q.sheet 5, using G compact. ■

FIXED POINTS

Def $v \in V$ is a fixed point of the G -action if $g \cdot v = v$ for all $g \in G$.

$$\Rightarrow V^G = \{\text{fixed points}\} \subseteq V \quad \text{subrep}$$

For finite groups G you build fixed points by averaging:

$$V^G = \left\{ \frac{1}{|G|} \sum_{g \in G} g \cdot w : w \in V \right\}$$

Proof: $h \in G \Rightarrow h \cdot \left(\frac{1}{|G|} \sum g \cdot w \right) = \frac{1}{|G|} \sum hg \cdot w = \frac{1}{|G|} \sum gw$ since $\phi_h : G \xrightarrow{\sim} G$ bijection.

Conversely: v fixed $\Rightarrow \frac{1}{|G|} \sum_{g \in G} g \cdot v \underset{=v}{=} v = \underbrace{\frac{1}{|G|} \sum_{g \in G} v}_{=1} = v$ ■

Thm For compact Lie group G ,

$$\boxed{V^G = \left\{ \int_{g \in G} g \cdot w : w \in V \right\} \subseteq V}$$

Pf:

\supseteq : For $h \in G$, $\begin{matrix} V & \longrightarrow & V \\ v & \mapsto & h \cdot v \end{matrix}$ is linear $\Rightarrow h \int_{g \in G} g \cdot w = \int_{g \in G} h \cdot g \cdot w = \int_{g \in G} g \cdot v$.

\subseteq : v fixed $\Rightarrow \int_{g \in G} g \cdot v = \int_g v = \left(\int_G 1 \right) \cdot v = v$ ■

LINEARITY
(RMK 3)

Lemma 2 $\dim V^G = \sum_{g \in G} \chi_v(g)$

$$\underline{\text{Pf}} \quad \int \chi_v(g) = \int \text{Tr}(p(g)) = \underbrace{\text{Tr} \int p(g)}_{\substack{\text{LINEARITY} \\ (\text{Rmk 3})}} \underbrace{\int}_{\substack{\text{integrate each} \\ \text{matrix entry}}} p(g)$$

TRICK: averaging operator $\varphi: V \rightarrow V$, $\varphi(v) = \int g \cdot v$ is a projection onto V^G meaning $\varphi^2 = \varphi$ (indeed if $v \in V^G = \text{Image}(\varphi)$ then $\varphi(v) = \int \underbrace{gv}_{=v} = \int 1 = v$)

For projection maps, $\text{Tr}(\varphi) = \dim_{\mathbb{F}} \text{Im}(\varphi)$, since $V = \text{Im}(\varphi) \oplus \text{Ker}(\varphi)$
 $v \mapsto \varphi(v) + (v - \varphi(v))$

$$\Rightarrow \text{Tr} \int p(g) = \text{Tr}(\varphi) = \dim_{\mathbb{F}} \text{Im } \varphi = \dim_{\mathbb{F}} V^G \blacksquare$$

$$\begin{array}{c} \uparrow \\ \varphi = \text{Id} \end{array} \quad \begin{array}{c} \uparrow \\ \varphi = 0 \end{array}$$

ORTHOGONALITY RELATIONS

Theorem For compact Lie group G ,

$$\langle \chi_v, \chi_w \rangle \underset{\text{define}}{=} \int_{g \in G} \overline{\chi_v(g)} \chi_w(g) = \dim \text{Hom}_G(V, W)$$

Cor • $\langle \chi_v, \chi_w \rangle$ defines an inner product on $\text{span}_{\mathbb{F}} \{ \chi_v : V \text{ rep} \}$

• V_i non-equivalent irreps $\xrightarrow{\text{Schur}}$ χ_{V_i} are orthogonal, so linearly independent

• Lecture 11 : $V \cong W$ irreps $\Rightarrow \int \overline{\chi_v} \cdot \chi_w \begin{cases} = 1 & \text{if } \mathbb{F} = \mathbb{C} \\ \geq 1 & \text{if } \mathbb{F} = \mathbb{R} \end{cases}$
 (Cor of Schur) (equivalent)

Pf Thm Work over $\mathbb{F} = \mathbb{C}$ (if $\mathbb{F} = \mathbb{R}$ just think of $G \rightarrow \text{Aut}(\mathbb{R}^n) \subseteq \text{Aut}(\mathbb{C}^n)$)

$$\text{TRICK} \quad \text{Hom}_G(V, W) = \text{Hom}_{\mathbb{C}}(V, W)^G$$

recall $(g\varphi)(v) = (g \circ \varphi \circ g^{-1})(v)$ so $g\varphi = \varphi \Leftrightarrow \varphi(gv) = g\varphi(v)$ all $v \in V$ all $g \in G$

$$\begin{aligned} \Rightarrow \dim \text{Hom}_G(V, W) &= \int \chi_{\text{Hom}_{\mathbb{C}}(V, W)}(g) && \text{(Lemma 2)} \\ &= \int \chi_{V^*}(g) \chi_W(g) && \text{(Lecture 11 : } \text{Hom}_{\mathbb{F}}(V, W) \cong V^* \otimes W \text{)} \\ &= \int \overline{\chi_V(g)} \chi_W(g) && \text{(and use properties of characters)} \end{aligned}$$

(Lemma 1) \blacksquare

Thm G compact \Rightarrow any rep is determined uniquely (up to equivalence) by character

$$\underline{\text{Pf}} \quad V \cong \bigoplus n_i \underbrace{V_i}_{\text{irreps}} \Rightarrow \chi_V = \sum n_i \chi_{V_i} \Rightarrow n_i = \frac{\langle \chi_V, \chi_{V_i} \rangle}{\langle \chi_{V_i}, \chi_{V_i} \rangle} \begin{cases} = 1 & \mathbb{F} = \mathbb{C} \\ \geq 1 & \mathbb{F} = \mathbb{R} \end{cases} \blacksquare$$

LECTURE 12

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**NON-EXAMINABLE
HANDOUT**

VOLUME FORMS

Def An n -form ω on an n -dimensional mfd M is a function

$$\omega|_p : \underbrace{T_p M \times \dots \times T_p M}_{n \text{ copies}} \longrightarrow \mathbb{R} \quad \begin{matrix} \text{eats } n \text{ vectors at } p \\ \text{and spits out a number} \end{matrix}$$

(1) MULTI-LINEAR: linear in each entry

(2) ALTERNATING: switches sign if you transpose two entries

(3) SMOOTH in p : if X_1, \dots, X_n are smooth v.f. then

$$M \rightarrow \mathbb{R}, p \mapsto \omega(X_1, \dots, X_n)|_p \text{ is smooth.}$$

Lemma Locally, in coordinates x_1, \dots, x_n after identifying $T_x \mathbb{R}^n \equiv \mathbb{R}^n$:

$\omega = g(x) dx_1 \wedge \dots \wedge dx_n$ where $g(x) = \omega\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ is a smooth function
and $dx_1 \wedge \dots \wedge dx_n : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$, $(v_1, \dots, v_n) \mapsto \det(v_1, \dots, v_n)$

Pf Recall \exists unique $\delta : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{=n \times n \text{ matrices}} \rightarrow \mathbb{R}$ satisfying (1), (2), $\delta(e_1, e_2, \dots, e_n) = 1$ \blacksquare
(namely $\delta = \det$) $\underbrace{\delta(e_1, e_2, \dots, e_n)}_{\text{identity matrix}}$

Why that notation? (\leftarrow more on this in the Appendix)

$$\mathbb{R}^n \equiv T_x \mathbb{R}^n \quad \text{dual vector space} \quad (\mathbb{R}^n)^* \equiv T_x^* \mathbb{R}^n$$

$$e_i \equiv \frac{\partial}{\partial x_i} \quad e_i^* \equiv dx_i$$

$$\Rightarrow dx_i\left(\frac{\partial}{\partial x_j}\right) = e_i^*(e_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\Rightarrow X = \sum \underbrace{dx_i(X)}_{\text{coefficients } a_i} \cdot \frac{\partial}{\partial x_i} \quad \text{for any vector field } X$$

$$\stackrel{(1)}{\Rightarrow} (dx_1 \wedge \dots \wedge dx_n)(X_1, \dots, X_n) = \det(dx_i(X_j))$$

Example \mathbb{R}^2 : $dx_1 \wedge dx_2 \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) = 1$ (identity matrix)

$$dx_1 \wedge dx_2 \left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \quad (\text{as expected by (2)})$$

Def If $\varphi: M \rightarrow N$ smooth then can pull-back n -forms from N to M :

$$(\varphi^* \omega_N) \Big|_{\mathbf{x}} (v_1, \dots, v_n) = \omega_N \Big|_{\varphi(\mathbf{x})} (D\varphi \cdot v_1, \dots, D\varphi \cdot v_n)$$

$v_i \in T_{\mathbf{x}} M$

Rmk $\varphi_1^* \circ \varphi_2^* = (\varphi_2 \circ \varphi_1)^*$ by the chain rule.

Lemma 2 Locally $\varphi^* (g(y) dy_1 \wedge \dots \wedge dy_n) \Big|_{\mathbf{x}} = g \circ \varphi(\mathbf{x}) \cdot \det D\varphi \cdot dx_1 \wedge \dots \wedge dx_n$

$$\begin{aligned} \text{Pf } & (\varphi^* g(y) dy_1 \wedge \dots \wedge dy_n) \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \Big|_{\mathbf{x}} = g(\varphi(\mathbf{x})) \det dy_j \left(D\varphi \cdot \underbrace{\frac{\partial}{\partial x_j}}_{\parallel} \right) \\ & \qquad \qquad \qquad \underbrace{\sum_k (D\varphi)_{jk} \frac{\partial}{\partial y_k}}_{(D\varphi)_{ji}} \\ & = g \circ \varphi(\mathbf{x}) \cdot \det D\varphi \quad \blacksquare \end{aligned}$$

Def A volume form Ω on M^n is an n -form with $\Omega_p \neq 0 \forall p \in M$
 \Rightarrow locally, $g(x) \neq 0 \forall x$ in Lemma

Consequences:

4) either $g > 0 \forall x$ or $g < 0 \forall x$.

\Rightarrow can always ensure $g > 0$ by composing the parametrization with $\mathbb{R}^n \cong \mathbb{R}^n$
 $(x_1, x_2, \dots) \mapsto (-x_1, x_2, \dots)$

only use parametrizations for which $g > 0$

5) All n -forms are given by

$$\boxed{\omega = f \cdot \Omega}$$

smooth function $f: M \rightarrow \mathbb{R}$
locally: $f(x) = \frac{\omega \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)}{\Omega \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)}$

Lemma 3 transition $\tau = \tilde{\varphi}^{-1} \circ \varphi$ between parametrizations as in (4) $\Rightarrow \det(D\tau) > 0$

$$\text{Pf } \tau^* = \varphi^* \circ (\tilde{\varphi}^{-1})^*: dx_1 \wedge \dots \wedge dx_n \xrightarrow{(\tilde{\varphi}^{-1})^*} \frac{\Omega}{\tilde{g}(x)} \xrightarrow{\varphi^*} \frac{g(x)}{\tilde{g}(x)} dx_1 \wedge \dots \wedge dx_n$$

Cor (4) \Rightarrow Locally can integrate n -forms

Pf For a parametrization $\varphi: V \xrightarrow{\cong} U \subseteq M$ as in (4),

$$\varphi^* \omega = g(x) dx_1 \wedge \dots \wedge dx_n$$

$\uparrow \det(D\tau) \neq 0 \blacksquare$
(Lemma 2)

$$\Rightarrow \text{define } \int_U \omega = \int_V \psi^* \omega := \int_{V \subseteq \mathbb{R}^n} g(x) dx_1 \dots dx_n \quad \leftarrow \text{Known by multivariable calculus}$$

\star Check independent of choice of ψ by comparing with $\tilde{\psi}: \tilde{V} \xrightarrow{\cong} U$:

$$\int_{\tilde{V} \subseteq \mathbb{R}^n} \tilde{g}(\tilde{x}) d\tilde{x}_1 \dots d\tilde{x}_n = \int_{V \subseteq \mathbb{R}^n} g(x) |\text{Det } \tau| dx_1 \dots dx_n$$

multivariable calculus:
change of variables: $\tilde{x} \mapsto x = \tau(\tilde{x})$

Problem: $\tau^*(\tilde{g}(\tilde{x}) d\tilde{x}_1 \wedge \dots \wedge d\tilde{x}_n) \stackrel{\text{Lemma 2}}{=} g(x) \cdot \text{Det } \tau \cdot dx_1 \wedge \dots \wedge dx_n$

Solution: Lemma 3 $\Rightarrow \text{Det } \tau = |\text{Det } \tau|$, so O.K. \blacksquare

Cor (4) & M compact \Rightarrow Globally can integrate n-forms

IDEA: add up local contributions:

$$\int_M \omega = \sum_i \int_{U_i} \omega \quad \leftarrow \begin{array}{l} \text{see Appendix for details: only want to count} \\ \text{integral on overlaps } U_i \cap U_j \text{ once} \end{array}$$

IDEA: a tiny cube $\varepsilon e_1 \wedge \dots \wedge e_n$ of volume ε^n
maps approximately to the parallelepiped $\varepsilon \cdot \text{Det } \tau \cdot e_1 \wedge \dots \wedge e_n$ which has volume: $|\text{Det}(\varepsilon \cdot \text{Det } \tau \cdot e_1, \dots, \varepsilon \cdot \text{Det } \tau \cdot e_n)| = \varepsilon^n \cdot |\text{Det } \tau|$

Corollary If a manifold M has a volume form (such mfds are called orientable) and M compact then one can integrate functions $f: M \rightarrow \mathbb{R}$:

$$\int_M f = \int_M f \cdot \Omega \quad \left(\text{sometimes write } \int_{p \in M} f(p) \cdot \Omega|_p \right)$$

Satisfies:

LINEARITY : $\int_M (\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \int_M f_1 + \lambda_2 \int_M f_2 \quad \lambda_1, \lambda_2 \in \mathbb{R}$

POSITIVITY : $f \geq 0 \Rightarrow \int_M f \geq 0$ (and similarly with " $>$ ")

MONOTONE : $f_1 \geq f_2 \Rightarrow \int_M f_1 \geq \int_M f_2$

CHANGE OF VARIABLES : $\int_M \varphi^*(f \cdot \Omega) = \pm \int f \cdot \Omega \quad \text{for } M \text{ connected}$
 $\varphi: M \rightarrow M$ diffeo

Proof $\int_M \varphi^*(f \cdot \Omega) = \sum_i \int_{V_i} \varphi_i^* \varphi^*(f \cdot \Omega) = \sum_i (\varphi \circ \varphi_i)^*(f \cdot \Omega) = \pm \int_M f \cdot \Omega$

because: $\varphi \circ \varphi_i: V_i \rightarrow U_i$ are parametrizations and since M is connected either $(\varphi \circ \varphi_i)^* \Omega = F_i(x) dx_1 \wedge \dots \wedge dx_n$ for $F_i > 0 \forall i$ or $F_i < 0 \forall i$. In the first case call φ orientation-preserving: by \star can use $\varphi \circ \varphi_i$ instead of φ_i to compute integral. So get "+" in change of variables formula. In second case, get "-" since applying φ^* has the same effect as changing Ω to $-\Omega$. \blacksquare φ called orientation-reversing

INTEGRATION ON COMPACT GROUPS

For a Lie group G , $\dim G = n$,

$$\begin{array}{ccc}
 \left\{ \begin{array}{l} \text{multilinear alternating} \\ \underbrace{\text{Lie } G \times \dots \times \text{Lie } G}_{n \text{ copies}} \xrightarrow{\delta} \mathbb{R} \end{array} \right\} & \xleftrightarrow{\text{iso of vector spaces}} & \left\{ \begin{array}{l} \text{left-invariant} \\ n\text{-forms } \omega \text{ on } G \end{array} \right\} \\
 \omega|_1 & \longleftarrow & \omega \\
 \delta & \longleftarrow & (\omega|_g = \phi_{g^{-1}}^* \delta) \\
 & & \phi_g^* \cdot \omega|_h = \omega|_{g^{-1}h}
 \end{array}$$

Cor G Lie group $\Rightarrow \exists$ left-invariant volume form Ω and it is unique up to constant factor $c \neq 0 \in \mathbb{R}$.

Pf Pick basis for $\text{Lie } G$, so $\text{Lie } G \cong \mathbb{R}^n$. Then take $\delta = \det$ above:

$$\omega|_g = \phi_{g^{-1}}^*(\det).$$

Changing basis by matrix A changes ω by factor $c = (\det A)^{-1}$ ■

Cor G compact Lie group
 $\Rightarrow \exists$ unique left-invariant normalized volume form

Pf $\int_G c \cdot \Omega = c \cdot \int_G \Omega = 1$ fixes the c above ■

Example For finite groups G , $\int_G f = \frac{1}{|G|} \sum_{g \in G} f(g)$.

FROM NOW ON: G COMPACT, Ω LEFT-INVNT & NORMALIZED.

Lemma The integral is left-invariant $\int_G f = \int_G f \circ \phi_h$ ($\int_{g \in G} f(g) = \int_{g \in G} f(hg) \forall h \in H$)

Pf $\int_G f = \int_G f \Omega = \int_G \phi_h^*(f \Omega) = \int_G f \circ \phi_h \cdot \underbrace{\phi_h^* \Omega}_{\substack{\parallel \\ \Omega \text{ since left-invariant}}} = \int_G f \circ \phi_h$ ■

FACT Also right-invariant: $\int f = \int_{g \in G} f(gh)$

iso-invariant: for $\varphi: G_1 \rightarrow G_2$ Lie group iso, $\int f = \int_{G_2} f \circ \varphi$

inversion-invariant: $\int_{g \in G} f(g) = \int_{g \in G} f(g^{-1})$.

APPENDIX

Def An k -form ω on M^n is a map $\omega_p : \underbrace{T_p M \times \dots \times T_p M}_{k \text{ copies}} \rightarrow \mathbb{R}$

which is: 1) **MULTI-LINEAR**: linear in each entry

2) **ALTERNATING**: switches sign if you transpose two entries

3) **SMOOTH** in p : if X_1, \dots, X_k are smooth v.f. then

$M \rightarrow \mathbb{R}, p \mapsto \omega(X_1, \dots, X_k)|_p$ is smooth.

Examples

1) In local coordinates $(x_1, \dots, x_n) \in \mathbb{R}^n$, dx_i is a 1-form:

$$dx_i : T\mathbb{R}^n \rightarrow \mathbb{R}, dx_i\left(\frac{\partial}{\partial x_j}\right) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

2) Locally, $dx_1 \wedge \dots \wedge dx_k$ is a k -form defined by:

$$(dx_1 \wedge \dots \wedge dx_k)(X_1, \dots, X_k) = \det(dx_i(X_j)) = \sum_{\text{permutations}} \text{sign}(\sigma) dx_1(X_{\sigma(1)}) \dots dx_k(X_{\sigma(k)})$$

For example: $dx_1 \wedge dx_3\left(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1}\right) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

3) Forms can be added/scaled, so get vector space.

\Rightarrow Locally any k -form is:

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f_{i_1 \dots i_k}(x) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

smooth functions

Note: can reorder
 $dx_i \wedge dx_j = -dx_j \wedge dx_i$
(transposing 2 rows switches sign of det)
also $dx_i \wedge dx_i = 0$
(det = 0 if 2 rows are equal)

PULL-BACK: smooth maps $\varphi : M \rightarrow N$ pull-back forms:

$$\varphi^* : \Omega^k N \rightarrow \Omega^k M, \varphi^* \underline{\omega_N}(X_1, \dots, X_k) = \omega_N(D\varphi \cdot X_1, \dots, D\varphi \cdot X_k)$$

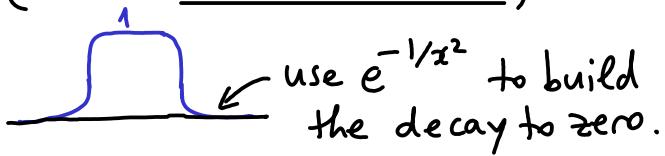
\uparrow \uparrow \uparrow
k-forms form on N vector fields on M

GLOBAL INTEGRATION

Given a countable cover U_1, U_2, \dots of M and parametrizations $\psi_i : V_i \xrightarrow{\cong} U_i$ one can construct functions $\rho_i : M \rightarrow [0, 1]$ (called partition of unity) such that

$$\rho_i = \begin{cases} 0 & \text{outside } U_i \\ 1 & \text{on } U_i \text{ except near boundary } \partial U_i \end{cases}$$

\leftarrow IDEA



such that $\sum_i \rho_i \equiv 1$. Then:

$$\int_M \omega = \int_M (\sum_i \rho_i) \omega = \sum_i \int_{U_i} \rho_i \omega = \sum_i \int_{V_i} \psi_i^*(\rho_i \omega)$$

OPTIONAL NON-EXAMINABLE EXERCISES

1) Find the left-invariant volume form for S' such that $\int_{S'} l = l$, and state an easy formula to compute $\int_{S'} f$ in practice. Do the same for T^n .

Hint For S' , $\omega = g(\theta)d\theta$ but left-inverse forces $g = \text{constant}$ and normalization forces $g = \frac{1}{2\pi}$ so $\int_{S'} f = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$.

2) G compact Lie group with the normalized left-invariant volume form. Show the following properties of the integral:

i) right-invariance: $\int f = \int_{g \in G} f(gh) \quad \forall h \in H$

ii) isomorphism-invariance: $\int f = \int_{G_1} f \circ \varphi \quad \text{if } \varphi: G_1 \rightarrow G_2 \text{ is}$

iii) inversion-invariance: $\int_{G_2} f(g) = \int_{G_1} f(g^{-1}) \quad \text{a Lie group isomorphism.}$

Hints Use the change of variables for integrals to reduce the problem to whether or not the new volume form you get is left-invariant and normalized.

More Hints call $\psi_h: G \rightarrow G$, $\psi_h(g) = gh$ (right-translation)

$$\int f(g) \Omega = \int \varepsilon_h \cdot \psi_h^*(f \cdot \Omega) = \int f(gh) \cdot \varepsilon_h \cdot \psi_h^* \Omega$$

↑
change
of vars sign ± 1 depending on whether
 ψ_h is orientation preserving or not

Now check that $\varepsilon_h \cdot \psi_h^* \Omega$ is left-invariant and normalized, hence by uniqueness $\varepsilon_h \cdot \psi_h^* \Omega = \Omega$, thus get (i).

This lecture: G COMPACT LIE GROUP, $\mathbb{F} = \mathbb{C}$

REPRESENTATION RING = CHARACTER RING

Def

Representation ring $R(G) = \left\{ \sum n_i V_i : n_i \in \mathbb{Z}, \text{finitely many } n_i \neq 0 \right\}$

using $+$ and \otimes

V_i : non-equivalent irreps of G .

$(V_i \otimes V_j = \sum m_k V_k \text{ in } R(G) \text{ if } V_i \otimes_{\mathbb{C}} V_j \simeq \bigoplus m_k V_k)$ virtual reps or virtual G -mods

"honest" reps: $\left\{ \sum n_i V_i \in R(G) : n_i \geq 0 \right\} \xleftrightarrow{1:1} \left\{ \text{equivalence classes of reps} \right\}$

(complete reducibility + last thm of Lecture 11)

Def $C\ell(G) = \text{ring of class functions}$: continuous $G \xrightarrow{\text{f}} \mathbb{C}$ satisfying $f(h^{-1}gh) = f(g)$

↑ pointwise addition & multiplication ↗ EXAMPLE characters $f = \chi_V$.

Thm $\chi: R(G) \rightarrow C\ell(G)$, $\chi(\sum n_i V_i) = \sum n_i \chi_{V_i}$ is an injective hom of rings

Pf Injective by orthogonality relns, hom since $\chi_{V_i} \otimes_{\mathbb{C}} \chi_{V_j} = \chi_{V_i} \cdot \chi_{V_j}$ ◻

Def Often identify $R(G)$ with $\chi(R(G)) \leftarrow$ called character ring

Thm 1 Any class function is uniformly approximated by $\sum z_i \chi_{V_i}$ ($z_i \in \mathbb{C}$)

(Q.Sheet 6) That is: $\text{span}_{\mathbb{C}}(\text{Image } \chi) \subseteq C\ell(G)$ dense.

Rmk fails for $\mathbb{F} = \mathbb{R}$: holds if use $\{f: G \rightarrow \mathbb{R} \text{ in } C\ell(G) \text{ with } f(g) = f(g^{-1})\}$

EXAMPLE $G = S^1 = \mathbb{R}/\mathbb{Z}$ $\rho_1: S^1 \rightarrow GL(1, \mathbb{C})$ $\rho_a = \rho_1^{\otimes a}: S^1 \rightarrow GL(1, \mathbb{C})$

(here $a \in \mathbb{Z}$, $\rho_a = \rho_1^{\otimes |a|}$ if $a < 0 \in \mathbb{Z}$) $\rho_1(x) = e^{2\pi i x} \cdot \text{Id}$ $\rho_a(x) = e^{2\pi i ax} \cdot \text{Id}$
 $\chi_1(x) = e^{2\pi i x}$ $\chi_a(x) = e^{2\pi i ax}$

Q.sheet 5 $\Rightarrow R(S^1) = \left\{ \sum_{a \in \mathbb{Z}} n_a \rho_a \text{ finite sum, } n_a \in \mathbb{Z} \right\} \cong \mathbb{Z}[t, t^{-1}] = \text{Laurent polys in } t$

$\chi: R(S^1) \rightarrow C\ell(S^1)$, $\chi(\sum n_a \rho_a) = \sum n_a e^{2\pi i ax} = \text{trigonometric polys with integer coeffs.}$

EXAMPLE $G = T^n = \mathbb{R}^n / \mathbb{Z}^n$ Q.sheet 6 using $T^n = S^1 x_1 \times \dots \times S^1$ (or directly by Q.sheet 5):

$R(G) = \left\{ \sum_{a \in \mathbb{Z}^n} n_a \rho_a \text{ finite sum, } n_a \in \mathbb{Z} \right\} \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \text{Laurent polys in } t_1, \dots, t_n$

$\rho_a: T^n \rightarrow GL(1, \mathbb{C})$, $\rho_a(x) = e^{2\pi i \langle a, x \rangle} \cdot \text{Id}$, $\chi_a(x) = e^{2\pi i \langle a, x \rangle}$
where $\langle a, x \rangle = a_1 x_1 + \dots + a_n x_n$

PETER-WEYL THEOREM

$\rho: G \rightarrow \text{Aut}(\mathbb{C}^n)$ gives a subset of $C(G) = \{ \text{continuous functions } f: G \rightarrow \mathbb{C} \}$

called matrix entries: $f(g) = ((i,j)\text{-entry of the matrix } \rho(g)) = \text{Trace } (\varphi \circ \rho(g))$

where $\varphi: \mathbb{C}^n \rightarrow \mathbb{C}^n$ linear map with $\varphi(e_i) = e_j$ and $\varphi(e_k) = 0$ all $k \neq j$.
 $(\varphi = \text{matrix with } 1 \text{ in position } (j,i) \text{ and } 0 \text{ elsewhere})$

EXAMPLE $G = S^1 \subset \mathbb{C}^2$, $\rho(x) = \begin{pmatrix} \cos 2\pi x & -\sin 2\pi x \\ \sin 2\pi x & \cos 2\pi x \end{pmatrix} \in \text{Aut}(\mathbb{C}^2)$. Take $(i,j) = (1,2)$
 $\varphi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{(2,1) \text{ entry}} \Rightarrow \varphi \circ \rho(x) = \begin{pmatrix} 0 & 0 \\ \cos 2\pi x & -\sin 2\pi x \end{pmatrix} \Rightarrow \text{Trace}(\varphi \circ \rho) = -\sin 2\pi x = \begin{pmatrix} (1,2) \text{ entry} \\ \text{of } \rho(x) \end{pmatrix}$

FACT (Schur) matrix entries of irreps are orthogonal using $\langle f_1, f_2 \rangle = \sum_{g \in G} \overline{f_1(g)} f_2(g)$.

Rmk holds for $\text{IF} = \mathbb{R}$ for two non-equivalent irreps, otherwise fails (e.g. example above).

EXAMPLE $G = S^1$ $\dim(\text{irreps}) = 1 \Rightarrow$ matrix entry $= X_\alpha = e^{2\pi i \alpha x}$ for $\alpha \in \mathbb{Z}$ are orthog.
 (indeed orthogonality relns: $\langle X_\alpha, X_\beta \rangle = 0$ for $\alpha \neq \beta$). Above example fails since reducible/ \mathbb{C} !

Def Representative function means any linear combination of matrix entries

They can always be written as $L \circ \rho$ where $\rho: G \rightarrow \text{Aut } V$, $L \in \text{Hom}_{\mathbb{C}}(V, V)^*$

Let $F(G) = \{\text{representative fns}\}$ " " " $\text{Tr}(\varphi \circ \rho)$ for $\rho: G \rightarrow \text{Aut } V$, $\varphi \in \text{Hom}_{\mathbb{C}}(V, V)$

- ① $L_1 \circ \rho_1 + \lambda L_2 \circ \rho_2 = (L_1 \oplus \lambda L_2) \circ (\rho_1 \oplus \rho_2)$ using $V = V_1 \oplus V_2 \Rightarrow F(G)$ is v.s. / \mathbb{C}
- ② Product of two matrix entries from ρ_1, ρ_2 is a matrix entry of $\rho_1 \otimes \rho_2$ (Q.1 Q.Sheet 5)
 $\Rightarrow F(G) \subseteq C(G)$ is subring and v.s./ \mathbb{C} , so it's \mathbb{C} -algebra.
- ③ If only allow rep V , get vector subspace $F_V(G) \subseteq F(G)$ of $\dim \leq (\dim V)^2 = \#(\text{matrix entries})$

Above Fact $\Rightarrow F(G) = \bigoplus F_{V_i}(G)$ orthogonal direct sum over the irreps V_i

Peter-Weyl Theorem (version 1) $F(G) \subseteq C(G)$ is dense (will not prove it.
Really mostly functional analysis)

Rmk also holds for $\text{IF} = \mathbb{R}$.

$G \xrightarrow{f} \mathbb{C}$ cts \Rightarrow can uniformly approximate f by representative fns
 (given $\varepsilon > 0$, $\exists \rho: G \rightarrow \text{Aut}(V)$, $\exists \varphi \in \text{Hom}_{\mathbb{C}}(V, V)$ with $\sup_{g \in G} |f - \text{Tr}(\varphi \circ \rho)| < \varepsilon$)

Fact $(f: G \rightarrow \mathbb{C}) \in \ell(G) \Rightarrow$ can choose $\varphi \in \text{Hom}_G(V, V) \subseteq \text{Hom}_{\mathbb{C}}(V, V)$
Stone-Weierstrass theorem (NON-EXAMINABLE)

- M compact mfd (more generally a compact Hausdorff topological space)
- $S \subseteq C(M) = \{\text{cts } M \rightarrow \mathbb{C}\}$ is *-subalgebra separating points with $1 \in S$
 Then $S \subseteq C(M)$ is dense!
 $f \in S \xrightarrow{f \in S} \bar{f} \in S \xrightarrow{S \subseteq C(M) \text{ subring}} \text{and vector subspace} \xrightarrow{m \neq m' \Rightarrow \exists f \in S \text{ with } f(m) \neq f(m')}$

Rmk also holds for $\text{IF} = \mathbb{R}$ so $S \subseteq C(M, \mathbb{R})$.

EXAMPLE $S = \text{Span}_{\mathbb{C}}\{f_\alpha(x) = e^{2\pi i \alpha x}: \alpha \in \mathbb{Z}\} \subseteq C(S^1)$ where $x \in S^1 = \mathbb{R}/\mathbb{Z}$

S is v.s. since it's a span, and a subalgebra since $f_a f_b = f_{a+b} \in S$
 $f_0 \in S \quad \bar{f}_a = e^{-2\pi i \alpha x} = f_{-\alpha} \in S \quad x \neq y \pmod{\mathbb{Z}} \Rightarrow e^{2\pi i x} \neq e^{2\pi i y}$
 $\Rightarrow S = \{\text{"trig polynomials"}\} \subseteq C(S^1)$ dense!

EXAMPLE for $\text{IF} = \mathbb{R}$ use $\text{Span}_{\mathbb{R}}\{\cos 2\pi ax, \sin 2\pi ax: a \in \mathbb{N}\} \subseteq C(S^1, \mathbb{R})$.

CONSEQUENCE: FOURIER ANALYSIS

$$\|f\|_{L^2}^2 = \int_S |f|^2 d\mu < \infty$$

Recall $C(S) \subseteq L^2(S) = \{\text{square-integrable fns } f: S \rightarrow \mathbb{C}\}$ dense (easy fact)
 \Rightarrow can L^2 -approximate any L^2 -function by trig polys. We knew this:

- $f = \left(\text{Fourier series } \sum_{a \in \mathbb{Z}} z_a e^{2\pi i ax} \right) \approx \sum_{a=-N}^N z_a f_a \text{ for } N \gg 0.$
- $z_a = \int_0^1 e^{-2\pi i ax} f(x) dx = \int_0^1 \overline{f_a(x)} f(x) dx = \langle f_a, f \rangle \in \mathbb{C}$ $\left(\begin{array}{l} z_a = \frac{1}{2\pi} \int_0^{2\pi} e^{-ia\theta} f(\theta) d\theta \\ \text{if parametrize by } \theta \in [0, 2\pi] \end{array} \right)$

PROOF OF PETER-WEYL FOR MATRIX GROUP

CLAIM $G = \text{compact matrix group} \Rightarrow F(G) \subseteq C(G)$ dense

Pf Already showed $S = F(G)$ is \mathbb{C} -algebra.

I $\in S$ since = (1,1) entry of trivial rep $G \rightarrow \text{Aut}(\mathbb{C})$, $g \mapsto \text{Id}$.

$f = L \circ \rho \in S \Rightarrow \overline{f} = L \circ \bar{\rho} \in S$ using dual rep $V^* \cong \overline{V}$ (Q.sheet 5, G compact)

Separate points: use standard rep $G \subseteq GL(n, \mathbb{R})$ acting on \mathbb{C}^n by matrix mult,
 so $\varphi: G \subseteq GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{C})$ inclusion, and if $g_1 \neq g_2$ are different
 matrices then some entry must be different $\Rightarrow f(g_1) \neq f(g_2)$ some $f = \text{Tr}(\varphi \circ \rho) \in S$ ■

COMPACT LIE GPS ARE MATRIX GROUPS

Peter-Weyl theorem (version 2)

G compact Lie gp $\Rightarrow \exists$ faithful rep $G \longrightarrow U(m)$ some m
 $(= \text{injective})$

We proved version 2 \Rightarrow version 1, Q.sheet 6: version 1 \Rightarrow version 2!

REGULAR REP, FOURIER ANALYSIS, ∞ -DIM REPS (NON-EXAMINABLE!)

Peter-Weyl Theorem (version 3)

\downarrow (v.s. with inner product, complete)

Any unitary rep $G \rightarrow U(H) \subseteq \text{Aut}(H)$ on a Hilbert space H
 is an orthogonal direct sum of finite dim'l unitary subreps

$$H = \widehat{\bigoplus} W_i \quad \leftarrow \text{(allow infinite sums if cgt in } H \text{ i.e. closure of the usual direct sum)}$$

Recall for finite gp G the regular rep is $H = \text{Functions}(G, \mathbb{C})$.
 H is a v.s. of $\dim = |G|$ with basis e_h indexed by $h \in G$ given by:

$$e_h = (\text{function } h \mapsto 1 \text{ and all other } g \mapsto 0)$$

G-action: $g \cdot e_h = e_{gh}$ (since on functions $(g \cdot e_h)(gh) = e_h(g^{-1}gh) = 1$)
 $\Rightarrow G \xrightarrow{\text{faithful}} \{\text{permutation matrices}\} \subseteq \text{Aut}(\mathbb{C}^{|G|})$.

FACT G finite \Rightarrow reg. rep. $H \cong \bigoplus (\dim V_i) \cdot V_i$ summing over all irreps

So in principle can find all irreps of G !

Can't work for compact Lie G unless allow ∞ -dim reps since for infinite G (compact) there are countably infinitely many finite dim'l irreps (for example for $SU(2)$, Q.sheet 5). This will follow from PW version 4 below.

Regular rep $H = L^2(G) = \{\text{square-integrable } f : G \rightarrow \mathbb{C}\}$

Hilbert space with $\langle f_1, f_2 \rangle = \int_{g \in G} \overline{f_1(g)} f_2(g)$ and G -action: $(h \cdot f)(g) = f(h^{-1}g)$

Peter-Weyl Theorem (version 4) $L^2(G) = \bigoplus W_i$

where $W_i \simeq (\dim V_i) \cdot V_i$ and all finite-dim irreps V_i arise

FACT An orthonormal analysis-basis for $L^2(G)$ is $\sqrt{\dim V_i} \cdot p_i^{(jk)}$ where $p_i^{(jk)}(g) = \langle p(g) \cdot v_j, v_k \rangle$ is the (j,k) matrix entry in o.n. basis v_j for V_i .
Analysis-basis e_n means linear combos can be infinite convergent sums so each $v \in H$ is uniquely $v = \sum_{n \in \mathbb{Z}} z_n e_n$ ($\dim L^2(G)$ countable Q.sheet 6)

EXAMPLE $L^2(S')$

Reg. rep. $\rho : G \rightarrow \text{Aut}(L^2(S'))$, $(\rho(x) \cdot f)(y) = f(y-x)$ ($x, y \in S' = \mathbb{R}/\mathbb{Z}$)

Claim $L^2(S') = \bigoplus_{a \in \mathbb{Z}} \mathbb{C} \cdot e^{-2\pi i ax}$ (if parametrize S' by $e^{i\theta}$
then $(\rho(e^{i\theta}) f)(e^{i\varphi}) = f(e^{-i\theta} e^{i\varphi}) = f(e^{i(\varphi-\theta)})$)

Pf By P.W. thm: $L^2(S') = \bigoplus_{a \in \mathbb{Z}} V_a$ (irreps of S' have $\dim=1$, $X_a = e^{2\pi i ax}$)

hence $V_a = \{f \in L^2(S') : \rho(x) \cdot f = e^{2\pi i ax} \cdot f\}$ (put $y=0$ replace x by $-x$)
 $= \mathbb{C} \cdot e^{-2\pi i ax}$ ■ (f(y-x) = $e^{2\pi i ax} f(y)$)
(f(x) = f(0) $\cdot e^{-2\pi i ax}$)

Rmk $F(G) \subseteq C(G) \subseteq L^2(G)$ are dense inclusions
 $\text{Span}_{\mathbb{C}}(X) \subseteq C(G)^G = C(G)^G \subseteq L^2(G)^G$ are dense inclusions. ← by Thm 1
What is character of $L^2(S')$? (conjugation invariant L^2 -functions)

$\chi_{L^2(S')} = \sum_{a \in \mathbb{Z}} e^{2\pi i ax}$ does not converge to a function.

It does however converge to a distribution (i.e. linear functional $C^\infty(S') \rightarrow \mathbb{C}$)

$f \in C^\infty(S') \mapsto \int \sum_{-N}^N e^{2\pi i ax} f(x) dx = \sum_{a=-N}^N z_{-a} \rightarrow \sum_{a \in \mathbb{Z}} z_a = f(0)$
 $f \mapsto f(0)$ is the delta-function! $\sum z_a e^{2\pi i ax} = f(x)$ at $x=0$ Fourier coeffs

General G: ∞ -dim rep ρ , the character is distribution $\chi_H(f) = \text{Tr} \left(\int_{g \in G} f(g) \rho(g) \right)$

For reg. rep $L^2(G)$: $\chi_{L^2(G)}(f) = f(1)$ is delta-function at 1

⇒ recover f : $\chi_{L^2(G)}(f \circ \phi_h) = f(\phi_h(1)) = f(h)!$

Over $\mathbb{F} = \mathbb{C}$

- G compact \Rightarrow Reps decompose $\rho = \bigoplus n_i \rho_i$ into irreps
- G abelian \Rightarrow Irreps are 1-dimensional, $G \rightarrow GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$
- G compact & abelian \Rightarrow Irreps must land in $S^1 \subseteq \mathbb{C} \setminus \{0\}$

Proof If $\rho(g) = \lambda \in \mathbb{C} \setminus \{0\}$ with $|\lambda| > 1$ then $\rho(g^n) = \lambda^n \rightarrow \infty$
 " " " " " $|\lambda| < 1$ then $\rho(g^{-n}) = \lambda^{-n} \rightarrow \infty$

But $\text{Image } (\rho : G \rightarrow \mathbb{C} \setminus \{0\})$ is compact hence bounded, so $|\lambda| = 1$ ■

Hence:

$$\begin{array}{ccc} \bullet G \text{ compact \& abelian} \Rightarrow & \left\{ \text{Irreps} \right\} & \xleftrightarrow{\substack{\text{1:1} \\ \text{equivalence}}} \left\{ \begin{array}{c} \text{Lie gp homs} \\ G \rightarrow S^1 \end{array} \right\} \\ & \rho = \chi_\rho \cdot \text{Id} & \longleftrightarrow \chi_\rho \end{array}$$

$\mathbb{F} = \mathbb{C}, G = S^1$ We know Lie gp homs $G = S^1 \rightarrow S^1$ so irreps are:

$$\rho_a : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow GL(1, \mathbb{C}), x \mapsto e^{2\pi i a x} \cdot \text{Id} \quad (a \in \mathbb{Z})$$

$\mathbb{F} = \mathbb{C}, G = T^n$ We know Lie gp homs $G = T^n \rightarrow S^1$ so irreps are:

$$\rho_a : T^n = \mathbb{R}^n/\mathbb{Z}^n \rightarrow GL(1, \mathbb{C}), \rho_a(x) = e^{2\pi i \langle a, x \rangle} \cdot \text{Id} \quad \left(\begin{array}{l} a \in \mathbb{Z}^n \\ \langle a, x \rangle = a_1 x_1 + \dots + a_n x_n \end{array} \right)$$

 $\mathbb{F} = \mathbb{R}$

G compact \Rightarrow Reps decompose $\rho = \bigoplus n_i \rho_i$ into irreps

G abelian \Rightarrow Irreps have $\dim_{\mathbb{R}} = 1$ or 2

Proof $G \rightarrow GL(n, \mathbb{R}) \subseteq GL(n, \mathbb{C})$

so as a complex rep, it has a complex subrep $\mathbb{C} \cdot v \overset{v \in \mathbb{C}^n}{\text{of dim}_{\mathbb{C}} = 1}$

Let $x = \text{Re}(v) = \frac{1}{2}(v + \bar{v})$ and $y = \text{Im}(v) = \frac{1}{2i}(v - \bar{v})$ then

$\text{span}_{\mathbb{R}}(x, y) = \text{Re}(\mathbb{C}v) + \text{Im}(\mathbb{C}v) \subseteq \mathbb{R}^n$ is real subrep of $\dim_{\mathbb{R}} = 2$ or 1

Indeed: for $A = \rho(g)$: $Ax = \frac{1}{2}(Av + A\bar{v}) = \frac{1}{2}(Av + \overline{Av}) = \text{Re}(Av)$

similarly $Ay = \text{Im}(Av) \in \text{Im}(\mathbb{C}v)$. ■ $\overset{\text{A real}}{\underset{\in \mathbb{C}v}{\text{Im}(\mathbb{C}v)}}$

$\mathbb{F} = \mathbb{R}, G = S^1$ Nontrivial irreps have $\dim_{\mathbb{R}} = 2$

Proof $\dim_{\mathbb{R}} = 1 \Rightarrow \dim_{\mathbb{C}} = 1$ irrep $\Rightarrow \rho = \chi_a \cdot \text{Id} \notin GL(1, \mathbb{R})$ ■

Claim $\rho_a^{\mathbb{R}} : G = S^1 \rightarrow GL(2, \mathbb{R})$, $\rho_a^{\mathbb{R}}(x) = \begin{pmatrix} \cos 2\pi a x & -\sin 2\pi a x \\ \sin 2\pi a x & \cos 2\pi a x \end{pmatrix}$ is irreducible. ($a \neq 0 \in \mathbb{Z}$)

Pf 1 $v \neq 0 \in \mathbb{R}^2 \Rightarrow v, \rho_a^{\mathbb{R}}\left(\frac{1}{4a}\right)v = (v \text{ rotated by } \frac{\pi}{2})$ are lin.indep/

\Rightarrow no subreps except $0, \mathbb{R}^2$ ■ $\overset{\text{not real!}}{\mathbb{R}^2}$

Pf 2 A $\dim_{\mathbb{R}} = 1$ subrep would be a common eigenspace of all $\rho_a^{\mathbb{R}}(x)$ ■

Claim $\rho_a^{\text{IR}} \cong \rho_b^{\text{IR}} \iff a = -b$

Pf " \Rightarrow ": $\chi_{\rho_a}(x) = 2 \cos(2\pi ax)$ and recall $\rho \cong \rho' \Rightarrow \chi_\rho = \chi_{\rho'}$.
 " \Leftarrow ": $s^{-1} \circ \rho_a \circ s = \rho_{-a}$ if s = reflection in x -axis ■

Claim The irreps_{IR} of S^1 are $\begin{cases} \rho_a^{\text{IR}} \text{ for } a=1,2,3,\dots \in \mathbb{N} \\ \text{trivial irrep } S^1 \rightarrow \text{GL}(1, \mathbb{R}), x \mapsto \text{Id} \end{cases}$

Pf 1 ρ irrep of $\dim_{\mathbb{R}} = 2 \Rightarrow$ irrep over \mathbb{C} of $\dim_{\mathbb{C}} = 2 \Rightarrow \rho \cong \rho_a \oplus \rho_b$ over \mathbb{C}
 ρ real $\Rightarrow \rho = \bar{\rho} \Rightarrow \rho_a \oplus \rho_b \cong \bar{\rho}_a \oplus \bar{\rho}_b = \rho_{-a} \oplus \rho_{-b}$
 But $a = -a, b = -b$ would imply $a = 0, b = 0$ so $\rho = \text{trivial}$. So $a = -b$.
 $\Rightarrow \rho \cong \rho_a \oplus \rho_{-a}$. So \exists subrep $\mathbb{C}\nu, \nu \in \mathbb{C}^2$, with $\rho(x)\nu = e^{2\pi i ax}\nu$
 Then $\rho(x)\bar{\nu} = \overline{\rho(x)\nu} = \overline{e^{2\pi i ax}\nu} = e^{-2\pi i ax}\bar{\nu}$ ($\Rightarrow \mathbb{C}\bar{\nu}$ = subrep ρ_{-a})

Let $\alpha = \frac{1}{2i}(\nu - \bar{\nu}) = \text{Im}(\nu)$, $\beta = \frac{1}{2}(\nu + \bar{\nu}) = \text{Re}(\nu) \in \mathbb{R}^2$. As above, for $A = \rho(x)$:
 $A\alpha = \text{Im}(A\nu) = \text{Im}(e^{2\pi i ax}\nu) = \cos(2\pi ax) \cdot \text{Im}(\nu) + \sin(2\pi ax) \cdot \text{Re}(\nu)$
 $A\beta = \text{Re}(A\nu) = \text{Re}(e^{2\pi i ax}\nu) = \cos(2\pi ax) \cdot \text{Re}(\nu) - \sin(2\pi ax) \cdot \text{Im}(\nu)$
 so in basis $\alpha, \beta \in \mathbb{R}^2$ get $A = \rho(x) = \begin{pmatrix} \cos 2\pi ax & -\sin 2\pi ax \\ \sin 2\pi ax & \cos 2\pi ax \end{pmatrix} = \rho_a^{\text{IR}}(x)$. ■

Pf 2 Pick a G -invariant inner product on $\mathbb{R}^n \Rightarrow$ (up to an equivalence since change to an orthonormal basis of G -invt i.p.) can assume

$\rho: G = S^1 \rightarrow O(2) \subseteq \text{GL}(2, \mathbb{R})$ indeed land in $SO(2)$ since S^1 connected
 \Rightarrow if identify $\mathbb{R}^2 = \mathbb{C}$, (rotation by θ on \mathbb{R}^2) = (multiplication by $e^{i\theta}$ on \mathbb{C})
 \Rightarrow get $\dim_{\mathbb{C}} = 1$ \times irrep $\underset{(\text{some } a \in \mathbb{Z})}{\rho(x) = \rho_a(x) = e^{2\pi i ax}}$. $\text{Id}_{\mathbb{C}} = \begin{pmatrix} \cos 2\pi ax & -\sin 2\pi ax \\ \sin 2\pi ax & \cos 2\pi ax \end{pmatrix} = \rho_a^{\text{IR}}(x)$

Pf 3 (using Peter-Weyl) matrix entries of ρ_a are $\cos(2\pi ax), \sin(2\pi ax)$
 As vary $a \in \mathbb{N}$, these generate a dense subalgebra of $C(S^1, \mathbb{R})$ (Fourier analysis)
 The matrix entries of two non-equivalent irreps are orthogonal, so if ρ is an irrep different from ρ_a , then $\langle \chi_\rho, f \rangle = 0$ for $f \in$ (that subalgebra). But such f are dense in $C(S^1)$ so $\langle \chi_\rho, f \rangle = 0$ all $f \in C(S^1)$ so $\chi_\rho = 0$ contradicting the orthogonality relation $\langle \chi_\rho, \chi_\rho \rangle \geq 1$ (over \mathbb{R}) ■

Real irreps of T^n : similar argument, get trivial irrep \mathbb{R} and 2-dim irreps:

$$\rho_a: T^n \rightarrow \text{GL}(2, \mathbb{R}), \rho_a(x) = \begin{pmatrix} \cos 2\pi \langle a, x \rangle & -\sin 2\pi \langle a, x \rangle \\ \sin 2\pi \langle a, x \rangle & \cos 2\pi \langle a, x \rangle \end{pmatrix}$$

for $a \in \mathbb{Z}^n \setminus \{0\}$ and $\rho_a \cong \rho_b \iff a = -b$.

LECTURE 14

HANDOUT ON ROOTS

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G	Maximal torus T	g
U(n)	$\left\{ \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} : \lambda_k = e^{2\pi i x_k} \right\}$ <p><u>Example calculation</u></p> <p>$A \in T$ then</p> $A z E_{kl} A^{-1} = e^{2\pi i x_k} z E_{kk} A^{-1}$ $= e^{2\pi i x_k} z E_{kl} e^{-2\pi i x_l}$ $= e^{2\pi i (x_k - x_l)} z E_{kl}$ <p>and on negative conjugate transpose</p> $(-z E_{kl})^T = -\bar{z} E_{lk} \text{ act by } e^{2\pi i (x_k - x_l)}$	<p>Will use matrices $E_{lk} = \begin{cases} 1 & \text{in position } (l,k) \\ 0 & \text{elsewhere} \end{cases}$</p> <p>$u(n) = \{B \in \text{Mat}_{n \times n}(\mathbb{C}) : \bar{B}^T = -B\}$</p> $= \bigoplus_k \mathbb{R} \cdot i E_{kk} \bigoplus \underbrace{\bigoplus_{k < l} \{z E_{kl} - \bar{z} E_{lk} : z \in \mathbb{C}\}}$ $\simeq \underbrace{n \cdot V_0}_{\text{ }} \bigoplus_{k < l} V_{x_k - x_l}$ $\simeq (g_0 = t) \bigoplus_{k < l} g_{x_k - x_l}$ <p>Roots = $\{x_k - x_l : 1 \leq k < l \leq n\}$</p>
SL(n)	as above except $\det = \prod \lambda_k = 1$ so $\sum x_k = 0$	$su(n) = \{B \in u(n) : \text{Tr}(B) = 0\}$ $= \bigoplus_{k \neq n} \mathbb{R} \cdot i (E_{kk} - E_{nn}) \bigoplus_{k < l} \text{ as above}$ $\simeq (n-1) V_0 \bigoplus_{k < l} V_{x_k - x_l}$ <p>Same roots as for $u(n)$.</p>
SO(2n)	$\left\{ \begin{pmatrix} R_1 & & & \\ & \ddots & & \\ & & R_n \end{pmatrix} : R_k = \begin{pmatrix} c(x_k) & -s(x_k) \\ s(x_k) & c(x_k) \end{pmatrix} \right\}$ $c(x) = \cos 2\pi x$ $s(x) = \sin 2\pi x$	$so(n) = \{B \in \text{Mat}_{n \times n}(\mathbb{R}) : B^T = -B\}$ $g_0 = \left\{ \begin{pmatrix} B_1 & & & \\ & \ddots & & \\ & & B_n \end{pmatrix} : B_k = \begin{pmatrix} 0 & x_k \\ -x_k & 0 \end{pmatrix}, x_k \in \mathbb{R} \right\} = t$ $g_{x_k - x_l} = \left\{ \begin{pmatrix} & R \\ -R^T & \end{pmatrix} : R = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right\}$ <p style="text-align: center;">↑ position (k,l) among 2×2 blocks position (l,k) as 2×2 block</p> $g_{x_k + x_l} = \left\{ \begin{pmatrix} & R \\ -R^T & \end{pmatrix} \text{ except now } R = \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \right\}$ <p>Roots = $\{x_k \pm x_l : 1 \leq k < l \leq n\}$</p> <p><u>Rmk</u> $U(n) \rightarrow SO(2n)$ embeds so $u(n) \rightarrow so(2n)$ embeds giving rise to $g_0 \bigoplus_{k < l} g_{x_k - x_l}$.</p>
SO(2n+1)	<p><u>Trick</u> $SO(2n) \subseteq SO(2n+1)$</p> $A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ $T \longmapsto \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}$ <p>max torus = $\begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}, T \in SO(2n)$</p>	$so(2n+1) \simeq so(2n) \bigoplus \bigoplus_k V_{x_k}$ $V_{x_k} = \left\{ \begin{pmatrix} x & y \\ -x-y & \end{pmatrix} : x, y \in \mathbb{R} \right\}$ <p style="text-align: center;">↑ in last column in k-th 2×1 block ↑ negative transpose</p> <p>Roots = $\{x_k \pm x_l : 1 \leq k < l \leq n\} \cup \{x_k : 1 \leq k \leq n\}$</p>

LECTURE 14

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ASSUME G COMPACT
 G CONNECTED

MAXIMAL TORI

Def A Lie subgp $T \subseteq G$ is called a torus if $T \cong \mathbb{R}^n / \mathbb{Z}^n$ (Lie gp iso)
T is a maximal torus if $T \subseteq T' \subseteq G \Rightarrow T = T'$ for any torus T' .

EXAMPLE $T = \{\text{diagonal matrices } \in U(n)\} \subseteq G = U(n)$ max torus

Lemma

- ① Max T exist (Pf $\{1\} \subsetneq T_1 \subsetneq T_2 \subsetneq \dots$ must stop since the $\dim \text{Lie}(T_j) < \dim g$ increase)
- ② Every torus lies in a max torus (Pf By Pf of ①.)
- ③ $T \text{ max} \Rightarrow$ conjugates gTg^{-1} are max tori (Pf $gTg^{-1} \subseteq T' \Leftrightarrow T \subseteq g^{-1}T'g$).
- ④ Tori are always embedded (Pf T compact $\Rightarrow \text{Image}(T \xrightarrow{\text{smooth}} G)$ compact so closed).

Lemma $\{\text{maximal tori}\} = \{\text{maximal connected abelian subgps } A \subseteq G\}$

Pf $T \subseteq A \subseteq \overline{A} \subseteq G \Rightarrow \overline{A}$ compact connected abelian \Rightarrow torus $\Rightarrow T = A = \overline{A}$ ■

Fact T max torus $\Rightarrow T$ is a maximal abelian subgrp \leftarrow (uses G connected)

Converse is false $SO(3) \supseteq A = \left\{ \begin{pmatrix} \pm 1 & \pm 1 \\ & \pm 1 \end{pmatrix} \right\}$ max abelian (disconnected).

Cor T max \Rightarrow Centralizer $Z(T) = \{z \in G : zx = xz \text{ all } x \in T\} = T$

Pf $T \subseteq Z(T)$. If $z \in Z(T) \setminus T$ then $\langle T \cup \{z\} \rangle$ is larger abelian subgp than T ■

EXAMPLE If $A \in U(n)$ commutes with diafonal matrices then A is diagonal!

T torus \Rightarrow rep: $\rho = \text{Ad}|_T : T \rightarrow \text{Aut}(g)$ $\rho(x) = \text{Ad}(x) = D, \xrightarrow{\text{Ad}} (\text{Ad}(h) = ghg^{-1})$

T compact \Rightarrow completely reduce $g \cong \bigoplus n_i V_i$ real irreps $V_i, n_i \in \mathbb{N}$

Irreps of tori: $V_0 = \mathbb{R}$ trivial rep T acts by Id
 $V_a = \mathbb{R}^2, \rho(x) = \text{rotation by } 2\pi\theta_a(x) = \begin{pmatrix} \cos 2\pi\theta_a(x) & -\sin 2\pi\theta_a(x) \\ \sin 2\pi\theta_a(x) & \cos 2\pi\theta_a(x) \end{pmatrix}$
where $\theta_a(x) = \langle a, x \rangle = a_1 x_1 + \dots + a_n x_n$ and $a \neq 0 \in \mathbb{Z}^n$

Since $V_a \cong V_b \Leftrightarrow a = -b$, don't distinguish $a, -a$ so don't distinguish $\theta_a, -\theta_a = \theta_{-a}$

$\Rightarrow g = g_0 \oplus \bigoplus_{a \neq 0} g_a$ $\text{Ad}(x)$ acts by Id on $g_0 = n_0 V_0$
 $\text{Ad}(x)$ " " rot'n by $2\pi\theta_a(x)$ on each V_a in $g_a = n_a V_a$.

Def For $n_a \neq 0$, $\theta_a : T \rightarrow S^1$ called roots of G

Rmks ① Lie alg rep $\text{ad}|_t : t = \text{Lie } T = \mathbb{R}^n \rightarrow \text{End}(g), \text{ad}(x) \cdot X = [x, X]$

$\tilde{\theta}_a : t = \mathbb{R}^n \rightarrow \mathbb{R} = \text{Lie}(S^1), \tilde{\theta}_a(x) = a_1 x_1 + \dots + a_n x_n$ roots of g ($n_a \neq 0$)

② Many books work with $g \otimes_{\mathbb{R}} \mathbb{C}$ instead of g (so $T \rightarrow \text{Aut}(g) \subseteq \text{Aut}(g \otimes_{\mathbb{R}} \mathbb{C})$)

$\Rightarrow \dim_{\mathbb{C}} (\text{irreps } V_a^{\mathbb{C}} \text{ of } T) = 1 \Rightarrow V_a \otimes_{\mathbb{R}} \mathbb{C} \cong V_a^{\mathbb{C}} \oplus V_{-a}^{\mathbb{C}}$ (diagonalize rot'n by $\begin{pmatrix} e^{2\pi i \theta_a} & 0 \\ 0 & e^{-2\pi i \theta_a} \end{pmatrix}$)

$\Rightarrow g \otimes_{\mathbb{R}} \mathbb{C} = n_0 V_0^{\mathbb{C}} \oplus \bigoplus_{a \in \mathbb{Z}^n \setminus 0} n_a V_a^{\mathbb{C}}$ with $n_a = n_{-a}$ since g real rep of T .

$V_0^{\mathbb{C}}, V_a^{\mathbb{C}}$ (for $n_a \neq 0$) called weight eigenspace with weight $0, 2\pi i \tilde{\theta}_a \in g^*$ since $\rho(x)v = 0 \cdot v, e^{2\pi i \tilde{\theta}_a(x)}v$ respectively for $v \in V_0, V_a$, similarly at Lie algebra level:
 $\text{ad}(x) \cdot v_a = [x, v_a] = (\text{differentiate}) = 0, 2\pi i \tilde{\theta}_a(x)v_a$ respectively.

③ Lie subalgebra $t = \text{Lie } T \subseteq g$ (after $\otimes_{\mathbb{R}} \mathbb{C}$) is called Cartan subalgebra since t abelian and ad_t acts diagonally (i.e. $g \otimes \mathbb{C} = \bigoplus \text{weight spaces}$) and maximal ($g_0 = t$)

Lemma T maximal $\iff t = g_0 \subseteq g$ $(t = \text{Lie}(T))$

Pf " \Rightarrow ": T abelian $\Rightarrow \text{Ad}_g = \text{Id}$ on $T \Rightarrow \text{Ad}(x) = D, \text{Ad}_g = \text{Id}$ on $t \Rightarrow t \subseteq g_0$

If $y \in g_0 \Rightarrow \text{Ad}(x)y = y \Rightarrow \exp(y) = \exp(\text{Ad}(x) \cdot y) = x \exp(y)x^{-1}$ all $x \in T$.

Same holds for $sy, s \in \mathbb{R}$ so T commutes with subgrp $H = \{\exp(sy) : s \in \mathbb{R}\}$

so $\langle TuH \rangle$ is a connected abelian subgrp larger than T unless $y \in T$ ■

" \Leftarrow ": $T \subseteq T' \subseteq G$ larger torus $\Rightarrow t \subseteq \text{Lie } T' \subseteq g'_0 \subseteq g_0$

Assumption $t = g_0 \Rightarrow t = \text{Lie } T' \Rightarrow T = T'$ \blacksquare $\begin{matrix} \uparrow \\ \text{1st part} \\ \text{of proof} \end{matrix} \quad \begin{matrix} \uparrow \\ \text{Ad}(x')y = y \quad \forall x' \in T' \\ \text{so also for } x' = x \in T \end{matrix}$
 $\begin{matrix} \uparrow \\ \text{decompose } g \text{ using } T' \end{matrix}$

EXAMPLE

Recall (Q. Sheet 3) for matrix groups G , $\text{Ad}(A) \cdot B = ABA^{-1}$.

$$G = U(2)$$

$$T = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} : \lambda_k = e^{2\pi i x_k}, (x_1, x_2) \in T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \right\}$$

$$\begin{aligned} U(2) &= \left\{ B \in \text{Mat}_{2 \times 2}(\mathbb{C}) : \bar{B}^T = -B \right\} = \left\{ \begin{pmatrix} ia & b+ic \\ -b+ic & id \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\} \\ &= \underbrace{\mathbb{R} \cdot \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}}_{g_0} \oplus \underbrace{\mathbb{R} \cdot \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}}_{g_a} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} : z \in \mathbb{C} \right\}}_{g_z} \end{aligned}$$

proof $\text{Ad} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_1 \lambda_1^{-1} i & 0 \\ 0 & \lambda_2 \lambda_2^{-1} i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$

$$\text{Ad} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \text{ similarly}$$

$$\text{Note: } g_0 = \left\{ \begin{pmatrix} ia & 0 \\ 0 & id \end{pmatrix} \right\} = t \text{ hence } T \text{ maximal.}$$

Now find $a \in \mathbb{Z}^2$:

$$\text{Ad} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \lambda_1 \lambda_2^{-1} z \\ -\bar{\lambda}_1 \bar{\lambda}_2^{-1} \bar{z} & 0 \end{pmatrix}$$

$$\Rightarrow g_a \cong \mathbb{C} \cong \mathbb{R}^2 \text{ is the rep } z \mapsto \lambda_1 \lambda_2^{-1} z = e^{2\pi i(x_1 - x_2)} z \\ \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \mapsto z \mapsto \begin{pmatrix} \operatorname{Re} z \\ \operatorname{Im} z \end{pmatrix} \quad = \text{rotation by } 2\pi i(x_1 - x_2)$$

$$\Rightarrow \theta_a(x) = a_1 x_1 + a_2 x_2 = x_1 - x_2 \text{ root}$$

$$\Rightarrow a = (1, -1) \in \mathbb{Z}^2$$

Typically write $g_{x_1 - x_2}$ instead of $g_{(1, -1)}$.

$\Rightarrow U(2) \cong g_0 \oplus g_{x_1 - x_2}$ has one root: $x_1 - x_2$. ■

Rmk Lie algebra approach is: abbreviate $y_1 = 2\pi i x_1, y_2 = 2\pi i x_2$ then:

$$\text{ad} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} = \left[\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}, \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \right] = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} - \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} = \begin{pmatrix} 0 & (y_1 - y_2)z \\ -\bar{(y_1 - y_2)}\bar{z} & 0 \end{pmatrix}$$

CONJUGATES OF MAX T COVER G

Theorem Any $h \in G$ lies in a conjugate of T ($h = g x g^{-1}$ some $x \in T$)

EXAMPLE Any $A \in U(n)$ is diagonalizable, so conjugate to some $(\lambda_1 \dots \lambda_n) \in T$.

Rmk Uses G connected since $gTg^{-1} \subseteq G_0$ cannot reach $h \in G \setminus G_0$.

Proof (Non-examinable)

Trick $f = \phi_h : G/T \rightarrow G/T$, $f(gT) = hgT$ (just a smooth map, G/T manifold)

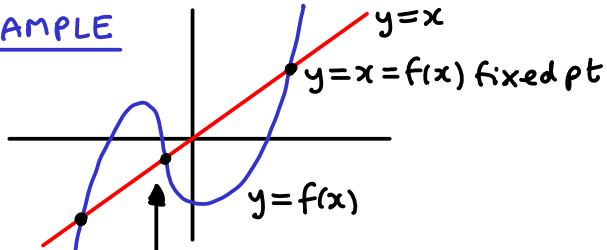
gT fixed point of $f \Leftrightarrow f(gT) = gT \Leftrightarrow hgT = gT \Leftrightarrow g^{-1}hg \in T$ ✓

Remains to show \exists fixed point.

Trick 2 For a smooth map $f: M \rightarrow M$ of a manifold, the number of fixed points (counted with multiplicity) does not change if we continuously deform f .

$f(x) = x$ count x as $+1$ if $\det(Id - D_x f) > 0$ (more complicated)
 -1 " " " < 0 if $\det = 0$: have multiplicity
Lefschetz fixed pt thm

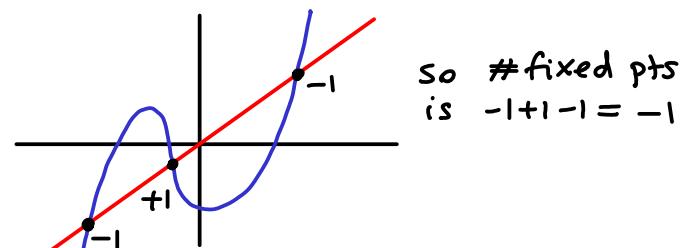
EXAMPLE



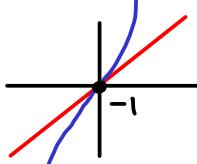
here $x - f(x)$ increases, so

$$\frac{d}{dx}(x - f(x)) = 1 - f'(x) > 0 \text{ so count as } +1$$

Now continuously deform f to



fixed pts is still -1 . ✓



Trick 3 Q.2 Sheet 3 : almost any $T \in T$ generates $\overline{\langle T \rangle} = T$. Pick such a T .
 \hookrightarrow (closure of subgp gen by T)

Since G connected mfd, it is path-connected, so deform f by moving h to T .

Remains to show : $f: G/T \rightarrow G/T$, $f(gT) = \tau gT$ has $\#(\text{fixed pts}) \neq 0$.

Claim $\{\text{fixed points}\} = \{nT : n \in N(T)\}$ ← Normalizer $N(T) = \{g \in G : gTg^{-1} = T\}$

Pf nT fixed $\Leftrightarrow n^{-1}\tau n \in T \Leftrightarrow \overline{\langle n^{-1}\tau n \rangle} = n^{-1}\overline{\langle \tau \rangle}n = n^{-1}Tn \subseteq T$ ■

Rmk $T \subseteq N(T) \subseteq G$ closed subgroup so compact Lie subgp. (equality since $n^{-1}Tn$ max torus)

Claim $N(T)_o = T$

Pf $n \in N(T)_o \Rightarrow T \rightarrow T$, $g \mapsto ngn^{-1}$ is a Lie gp hom depending on a continuous parameter n . But homs $T \rightarrow T$ are parametrized by a discrete parameter in $\mathbb{Z}^n \times \dots \times \mathbb{Z}^n$ (Q.2 Sheet 3). So ngn^{-1} independent of n up to deforming. So move n to 1 $\Rightarrow ngn^{-1} = g \Rightarrow ng = gn$. If $T \not\subseteq N(T)_o \Rightarrow$ could create connected abelian subgp larger than T by taking $\langle T \cup \{\exp(sy) : s \in \mathbb{R}\} \rangle$ for $y \in \text{Lie } N(T)_o \setminus T$ ■

Consequence fixed points are cosets nT of $T = N(T)_0$ in $N(T)$

\Rightarrow finitely many since $N(T)/N(T)_0$ is discrete + compact.

(or directly: $N(T)$ compact
 $N(T)_0 \subseteq N(T)$ open
and cosets cover $N(T)$)

Trick 4 $\det(I - D_{nT} f)$ is independent of n .

Pf Consider $\psi_n^{-1} \circ f \circ \psi_n$ where $\psi_n =$ right-multiplication by n on G/T (NOTE $gTn = gnT$ since $n \in N(T)$)

$$\psi_n^{-1} \circ f \circ \psi_n(gT) = \psi_n^{-1} f(gTn) = \psi_n^{-1}(\tau gTn) = \tau gT \cancel{\psi_n^{-1}} = f(gT)$$

$$\Rightarrow \psi_n^{-1} \circ f \circ \psi_n = f$$

$$\begin{aligned} \Rightarrow \det(I - D_T f) &= \det(I - D_{nT} \psi_n^{-1} \circ D_{nT} f \circ D_T \psi_n) = \det(D\psi_n^{-1} \circ (I - D_{nT} f) D\psi_n) \\ &\quad \text{chain rule} \qquad \qquad \qquad \nwarrow \psi_n(T) = Tn = nT \\ &= \det(I - D_{nT} f) \blacksquare \end{aligned}$$

Remains to calculate sign ($\det(I - D_T f)$).

Trick 5 $A_\tau : G \rightarrow G$, $A_\tau(g) = \tau g \tau^{-1}$

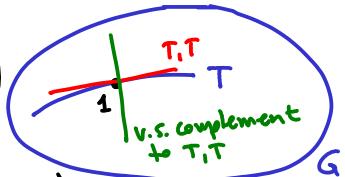
$$\Rightarrow A_\tau : G/T \rightarrow G/T, A_\tau(gT) = \tau gT \tau^{-1} = \tau gT = f(gT)$$

Since tangent space $T_1 G \cong T_1 T \oplus T_{T_1}(G/T)$,

$D_T f = D_1(A_\tau)$ restricted to a vector space complement of $T_1 T \subseteq T_1 G$

$$= D_1 A_\tau \Big|_{\bigoplus \sigma_a} = \text{Ad}(\tau) \Big|_{\bigoplus \sigma_a} = \bigoplus \left(\text{rotation by } 2\pi \theta_a(\tau) \text{ on } \sigma_a \right)$$

(omit $\sigma_0 = t \cong T_1 T$)



$$\begin{aligned} \det(I - D_T f) &= \prod_a \det \begin{pmatrix} 1 - \cos 2\pi \theta_a(\tau) & \sin 2\pi \theta_a(\tau) \\ -\sin 2\pi \theta_a(\tau) & 1 - \cos 2\pi \theta_a(\tau) \end{pmatrix} \\ &= \prod_a 2 \underbrace{(1 - \cos 2\pi \theta_a(\tau))}_{>0 \text{ since } \theta_a(\tau) \neq 0 \pmod{\mathbb{Z}}} \end{aligned}$$

otherwise $\theta_a(\tau) = \theta_a(T) = 0 \pmod{\mathbb{Z}}$
but $\theta_a \neq 0$ for $a \neq 0$.

\Rightarrow multiplicity of all nT is $+1$.

\Rightarrow # fixed points > 0 (if there is no σ_a , then $\sigma = t$, so $G = T$) ■

Corollaries

① All maximal tori are conjugate: T, T' max $\Rightarrow T' = gTg^{-1}$ some g

② Every element $h \in G$ lies in some max tori

③ Decomposition $\sigma \cong \bigoplus \sigma_0 \oplus \bigoplus_a \sigma_a$ is independent of choice of max T .

So roots θ_a do not depend on choice and $\dim(\text{max } T) = \dim(\sigma_0)$ called rank(G).

Pf ① $T' = \langle \tau' \rangle$ some τ' . By Thm, $\tau' = g \tau g^{-1}$ some $\tau \in T$. So $T' \subseteq \langle g \tau g^{-1} \rangle \subseteq gTg^{-1}$ ■

② $h \in gTg^{-1}$ some g , by Thm.

③ $T' = gTg^{-1} \Rightarrow$ Claim $\sigma'_a = \text{Ad}(g) \cdot \sigma_a \cdot \text{Ad}(g)^{-1}$ so $\sigma'_a \cong \sigma_a$ equivalent reps.

Pf $\text{Ad}(g \tau g^{-1}) \cdot \text{Ad}(g) \cdot y \cdot \text{Ad}(g)^{-1} = \text{Ad}(g) \text{Ad}(\tau) y \text{Ad}(g^{-1})$ and $\text{Ad}(x)y \in \sigma'_a$ iff $y \in \sigma_a$ ■

Similarly for σ_0 . Since $\sigma'_a \cong \sigma_a$ also $n'_a = n_a$ so characters $= 2 \cos(2\pi \theta_a(x))$ same

so $\theta'_a = \pm \theta_a$ (and recall we don't distinguish $\theta_a, -\theta_a = \theta_{-a}$). ■

LECTURE 15

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ASSUME: T MAX TORUS
 G COMPACT CONNECTED

WEYL GROUP

$$W(T) = \left\{ \begin{array}{l} A_n|_T : T \rightarrow T \\ x \mapsto nxn^{-1} \end{array} \quad \text{where } n \in N(T) \right\} \subseteq \text{Aut}(T)$$

Recall $N(T) = \text{normalizer} = \{n \in G : nTn^{-1} = T\}$

Rmk Independent (up to isomorphism) of choice of max torus T :
 $N(gTg^{-1}) = gN(T)g^{-1}$ and $W(gTg^{-1}) = Ag \circ W(T) \circ A_{g^{-1}}$

Lemma $W(T) \cong N(T)/T$ via $A_n|_T \mapsto nT$

Pf $N(T) \rightarrow W(T)$ surjective hom so done by 1st iso thm
 $n \mapsto A_n|_T$ if can show kernel = T .

kernel: $n \in N(T)$ with $A_n|_T = \text{id}$ so $nxn^{-1} = x$ all $x \in T$

$\Rightarrow n \in T$ otherwise $\langle T \cup \{n\} \rangle$ larger abelian subgp than T .
 (reproving $Z(T) = T$ see Lecture 14) ■

Rmk By Lecture 14 $T = N(T)_o$ so $N(T)/T = N(T)/N(T)_o$ is discrete and compact, hence finite. So $W(T)$ is a finite group

Cor Characters χ of G are determined by their restrictions $\chi|_T$ and the $\chi|_T$ are invariant under the Weyl group

Pf $\chi(h) = \chi(gxg^{-1}) \stackrel{\text{(recall trace is conjugation invariant)}}{\leftarrow} \chi(x) = \chi|_T(x)$ ✓

(recall thm: any $h \in G$ lies in a conjugate of T , say $h = gxg^{-1}$, $x \in T$)

$$(A_n \cdot \chi|_T)(x) = (\chi|_T \circ A_n^{-1})(x) = \chi|_T(n^{-1}x n) = \chi|_T(x) \quad \blacksquare$$

Theorem $R(G) \rightarrow R(T)^W = \{\text{Weyl invariant virtual characters}\}$
 $\chi \mapsto \chi|_T$ is an isomorphism!

Cor above shows injective hom. Surjectivity is harder (won't prove it). In practice, you don't use surjectivity: first you find reps of G giving all possible characters in $R(T)^W$ then by injectivity you know you have found all reps.

Example Representation theory of $U(n)$

$T = \{\text{diagonal matrices}\} \subseteq U(n)$

Claim 1 $W(T) = S_n$ = symmetric group acting on T by permuting diagonal entries.

Pf $A_n(x) = n x n^{-1}$ does not change the eigenvalues of $x \in T$ (diagonal entries).

Recall (Q.Sheet 3) $T = \langle \overline{x} \rangle$ if $x = \begin{pmatrix} e^{2\pi i x_1} & & \\ & \ddots & \\ & & e^{2\pi i x_n} \end{pmatrix}$ with $1, x_1, \dots, x_n$ lin. indep / \mathbb{Q}

A_n permutes the distinct entries $e^{2\pi i x_j} \rightarrow$

But $A_n(x)$ determines A_n on $T = \langle \overline{x} \rangle$ by continuity

$\Rightarrow A_n = \text{a permutation of the diagonal entries}$

Conversely: all permutations arise because all transpositions arise:

2×2 case: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$ so $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N(T)$

$n \times n$ case: use matrix with $\begin{cases} 1 & \text{on diagonal except in positions } (i,i), (j,j) \\ \text{(transposes } \lambda_i, \lambda_j \text{)} & \begin{cases} 1 & \text{in entries } (i,j), (j,i) \\ 0 & \text{else} \end{cases} \end{cases}$ (diagonal entries of T) ■

claim 2 $R(T)^{W(T)} = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{S_n} = \mathbb{Z}[P_1, \dots, P_n, P_n^{-1}]$ where the P_j are the elementary symmetric polynomials in n variables:

$$P_1 = \sum t_j, \quad P_2 = \sum_{i < j} t_i t_j, \quad \dots, \quad P_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} X_{i_1} X_{i_2} \dots X_{i_k}, \quad \dots, \quad P_n = X_1 X_2 \dots X_n$$

Pf t_j corresponds to rep: $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \in T \longleftrightarrow \lambda_j = e^{2\pi i x_j}$

$\Rightarrow S_n$ acts by permuting the t_1, \dots, t_n .

Fact from algebra: $\mathbb{Z}[t_1, \dots, t_n]^{S_n} = \mathbb{Z}[P_1, \dots, P_n]$ poly ring in elem. sym. polys

If $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{S_n}$ then (large power of P_n) $\cdot f$ will not have negative powers of t_j , so $f \in P_n^{-N} \cdot \mathbb{Z}[P_1, \dots, P_n] \subseteq \mathbb{Z}[P_1, \dots, P_n, P_n^{-1}]$.

Conversely $\mathbb{Z}[P_1, \dots, P_n, P_n^{-1}] \subseteq \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^{S_n}$, hence equality. ■

Claim 3 There exists a rep V_k of $U(n)$ with character $\chi_{V_k} = P_k$

Pf $U(n)$ acts on $V = \mathbb{C}^n$ (standard rep: $\rho(g) = g \in U(n) \subseteq GL(n, \mathbb{C})$) $\Rightarrow U(n)$ acts on $V \otimes V$. On generators: $g \cdot (v_1 \otimes v_2) = g v_1 \otimes g v_2$. But not irrep: has a subrep:

$\Lambda^2 V = \left\{ \text{tensors } \sum v_i \otimes w_i : \text{such that symmetric group } S_2 \text{ acting by permuting factors } v_i, w_i \text{ acts by sign(permulation)} \cdot \text{Id} \right\}$

$$\Rightarrow \Lambda^2 V = \text{span}_{\mathbb{C}} \{ v \otimes w - w \otimes v : v, w \in V \}$$

(transposition (12) acts by $w \otimes v - v \otimes w$ so by $- \text{Id} = \text{sign} \cdot \text{Id}$)

Convenient to abbreviate $v \wedge w = v \otimes w - w \otimes v$

$\Rightarrow \Lambda^2 V = \text{vector space over } \mathbb{C} \text{ with basis } (e_i \wedge e_j)_{1 \leq i < j \leq n}$
with $U(n)$ -action $g \cdot (e_i \wedge e_j) = g e_i \wedge g e_j$

Rmk we stipulate that the symbol \wedge is linear in each entry and antisymmetric: $e_j \wedge e_i = -e_i \wedge e_j$, so $g e_i \wedge g e_j$ makes sense.

alternating product $\Lambda^k V = \{z \in V^{\otimes k} : \sigma \cdot z = \text{sign}(\sigma) \cdot z \text{ for all } \sigma \in S_k\}$

act by permuting tensor factors

\cong vector space over \mathbb{C} with basis the symbols $(e_{i_1} \wedge \dots \wedge e_{i_k})_{1 \leq i_1 < \dots < i_k \leq n}$

$$(v_1 \wedge \dots \wedge v_k = \sum_{\sigma \in S_k} \text{sign}(\sigma) v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)})$$

In particular, $\Lambda^n V = \mathbb{C} \cdot e_1 \wedge e_2 \wedge \dots \wedge e_n$ is 1-dimensional.

Character restricted to T :

$$g = (\lambda_1, \dots, \lambda_n) \in T \Rightarrow g \cdot e_i = \lambda_i e_i \Rightarrow \chi_{V_n} = \sum \lambda_i;$$

$$\text{Also } g \cdot (e_i \wedge e_j) = g e_i \wedge g e_j = \lambda_i e_i \wedge \lambda_j e_j = \lambda_i \lambda_j (e_i \wedge e_j) \Rightarrow \chi_{V_2} = \sum_{i < j} \lambda_i \lambda_j$$

\wedge is multi-linear

$$g \cdot (e_{i_1} \wedge \dots \wedge e_{i_k}) = (\lambda_{i_1}, \dots, \lambda_{i_k}) e_{i_1} \wedge \dots \wedge e_{i_k} \Rightarrow \chi_{\Lambda^n V} = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k} = P_k(\lambda_1, \dots, \lambda_n)$$

$$\text{In particular: } \chi_{V_n}(\lambda_1, \dots, \lambda_n) = \lambda_1 \lambda_2 \dots \lambda_n = \det(g) \text{ (1-dimensional rep)}$$

Claim 4

$$R(U(n)) = \mathbb{C} [\chi_V, \chi_{\Lambda^2 V}, \dots, \underbrace{\chi_{\Lambda^n V}}_{\det}, \underbrace{\chi_{\Lambda^n V}^{-1}}_{(\det)^{-1}}]$$

Notice: we only use that $R(U(n)) \rightarrow R(T)^{W(T)}$ is injective (which we proved in general) and we deduced surjectivity in this example by Claim 3.

Claim 5 The $V, \Lambda^2 V, \dots, \Lambda^n V, \overline{\Lambda^n V}$ are irreps ($\Lambda^n V, \overline{\Lambda^n V}$ obviously since 1-dimensional)

Pf Suppose not: $\Lambda^k V = U_1 \oplus U_2$ sum of subreps.

If we knew that $e_1 \wedge e_2 \wedge \dots \wedge e_k \in U_1$, then in fact $U_1 = \Lambda^k V, U_2 = \{0\}$ because the matrix $g \in U(n)$ with columns $(e_{i_1}, |e_{i_2}| \dots |e_{i_k}| \text{ other basis vectors})$ acts by $g e_m = e_{i_m}$ so $g \cdot (e_1 \wedge \dots \wedge e_k) = e_{i_1} \wedge \dots \wedge e_{i_k}$ which is a basis for $\Lambda^k V$.

Trick Consider action of max tons $T \subseteq U(n) \rightarrow \text{Aut}(\Lambda^k V)$. Then

$\Lambda^k V = \bigoplus V_\alpha$ where $V_\alpha = \{v \in \Lambda^k V : x \cdot v = \chi_\alpha(x) v\}$ are weight spaces ($\alpha \in \mathbb{Z}^n$)

$V_\alpha = \text{sum of 1-dim irreps of } \mathbb{C}$ each with character $\chi_\alpha(x) = e^{2\pi i \langle x, \alpha \rangle}$. Similarly

$U_1 = \bigoplus V'_\alpha, U_2 = \bigoplus V''_\alpha$ with $V'_\alpha, V''_\alpha \subseteq V_\alpha$ (unique decomposition into irreps of T)

But we know $V_\alpha = \mathbb{C} e_{i_1} \wedge \dots \wedge e_{i_k}$ for $\alpha = (1, 1, \dots, 1, 0, \dots, 0)$ (1 in first k entries)
(indeed $V_\alpha = \mathbb{C} e_{i_1} \wedge \dots \wedge e_{i_k}$ where α has 1 in entries i_1, \dots, i_k and 0 elsewhere)

$\Rightarrow V'_\alpha = V_\alpha$ or $V''_\alpha = V_\alpha$ so $e_1 \wedge \dots \wedge e_k \in U_1$ or U_2 (as opposed to being $U_1 + U_2$)

Rmk There is a more systematic approach to finding irreps for G by looking for "highest weight vectors" in terms of a certain ordering of weights $\alpha \in \mathbb{Z}$. For $U(n)$, the lexicographic order works so $(1, 1, \dots, 1, 0, \dots, 0)$ is the highest weight arising in $\Lambda^k V$ and $e_1 \wedge \dots \wedge e_k$ the highest weight vector.

More examples

since $\det = 1$, lose $\det, \overline{\det}$

$$R(SU(n)) = \mathbb{Z} [X_V, X_{\lambda^2 V}, \dots, X_{\lambda^{n-1} V}] \text{ where } V = \mathbb{C}^n \text{ standard}$$

$$R(SO(2n+1)) = \mathbb{Z} [X_V, X_{\lambda^2 V}, \dots, X_{\lambda^n V}], V = W \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{2n+1}, W = \mathbb{R}^{2n+1} \text{ standard}$$

Some more facts

$$1) \quad Cl(G) \xleftarrow{1:1} Cl(T)^W = C(T)^W \quad (\text{since } T \text{ is abelian, } Cl(T) = C(T))$$

$$(f: G \rightarrow \mathbb{C}) \longleftrightarrow (f|_T: T \rightarrow \mathbb{C})$$

$$\begin{cases} f(h) = f(x) \\ \text{if } h = g \cdot x \cdot g^{-1} \end{cases} \longleftrightarrow (f: T \rightarrow \mathbb{C})$$

Rmk need check $f(h)$ well-defined and continuous. The key step to show well-definedness is fact:

$$x_1, x_2 \in T \text{ are conjugate in } G \iff w \cdot x_1 = x_2 \text{ some } w \in W(T)$$

(i.e. they are conjugate in $N(T)$)

Therefore $f(x_1) = f(w \cdot x_1) = f(x_2)$ since f is Weyl-invariant.

Example For $SU(2)$, $T = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\} \cong S^1$. Then $W(T) = S_2$ (just like for $U(2)$)

$$Cl(SU(2)) = \mathbb{C}[e^{2\pi i x}, e^{-2\pi i x}]^{S_2} = \mathbb{C}[2 \cos(2\pi x)] = \text{span}_{\mathbb{C}} \{ \cos(2\pi ax) : a \in \mathbb{Z} \}$$

2) Weyl group action on T permutes the roots (compare Q-Sheet 6)

meaning: $(w \cdot \theta_a)(x) = \theta_a(w^{-1} \cdot x) = \theta_{a'}(x)$ all $x \in T$, some root $\theta_{a'}$.

Pf $Ad(w^{-1} \cdot x) = Ad(n^{-1} x n) = Ad(n)^{-1} Ad(x) Ad(n)$ hence the rep $Ad|_T \circ w^{-1} \cong Ad|_T$ and $Ad(n)^{-1} \cdot V_a \cdot Ad(n)$ is an irred summand so $\cong V_{a'}$. ■

Example $U(n)$ roots are $x_k - x_\ell$ where $\lambda_j = e^{2\pi i x_j}$ are diag entries of T . $W(T) = S_n$ permutes the diagonal entries, so permutes roots.

3) The permutation action on the roots uniquely determines elements of $W(T)$

(So $W(T) \cong$ subgroup of S_m where $m = \# \text{roots of } G$)

Pf G connected Lie grp \Rightarrow Lie grp hom $w = A_n|_T: T \rightarrow T$ is determined by $D_w = D, A_n|_t = Ad(n): t \rightarrow t$ ($t = \text{Lie } T$). Since $A_n|_T = id$ on $\text{Centre}(G) \leq T$ have $D_w = id$ on $\text{Centre}(G) \leq t$. In Lecture 16 will show:

$$\text{dual } \mathfrak{g}^* = \text{span}_{\mathbb{R}}(\text{roots } \theta_a) \oplus \text{Centre}(G)^*$$

hence $(D_w)^*$ determined by action on roots, so result follows. ■

Weyl integration formula (Non-examinable)

Would like to recover $\langle X_1, X_2 \rangle = \int_{g \in G} \overline{X_1(g)} X_2(g)$ from an inner product on T

$$f \in Cl(G) \Rightarrow \int_{g \in G} f(g) = \int_{x \in T} f(x) \delta(x) \text{ where } \delta(x) = \frac{1}{|W(T)|} \prod_{\text{roots}} |e^{\pi i \theta_a(x)} - e^{-\pi i \theta_a(x)}|^2$$

$$\text{Example } G = U(n): \int_0^1 \dots \int_0^1 f(x_1, \dots, x_n) \frac{1}{n!} \prod_{k < l} |e^{2\pi i x_k} - e^{2\pi i x_l}|^2 dx_1 \dots dx_n$$

$$G = SU(2): \int_0^1 f(x) \frac{1}{2} |e^{\pi i (2x)} - e^{-\pi i (2x)}|^2 dx = 2 \int_0^1 f(x) \sin^2(2\pi x) dx$$

LECTURE 16

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ASSUME: T MAX TORUS
 G COMPACT CONNECTED

Weyl-invariant inner product on $t = \text{Lie}(T)$

Recall $G \rightarrow \text{Aut}(V) \Rightarrow \exists G\text{-invariant inner product on } V$
 $\text{Ad}: G \rightarrow \text{Aut}(g) \Rightarrow \quad " \quad " \quad " \quad (\cdot, \cdot) \text{ on } g$

Cor $\exists W(T)\text{-invariant inner product on } t = \text{Lie}(T)$

hence $W(T) \cong$ finite subgroup of $O(n)$ action on $(\mathbb{R}^n, (\cdot, \cdot)_{\text{standard}}) \cong t$

Pf $w = A_n|_T \in W(T)$ acts by $D_w = D, A_n = \text{Ad}(n)$ on $t \subseteq g$ ($n \in N(T)$) ■

Rmk Recall Lie group hom $w = A_n|_T$ is determined by Lie algebra hom $D_w = \text{Ad}(n)$.

Example $G = U(n)$ $(X, Y) = \text{Trace}(\bar{X}^T Y) = -\text{Trace}(XY)$ on $g = \{X: \bar{X}^T = -X\}$
 $t = \left\{ \begin{pmatrix} ix_1 & & \\ & \ddots & \\ & & ix_n \end{pmatrix} : x_i \in \mathbb{R} \right\} \Rightarrow (X, Y) = \sum x_j y_j$ standard inner product on $t \cong \mathbb{R}^n$

Finding the Weyl group combinatorially

$$\begin{aligned} T_\alpha &= \text{Ker}(\text{root } \theta_\alpha: T \rightarrow S' = \mathbb{R}/\mathbb{Z}) \\ t_\alpha &= \text{Lie } T_\alpha = \text{Ker}(\theta_\alpha: t \cong \mathbb{R}^n \rightarrow \mathbb{R}) \\ &= \text{the hyperplane orthogonal to } \alpha \in \mathbb{R}^n \cong t \quad (\text{equation: } \langle x, \alpha \rangle = 0) \end{aligned} \quad \left. \begin{array}{l} \theta_\alpha(x) = \langle x, \alpha \rangle \\ = x_1 \alpha_1 + \dots + x_n \alpha_n \end{array} \right\}$$

Example $U(n)$ for $\theta_\alpha = \theta_{x_k - x_\ell}: T_\alpha = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_k = \lambda_\ell \right\} \quad t_\alpha = \{x \in \mathbb{R}^n : x_k = x_\ell\}$

Rmk (exercise) $\text{Centre}(G) = \bigcap_{\text{roots}} T_\alpha$, $\text{Centre}(g) = \bigcap_{\text{roots}} t_\alpha = \bigcap \text{Ker } \theta_\alpha$

Example $\text{Centre}(U(n)) = \left\{ \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} \right\} = S^1 \cdot \text{Id}$, $\text{Centre}(u(n)) = \left\{ \begin{pmatrix} ix_1 & & \\ & \ddots & \\ & & ix_n \end{pmatrix} \right\} = i\mathbb{R} \cdot \text{Id}$

Fact (tricky) For any root θ_α , there exists $w \neq \text{Id}$ in W fixing each point of T_α

Cor $W(T)$ acting on t contains all reflections in the hyperplanes t_α

Pf $W(T)$ acts by orthogonal matrices $\Rightarrow w$ from Fact must be the reflection in t_α ■

Fact $W(T) \subseteq \text{Aut}(t)$ is generated by the reflections in the hyperplanes t_α

Example $U(n)$ reflection in $t_\alpha = \{x \in \mathbb{R}^n : x_k = x_\ell\}$ swaps x_k, x_ℓ coordinates
 $\Rightarrow W(T) \subseteq \text{Aut}(\mathbb{R}^n)$ is group S_n = permutation matrices which permute coords

2x2 Example $w = \text{conjugation by } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $w \cdot \begin{pmatrix} ix_1 & 0 \\ 0 & ix_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} ix_1 & 0 \\ 0 & ix_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} ix_2 & 0 \\ 0 & ix_1 \end{pmatrix}$

Rmk Generating reflections have formula: $S_\alpha: t \rightarrow t, S_\alpha(x) = x - \frac{2\theta_\alpha(x)}{\langle \theta_\alpha^*, \theta_\alpha^* \rangle} \theta_\alpha^*$

where $\theta_\alpha^* \in t^*$ are duals of $\theta_\alpha \in t^*$ via inner product (\cdot, \cdot) so: $(\theta_\alpha^*, Y) = \theta_\alpha(Y)$.

Rmk $t = \text{centre}(g) \oplus \text{span}_{\mathbb{R}} \theta_\alpha^*$, and the dual $t^* = \text{centre}(g)^* \oplus \text{span}_{\mathbb{R}} (\text{roots } \theta_\alpha)$

Pf Orthog. complement to $\text{span}_{\mathbb{R}} \theta_\alpha^*$ is $\{x \in t : (x, \theta_\alpha^*) = 0\} = \{x \in t : \theta_\alpha(x) = 0\} = \bigcap \text{Ker } \theta_\alpha$ ■

KILLING FORM

Recall (Q. sheet 2) :

$$\langle x, y \rangle = \text{Trace}(\text{ad}(x) \text{ad}(y))$$

recall $\text{ad}(x) \in \text{End}(V)$
 $\text{ad}(x)(y) = [x, y]$

is bilinear map $V \times V \rightarrow \mathbb{F}$
 any Lie algebra V .

Thm For G compact,

$$\begin{aligned} \langle x, x \rangle &\leq 0 \quad \text{all } x \in g \\ &= 0 \quad \text{if and only if } x \in \text{Centre}(g) \end{aligned} \quad \xrightarrow{\text{Q. sheet 4}} \quad \text{Centre}(g) = \text{Ker}(\text{ad}) = \text{Lie}(\text{Centre } G)$$

Pf Using $\text{Ad}(G)$ -invariant metric (\cdot, \cdot) from above, $g \cong \mathbb{R}^m$ and

$$\begin{aligned} \text{Ad}(g) &\in O(m) \quad \text{so } \text{ad}(g) \in \sigma(m) \text{ by naturality:} \quad g \xrightarrow{\text{ad}} \sigma(n) \\ \Rightarrow A &= \text{ad}(x) = \text{skew-symmetric matrix} \quad \downarrow \exp \quad \downarrow \\ \Rightarrow \langle x, x \rangle &= \text{Tr}(AA) = \text{Tr}(-ATA) = \sum - (A^T)_{ij} A_{ji} = \sum - A_{ji}^2 \leq 0 \\ &= 0 \quad \text{iff all } A_{ij} = 0, \text{ that is } A = \text{ad}(x) = 0. \quad \blacksquare \end{aligned}$$

Cor For G compact, $g = \text{Centre}(g) \oplus g'$ where $g' \subseteq g$ is an ideal on which the Killing form is negative definite. $\begin{array}{l} \text{ideal } W \subseteq V: \bullet \text{vector subspace } W \subseteq V \\ (\text{Q. sheet 4}) \quad \bullet [v, w] \in W \text{ for all } v \in V, w \in W \end{array}$

Pf $g' = \text{Centre}(g)^\perp \leftarrow$ using Ad-invariant metric (\cdot, \cdot)

$\text{Ad}(g)$ acts by orthogonal matrices so sends $g' \rightarrow g'$, so $\text{ad} = D, \text{Ad} : g \rightarrow \text{End}(g')$

$\Rightarrow \text{ad}(x)(y) = [x, y] \in g'$ for $x \in g, y \in g'$ so g' is an ideal.

By above Thm, $\langle y, y \rangle < 0$ for $y \neq 0 \in g'$ since $y \notin \text{Centre}(g)$ \blacksquare

CLASSIFICATION OF COMPACT LIE GROUPS

NON-EXAMINABLE

Recall (Q. Sheet 4)

Lie algebra V called simple if V not abelian and only ideals are $0, V$
 V semi-simple if $V = \bigoplus$ simple Lie algebras.

Q. Sheet 4 : for connected Lie group G ,

g simple $\Leftrightarrow G$ simple (meaning: not abelian and $\{1\}$ is the only non-trivial connected normal Lie subgp)

Fact Killing form on V is non-degenerate $\Rightarrow V$ semi-simple

Cor G compact $\Rightarrow g = (\text{abelian summand}) \oplus (\text{semi-simple summand})$

Rmk • So don't have nasty summands ("solvable summands") in Lie algebra.
 • Lie algebra theory can classify all simple Lie algebras!

Consequence: can classify compact Lie groups!

G compact connected

FACT 1 If G simply-connected and of simple, then G can be:

$SU(n)$

$Spin(n) = \text{universal cover of } SO(n)$

$Sp(n) = \text{symplectic group}$ (Lecture 2)

$G_2, F_4, E_6, E_7, E_8 \leftarrow \text{exceptional Lie groups.}$

(Example
 $SU(2) = Spin(3)$
= univ. cover of $SO(3)$)

Rmk If omit assumption of simple, then G is a product of above gps

FACT 2 G has a finite cover $G' \xrightarrow{\pi} G$ (meaning $\ker \pi$ finite)

with $G' \cong \text{torus} \times \text{finite product of groups from above list.}$

$\cong \text{finite product of copies of } S^1 \text{ and groups from above list.}$

Rmk By Q1 of Q.sheet 5, $G \cong G'/\Gamma$ where $\Gamma \subseteq \text{Centre}(G')$ is a finite group (since discrete and compact).

The wikipedia page "List of simple Lie groups" has lots of info and examples about this classification.

———— END OF COURSE ———