LECTURE 1: ADVERTISEMENT LECTURE.

PART III, MORSE HOMOLOGY, 2011

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What are nice functions?

We will consider the following setup:

M = closed[1] (smooth) m-dimensional manifold,
$f : M \to \mathbb{R}$ a smooth function.

Locally, $f : \mathbb{R}^m \to \mathbb{R}$ near $p = 0$:

$$\begin{aligned} f(x) &= f(0) + \sum_i \frac{\partial f}{\partial x_i}(0) \cdot x_i + \frac{1}{2!} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} + \cdots \\ &= f(0) + df_0 \cdot x + \frac{1}{2} x^T \cdot \mathrm{Hess}_0(f) \cdot x + \cdots \qquad \text{(matrix notation)} \end{aligned}$$

Hope:

- Want few $p \in M$ with

  $df_p = 0 \qquad \longleftarrow p$ **critical point** (e.g. max,min)

  Note: these are m conditions, so we hope

  (i) finite $\mathrm{Crit}(f) = \{\text{critical points of } f\}$

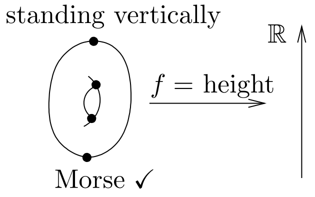
- At critical p, we want a good next order term:

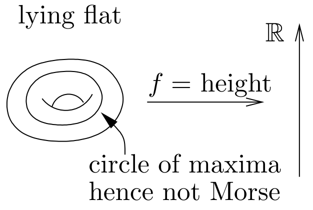
  (ii) $\det \mathrm{Hess}_p(f) \neq 0 \qquad \longleftarrow p$ **nondegenerate**

Fact. *(ii)* $\Rightarrow$ *(i)*

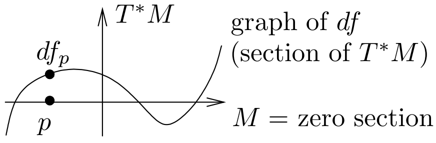
Def. $\boxed{f : M \to \mathbb{R} \textit{ is Morse if all critical points are nondegenerate}}$

Example. M = *torus:*





Modern perspective

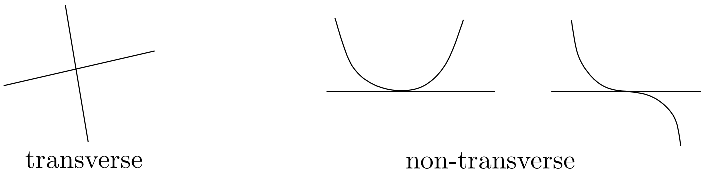


$\boxed{f \text{ is Morse} \Leftrightarrow \begin{array}{l}\text{the section } df \text{ is transverse} \\ \text{to the zero section of } T^*M\end{array}}$

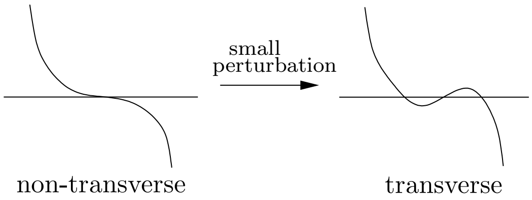
Date: May 1, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1] closed = compact and no boundary.

Idea of what "transverse" means:



Transverse objects are the "generic" ones in geometry:



$$\Rightarrow \boxed{\text{almost all functions are Morse}}$$

Fact. *All manifolds arise as submanifolds of some $\mathbb{R}^k$. We'll prove that: almost any "height function"[2] is Morse on $M \subset \mathbb{R}^k$.*

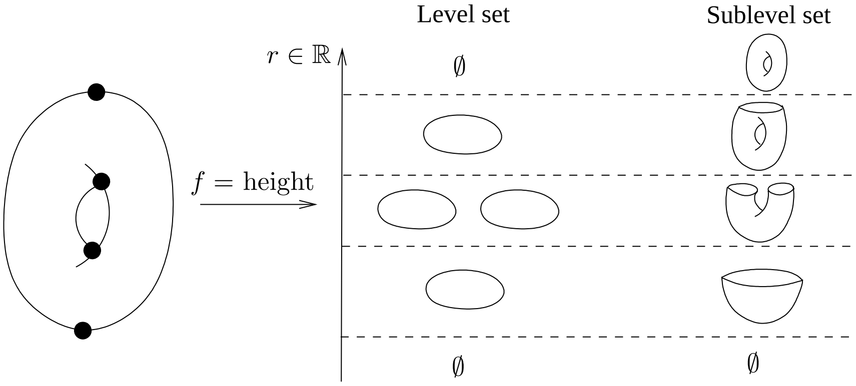
$$\Rightarrow \boxed{\text{it is easy to find Morse functions}}$$

How do you relate Morse f to the topology?

Two natural geometrical objects to look at, given a function f:

level sets $f = r$
sublevel sets $f \leq r$

Consider the torus standing vertically:



Observe that the topology changes when you cross the **critical values** $f(p)$, where $df_p = 0$. Otherwise the topology does not change:

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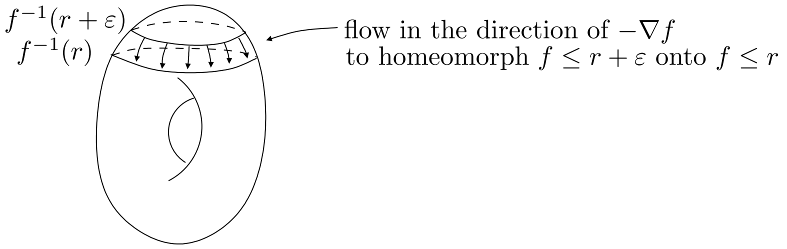
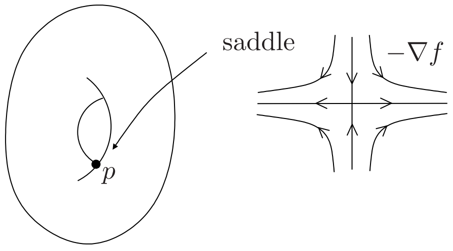
[2] height function = linear functional $\mathbb{R}^k \to \mathbb{R}$.



We will prove: passing through a critical point changes the topology by attaching a k-cell, where

$$k = \#\text{negative eigenvalues of } \mathrm{Hess}_p f \quad \longleftarrow \textbf{index } |p|$$

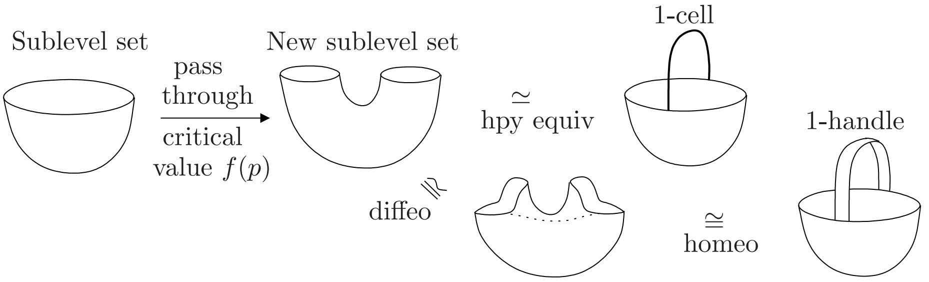
Example: $M =$ torus, $f =$ height



locally $f(x,y) = -x^2 + y^2$

$$\mathrm{Hess}_0 f = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

so $|p| = 1$



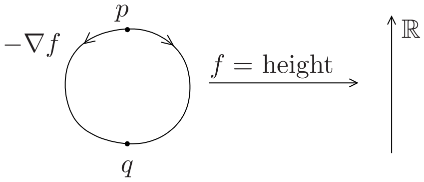
$\Rightarrow$ $\boxed{\text{Can reconstruct the mfd up to hpy equivalence}}$

Hwk 8: f Morse with 2 critical points $\Rightarrow$ $M \cong S^m$ homeomorphic.
Warning. Not diffeomorphic, $\exists$ "exotic" S^7 homeo but not diffeo to the usual $S^7 \subset \mathbb{R}^8$ (proved by Milnor, using the result of Hwk 8).

Is it easy to recover the homology from f?

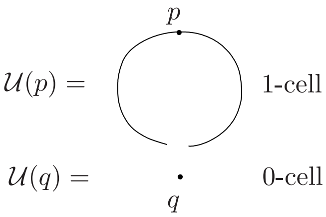
Classical approach (Morse $\sim$ 1930, Thom, Smale, Milnor $\sim$ 1960, . . .)
Pick a self-indexing Morse function (meaning index$(p) = f(p)$).
$\Rightarrow$ the above cell-attachments define a CW structure on M.
$\Rightarrow$ recover cellular homology of M

Example: $M = S^1$



Consider the unstable cells

$$x \mapsto \mathcal{U}(x) = \{\text{points flowing down from the critical point } x\}$$



The cellualar boundary is:[3]

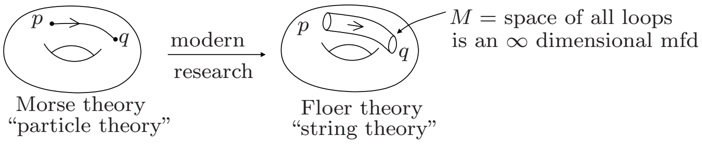
$$\partial\mathcal{U}(p) = \mathcal{U}(q) - \mathcal{U}(q) = 0$$

so $H_*^{\text{cell}}(S^1)$ is generated by cells $\mathcal{U}(q)$,$\mathcal{U}(p)$ in degrees $*$ =0,1, as expected.

Why is the classical approach bad?

If M is ∞-dimensional, then $\mathcal{U}(p)$ is usually ∞-dimensional, hence not a cell. Also, the "flow" is often not defined, so $\mathcal{U}(p)$ is not even well-defined.

You may ask: who cares about ∞-dimensional manifolds? Actually, these nowadays arise quite naturally in geometry. For example:

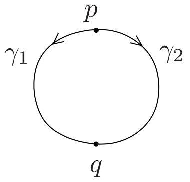


Modern approach (Witten, Floer, ... ∼ 1980)

Consider the **moduli space**[4]

$$\mathcal{M}(p,q) = \{-\nabla f \text{ flowlines from } p \text{ to } q\}/\text{reparametrization}$$

Example: $M = S^1$



$$\mathcal{M}(p,q) = \{\gamma_1, \gamma_2\}$$

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[3] Note that orientation signs are a subtle issue: if we got the sign wrong, then suddenly $\partial\mathcal{U}(p)$ would no longer be zero. To avoid such technical subtleties, we will work over $\mathbb{Z}/2$ in this course.

[4] these flowlines are called *instantons* or *tunneling paths* in physics.

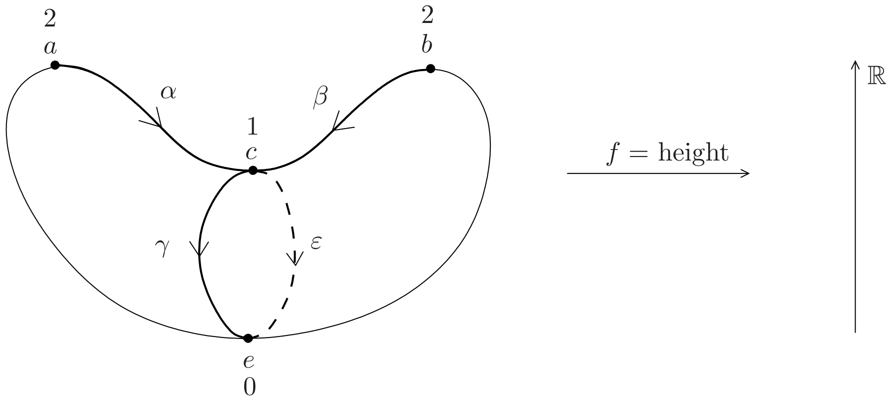
Define a chain complex, called **Morse complex**,

$$MC_*(f) = \mathbb{Z}_2 \cdot p \oplus \mathbb{Z}_2 \cdot q \oplus \cdots$$

where $p, q, \ldots$ are the critical points, and we work over $\mathbb{Z}_2 = \mathbb{Z}/2$ to avoid signs.

$$\begin{aligned} &d : MC_k \to MC_{k-1} \\ &dp = \textstyle\sum_q (\#\text{elements in } \mathcal{M}(p,q)) \cdot q \end{aligned}$$

Example: $M =$ hot-dog ($\cong S^2$)



Then:

$$\begin{array}{rcl@{\qquad}l} MC_2 & = & \mathbb{Z}_2 \cdot a \oplus \mathbb{Z}_2 \cdot b & da = \#\{\alpha\} \cdot c = c,\ db = \#\{\beta\} \cdot c = c \\ MC_1 & = & \mathbb{Z}_2 \cdot c & dc = \#\{\gamma, \varepsilon\} = 0 \ (\text{mod } 2) \\ MC_0 & = & \mathbb{Z}_2 \cdot e & de = 0 \end{array}$$

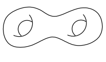
$$\begin{array}{lcccccccc} \textbf{Morse homology} = & \dfrac{\ker \partial}{\operatorname{im} \partial} & = & MH_*(f) & = & \mathbb{Z}_2 \cdot e & \oplus & \mathbb{Z}_2(a-b) \\ & & & * & = & 0 & & 1 \end{array}$$

Observe this is the same as $H_*(S^2)$ (over $\mathbb{Z}/2$).

$$\boxed{\textbf{Theorem. } MH_*(f) \cong H_*(M)}$$

Cor.

$$\begin{array}{rcl} \#(\textit{critical points of a Morse function}) & = & \#(\textit{generators of } MC_*(f)) \\ & \geq & \#(\textit{generators of } MH_*(f)) \\ & = & \sum \dim H_i(M) \quad (\textit{Betti numbers}) \end{array}$$

Example. A generic f :  $\to \mathbb{R}$ has $\geq 2 + 2 \cdot \text{genus} = 6$ critical points.[5]

Geometry is functional analysis

We made two tacit assumptions when defining MC_*, MH_*:

(1) need $\#\mathcal{M}(p,q)$ finite for $|q| = |p| - 1$. Rephrasing:

$$\boxed{\mathcal{M}(p,q) \text{ is a compact 0-dimensional manifold}}$$

(2) need $d^2 = d \circ d = 0$ to define homology.

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[5] *Non-examinable:* Algebraic topology tells you $\geq |\chi(M)| = |\sum(-1)^i \dim MC_i(f)|$, via the intersection number: $\text{graph}(df) \cdot 0_{T^*M} = -\chi(M)$. So for a torus it just predicts ≥ 0.

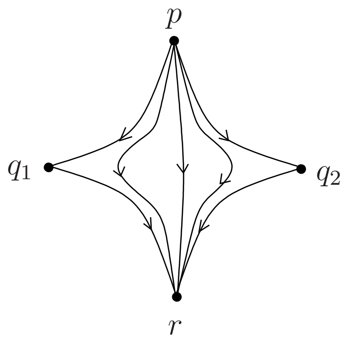
Idea of proof of (2):

$$\begin{aligned} d^2p &= d(\textstyle\sum_q \#\mathcal{M}(p,q)\cdot q) \\ &= \textstyle\sum_{q,r} \#\mathcal{M}(p,q)\cdot \#\mathcal{M}(q,r)\cdot r \end{aligned}$$

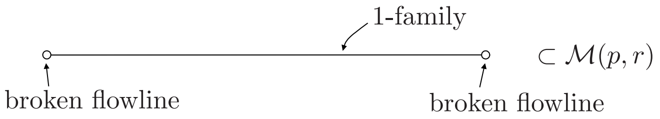
Now $\#\mathcal{M}(p,q)\cdot\#\mathcal{M}(q,r)$ counts "broken" flowlines from p to q to r. Hence:

$$d^2 = 0 \Leftrightarrow \text{once-broken flowlines arise in pairs}$$

Hope: $\exists$ a 1-family of flowlines joining two broken flowlines:



View the flowlines as points in the moduli space, then:



Hope:

- $\mathcal{M}(p,r)$ is a non-compact 1-mfd
- $\exists$ natural way of making it compact:

$$\overline{\mathcal{M}}(p,r) = \mathcal{M}(p,r) \cup \text{``}\partial\mathcal{M}(p,r)\text{''} \quad (\partial\mathcal{M}(p,r) = \{\text{broken flowlines}\})$$

$\Rightarrow \overline{\mathcal{M}}(p,r)$ compact 1-mfd
$\Rightarrow \overline{\mathcal{M}}(p,r) =$ disjoint union of circles and compact intervals
$\Rightarrow \partial\overline{\mathcal{M}}(p,r) =$ even number of points, so $= 0 \bmod 2$
$\Rightarrow d^2p = 0$
$\Rightarrow d^2 = 0 \quad \square$

Idea of proof of (1): Functional Analysis

(3) **transversality problem:** $\mathcal{M}(p,q)$ are smooth manifolds for a "generic" metric g (which defines ∇f by $g(\nabla f,\cdot) = df$), and

$$\dim\mathcal{M}(p,q) = |p| - |q| - 1.$$

(4) **compactness problem:** $\mathcal{M}(p,q)$ can be compactified by broken flowlines.

Most modern homology theories involve these two problems

$$\Rightarrow \boxed{\text{Morse homology is a perfect playground!}}$$

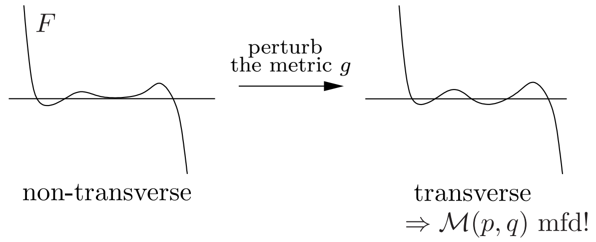
Idea to solve (3): consider the "Banach" vector bundle

$$\begin{tikzcd} \{\text{smooth vector fields along } u\} \arrow[d] \\ \{\text{smooth paths } u : \mathbb{R} \to M, u(s) \to p, q \text{ as } s \to -\infty, +\infty\} \arrow[u, bend right, "\text{section } F = \partial_s u - \nabla f(u)"'] \end{tikzcd}$$

Observe:

$$\boxed{F = 0 \Leftrightarrow \partial_s u = -\nabla f \Leftrightarrow u \in \mathcal{M}(p,q)}$$

$$\Rightarrow \boxed{\mathcal{M}(p,q) = \text{intersection of a section of a Banach vector bundle with 0 section}}$$



Geometry is algebra

Define the Morse cohomology $MH^*(f) \cong H^*(M)$ using

$$MC^* = \mathbb{Z}_2 \cdot p \oplus \mathbb{Z}_2 \cdot q \oplus \cdots$$
$$\delta : MC^k \to MC^{k+1}$$
$$\delta p = \sum_q \#\mathcal{M}(q,p) \cdot q \qquad (\text{where } |q| = |p| + 1)$$

Then *Poincaré duality* $H_*(M) \cong H^{m-*}(M)$ (over $\mathbb{Z}_2$)[6] is just the symmetry:

$$\begin{array}{rcl} MH_*(f) & \cong & MH^{m-*}(-f) \\ p & \mapsto & p \\ \mathcal{M}(p,q;f) & \cong & \mathcal{M}(q,p;-f) \\ -\nabla f \text{ flowline } u(s) & \mapsto & \nabla f \text{ flowline } u(-s) \end{array}$$

The switch in grading is because flipping the sign of f flips the sign of the Hessian.

Poincaré duality is just reversal of flowlines in Morse theory! If you use a height function, Poincaré duality is the intuitive idea "look at the manifold upside down!".

The *Künneth isomorphism* $H_*(M_1 \times M_2) \cong H_*(M_1) \otimes H_*(M_2)$ can be proved quite simply now by the observation: Morse functions $f_1 : M_1 \to \mathbb{R}$, $f_2 : M_2 \to \mathbb{R}$ give naturally rise to the Morse function $f_1 + f_2 : M_1 \times M_2 \to \mathbb{R}$, and the flowlines are just the combined flowline for f_1, f_2 on the respective factors of $M_1 \times M_2$.

The *cup product* $H^a(M) \otimes H^b(M) \to H^{a+b}(M)$ can also be described Morse theoretically: you count flows along a Y-shaped Feynman graph, flowing by a Morse function along each of the three edges of the graph and you require that the flow converges to the inputs p, q at the top, and to r at the bottom. This solution then contributes $p \cdot q = r + \cdots$ to the product.

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[6] this also works over $\mathbb{Z}$, but then one needs to assume M is orientable, which is secretly hidden in the orientation signs that define MC_*, MC^*.

LECTURE 2.

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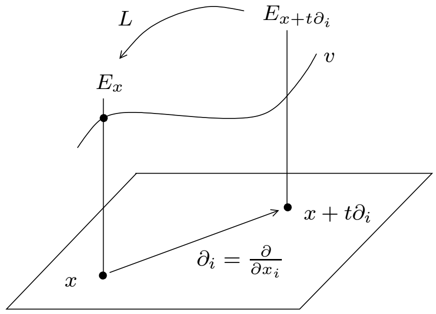
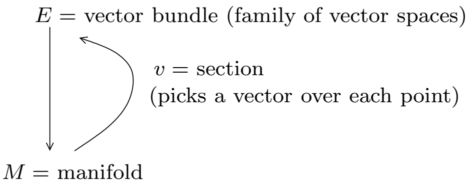
0. REVIEW OF DIFFERENTIAL GEOMETRY

0.1. **Connections: motivation.** How do you differentiate on vector bundles?





In very loose notation, we want

$$\nabla_{\partial_i} v = \text{``}\lim_{t\to 0} \frac{[L \cdot v(x+t\partial_i)] - v(x)}{t}\text{''}$$

where we need a linear identification $L : E_{x+t\partial_i} \to E_x$. Expanding

$$L = I + tA + \text{order } t^2,$$

since at $t = 0$ we definitely want L = identity. Expanding v:

$$v(x + t\partial_i) = v(x) + t\partial_i v + \text{order } t^2.$$

$$\begin{aligned} \text{So we want} \quad \nabla_{\partial_i} v &= \lim_{t\to 0} \frac{\cancel{v}(x) + t\partial_i v + tAv(x) - \cancel{v}(x) + \text{order } t^2}{t} \\ &= \partial_i v + Av \end{aligned}$$

0.2. **Connections.**

Notation.[1] $C^\infty(M)$ = smooth functions $M \to \mathbb{R}$
$C^\infty(E)$ = smooth sections $M \to E$ of the vector bundle $E \to M$.

Def. *A* connection ∇ *is a bilinear map*

$$\begin{aligned} \nabla : \quad & C^\infty(TM) \otimes C^\infty(E) \to C^\infty(E) \\ & (X, v) \mapsto \nabla_X v \end{aligned}$$

1. $C^\infty(M)$*-linear in* X*:* $\nabla_{fX} v = f \nabla_X v$
2. $C^\infty(M)$*-Leibniz in* v*:* $\nabla_X fv = (X \cdot f)v + f\nabla_X v$

Date: May 1, 2011,

[1] The space of sections is also written $\Gamma(E)$, but we will need to keep track of the topology.

Rmk. *X plays the same role as ∂_i, and $X \cdot f$ is differentiation of f just like $\partial_i f$ would be. Check that $X \cdot f = df(X)$, where the 1-form df "eats" the vector X.*

Equivalent Definition[2] (by defining $\nabla v \cdot X = \nabla_X v$),

$$\nabla : C^\infty(E) \to C^\infty(T^*M) \otimes C^\infty(E)$$

such that it is $C^\infty(M)$-Leibniz: $\nabla(fv) = df \otimes v + f\nabla v$.

Locally

$C^\infty(E)$ has basis $s_1, \ldots, s_r$ $\quad (E|_U \cong U \times \mathbb{R}^r)$

$v = \sum v^j s_j$ $\quad (v^j \in C^\infty(M))$

$\nabla v = \sum dv^j \otimes s_j + v^j \nabla s_j$

$$\nabla \begin{pmatrix} v^1 \\ \vdots \\ v^r \end{pmatrix} = d \begin{pmatrix} v^1 \\ \vdots \\ v^r \end{pmatrix} + A \begin{pmatrix} v^1 \\ \vdots \\ v^r \end{pmatrix} \quad (A = \text{"matrix of 1-forms" with } \nabla s_j = \textstyle\sum_k A_j^k s_k)$$

$\boxed{\nabla = d + A}$ locally, as expected!

0.3. **Pull-back bundle.**

$$\begin{tikzcd} u^*E \arrow[r] \arrow[d] & E \arrow[d] \\ {[0,1]} \arrow[r, "u"] & M \end{tikzcd}$$

For $u : [0,1] \to M$ smooth, define $(u^*E)_s = E_{u(s)}$. We will define $u^*\nabla : C^\infty(T[0,1]) \otimes C^\infty(u^*E) \to C^\infty(u^*E)$. Let s be the coordinate on $[0,1]$, v an E-field along u. We abbreviate $(u^*\nabla)(\partial_s \otimes v)$ by $\nabla_s v = \frac{\nabla}{\partial s} v$

Locally $\boxed{\nabla_s v = \frac{\partial v}{\partial s} + A(\partial_s u) \cdot v}$ (1-forms in A eat $\partial_s u$, $\cdot$ is matrix multiplication)

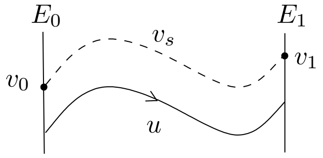
Rmk. *Some abusively write $\nabla_{\partial_s u} v$ because if u is an embedding,*[3] *then any $\widetilde{v} \in C^\infty(E)$ with $\widetilde{v}(u(s)) = v(s)$ satisfies*[4]

$$(\nabla_s v)_s = (\nabla_{\partial_s u} \widetilde{v})_{u(s)} = d\widetilde{v}(\partial_s u) + A(\partial_s u) \cdot \widetilde{v}$$

0.4. **Parallel transport along a path.** For smooth $u : [0,1] \to M$ define

$$\boxed{P_u : E_{u(0)} \to E_{u(1)}}$$

For $v_0 \in E_{u(0)}$ solve the linear ODE $\begin{cases} \nabla_s v = 0 \\ v(0) = v_0 \end{cases}$ in the unknown $v \in C^\infty(u^*E)$. Then $P_u(v_0) = v(1)$.



- ODE theory $\Rightarrow$ $\exists$ unique solution, depending smoothly on u $\Rightarrow$ P_u smooth.
- Linear ODE $\Rightarrow$ P_u linear.

$\Rightarrow$**Hwk 2.** can make Motivation 0.1 rigorous: for a path u with $u(0) = x, u'(0) = X$,

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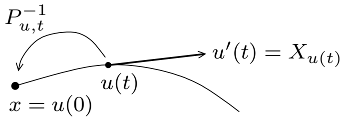
[2] $C^\infty(T^*M) = \Omega^1(M) =$ 1-forms on M, and $C^\infty(T^*M) \otimes C^\infty(E) = \Omega^1(M,E) =$ E-valued 1-forms on M.

[3] $f : M \to N$ is an *embedding* if f is an immersion with $f : M \to f(M)$ a homeomorphism. *Immersion* = the differential df is injective at each point. *Homeomorphism* = continuous bijection having a *continuous* inverse. Often require *proper* = preimages of compact sets are compact (if M, N are compact manifolds this is superfluous, and in this case embedding = injective immersion).

[4] $d\widetilde{v}(\partial_s u) = \frac{\partial}{\partial s}\widetilde{v}(u(s)) = \frac{\partial}{\partial s} v(s)$: so you only need to know how $\widetilde{v}$ varies along $u(s)$.

$$\begin{aligned}(\nabla_X v)_x &= \lim_{t\to 0} \frac{P_{u,t}^{-1}\cdot v(u(t)) - v(x)}{t} \\ &= \frac{\partial}{\partial t}\Big|_{t=0} P_{u,t}^{-1}\cdot v \quad \in E_x\end{aligned}$$



0.5. **Levi-Civita connection.**

$E = TM$, g Riemannian metric[5] $\Rightarrow \exists$ unique ∇ such that ∇ is

(1) symmetric:[6] $\nabla_X Y - \nabla_Y X = [X, Y]$
(2) g-compatible: $X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

Remarks.

Locally: $s_j = \partial_j = \frac{\partial}{\partial x_j}$, $\quad \nabla \partial_j = \sum A_j^k \otimes \partial_k$, $\quad \nabla_{\partial_i}\partial_j = \sum \Gamma_{ij}^k \partial_k$
(Γ_{ij}^k = Christoffel symbols, locally $C^\infty(M)$: $A_j^k = \sum \Gamma_{ij}^k dx^i$)
(1) $\Leftrightarrow$ locally $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i \Leftrightarrow \Gamma_{ij}^k = \Gamma_{ji}^k$.
(1) + (2) $\Leftrightarrow$ an equation[7] for Γ_{ij}^k in terms of $g^{-1}, \partial g$ holds.

Lemma. *P_u is an isometry for Levi-Civita ∇*

Proof. $\partial_s |v|^2 = \partial_s g(v, v) = g(\nabla_s v, v) + g(v, \nabla_s v) = 0$ (since $\nabla_s v = 0$). $\square$

0.6. **Geodesics.**

Consider $E = TM$, ∇ = Levi-Civita.

Def. $u : [0, 1] \to M$ is a *geodesic* $\Leftrightarrow$ $\partial_s u$ is parallel along u
$\Leftrightarrow$ $\nabla_{\partial_s u}\partial_s u = 0$
$\Leftrightarrow$ (nonlinear) 2^{nd} order ODE[8] in u.

Cor. *Geodesics have constant speed $|\partial_s u|$. Length$(u) = \int_0^1 |\partial_s u|\, ds = |\partial_s u|$.*

Proof. P_u is an isometry. $\square$

The space of sections is also written $\Gamma(E)$, but we will need to keep track of the topology. **Fact.** Geodesics minimize length *locally.*[9] So they can be used to measure distances between closeby points.

0.7. **Exponential map.**

ODE theory $\Rightarrow$ geodesics exist for small time
ODE theory $\Rightarrow$ geodesic is uniquely determined by initial conditions
$\Rightarrow$ call $\exp_p(s, v) = u(s)$ geod with $u(0) = p \in M$, $u'(0) = v \in T_pM$
Easy exercise. $\exp_p(st, v) = \exp_p(s, tv)$
$\Rightarrow$ can write $\exp_p(s \cdot v)$.

Denote $D_\varepsilon TM = \{v \in TM : |v| \leq \varepsilon\}$, called a *disc bundle.*

Cor. *For a closed manifold M,*

$$\exp : D_\varepsilon TM \overset{smooth}{\longrightarrow} M, \ (p, v) \mapsto \exp_p v$$

---

[5] $g \in C^\infty(\mathrm{Sym}^2(\mathrm{T^*M}))$ fibrewise positive definite (= g is an inner product in each fibre, varying smoothly over M). $\mathrm{Sym}^2(T^*M)$ = bundle of bilinear forms on TM. The norm is $|v| = \sqrt{g(v, v)}$.

[6] locally $X = \sum X^i \partial_i, Y = \sum Y^i \partial_i$ (X^i, Y^i functions in x), $[X, Y] = \sum (X^i \partial_i Y^j - Y^i \partial_i X^j)\, \partial_j$.

[7] Non-examinable: $\Gamma_{ij}^\ell = \frac{1}{2}\sum g^{\ell k}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})$, where $g^{\ell k}$ = matrix entries of g^{-1}.

[8] Non-examinable: $\partial_s^2 u^k + \sum \Gamma_{ij}^k \cdot \partial_s u^i \cdot \partial_s u^j = 0$.

[9] great circles are geodesics on the sphere, but they don't minimize length after half a circle.

is defined[10] *and distance preserving*[11] *for small* $\varepsilon > 0$.

Proof. Take $\varepsilon =$ smallest t such that $\exp_p(t, v)$ is defined for all $|v| = 1$. Smoothness follows from ODE theory: smooth dependence of solutions on initial conditions. □

Cor. *For* M *compact,* $\exists \varepsilon > 0$ *such that any two points in* M *at distance* $< \varepsilon$ *are connected by a unique*[12] *geodesic. So any two sufficiently close continuous paths are homotopic by following geodesic arcs.*

0.8. **Differential.** For a (smooth) map of mfds $f : M^m \to N^n$, the derivative map

$$df(p) = d_p f : T_p M \to T_{f(p)} N$$

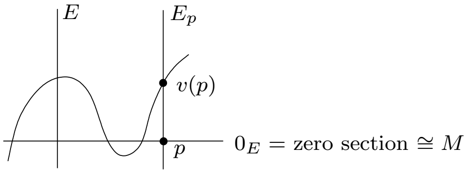
is the best linear approximation of f. Locally $f : \mathbb{R}^m \to \mathbb{R}^n$, $df = \left(\frac{\partial f_i}{\partial x_j}\right)$ (matrix).

Trick. Any vector v at p gives rise to a smooth curve c defined near $c(0) = p$, with $c'(0) = v$ (and vice-versa any c defines a vector $c'(0) = v$ at p). Then

$$\boxed{d_p f \cdot v = \left.\frac{\partial}{\partial s}\right|_{s=0} f(c(s))}$$

0.9. **Vertical differential.**

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \arrow[u, bend left, "\text{section } v\ (\pi v = \mathrm{id})"] \end{tikzcd}$$



A section $v : M \to E$ has derivative $dv : TM \to TE$. This is ugly because TE has double the dimension of E!

Key Idea:

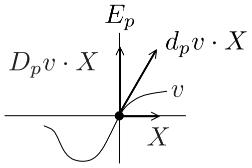
$$T_e^{\text{vertical}} E = \ker d\pi_e \cong E$$

since it is a vector space (analogue of $T_x \mathbb{R}^m \cong \mathbb{R}^m$).

Def. *For* ∇ *on* E*, there exists*[13] *a* vertical differential (*or* linearization)*,*

$$D_p : T_p M \to E_p$$
$$D_p v \cdot X = (\nabla_X v)_p$$

Lemma. *$D_p v$ is independent of ∇ if $v(p) = 0$, indeed it is the vertical projection composed with $d_p v$:*



$$\boxed{\begin{aligned} D_p v \cdot X &= d_p v^{loc} \cdot X \\ &= d_p v \cdot X - X \\ &= (\textit{projection to } E_p) \circ d_p v \cdot X \end{aligned}}$$

---

[10] $\exp : TM \to M$ is *defined* on all of TM (for closed M): patch local solutions of the ODE. But not necessarily on non-closed M: consider $\mathbb{C}\backslash$point and any straight line intersecting 0.

[11] meaning $\mathrm{dist}(p, \exp_p v) = |v|$, using the above Fact.

[12] uniqueness will be explained in Example 1.1.

[13] One could write $D_p = \nabla$. The emphasis is that the definition depends on the *vector* X at p, rather than a *vector field* X defined near p. Also, D_p reminds you about the Lemma.

Proof/Explanation.
Locally: $E|_U \cong U \times E_p$, $\quad T_{(p,v)}E \cong T_pU \times T_vE_p \equiv T_pU \times E_p$, write

$$\boxed{v(x) = (x, v^{\mathrm{loc}}(x)) \in U \times E_p}$$

$$\Rightarrow d_pv \cdot X = (X, d_pv^{\mathrm{loc}} \cdot X)$$

$$\Rightarrow (\nabla_X v)_p = d_pv^{\mathrm{loc}} \cdot X + A(X) \cdot v^{\mathrm{loc}}(p)$$

$$= d_pv^{\mathrm{loc}} \cdot X \qquad\qquad \text{(since } v^{\mathrm{loc}}(p) = 0)$$

In particular,

$$T_pM \equiv T_p0_E \subset TE,$$

so $d_pv \cdot X - X$ makes sense and $\equiv d_pv^{\mathrm{loc}} \cdot X$.

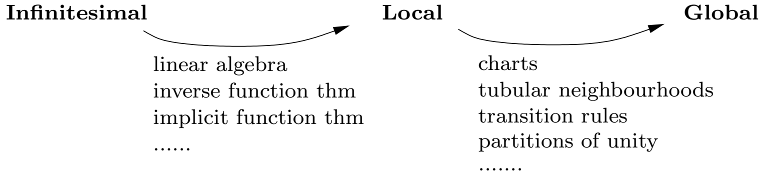
Also

$$T_{(p,0)}E \equiv T_pM \oplus T^{\mathrm{vertical}}_{(p,0)}E \equiv T_pM \oplus E_p$$

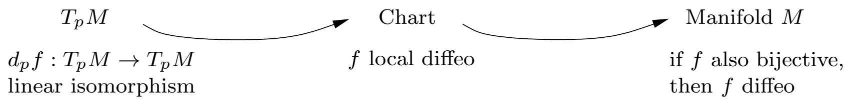
independently of coordinates, so vertical projection is defined.[14] □

1. DIFFERENTIAL TOPOLOGY

1.0. **Motivation.** The aim is to relate the infinitesimal, the local and the global:



Example:



1.1. Diffeomorphisms.

Def. *A smooth map $f : M^m \to N^n$ is a* diffeo *if*

$$\begin{cases} f \text{ is bijective} \\ f^{-1} \text{ is smooth} \end{cases}$$

f is a local *diffeo if $\exists$ open $p \in U \subset M$ such that*

$$\begin{cases} f(U) \text{ is open} \\ f|_U : U \to f(U) \text{ is a diffeo} \end{cases}$$

Rmk. *This forces $m = n$ since*

$$f \text{ local diffeo at } p \Rightarrow d_pf \text{ isomorphism with inverse } d_{f(p)}f^{-1}$$

Thm (Inverse Function Theorem)**.**

$$d_pf \text{ isomorphism } \Rightarrow f \text{ local diffeo at } p$$

---

[14] at $(p, v) \neq (p, 0)$ the iso $T_{(p,v)}E \cong T_pM \oplus E_p$ depends on ∇: ∇ decides which vectors are horizontal. *Non-examinable:* $T^{\mathrm{horiz}}_{(p,v)}E = ds_p(T_pM)$ for local sections s with $(\nabla_X s)_p = 0, \forall X \in T_pM$.

Ideas in the proof. [15]

$$\text{“}x = f^{-1}(y)\text{”} \Leftrightarrow y = f(x) \Leftrightarrow \text{fixed point } g_y(x) = x$$

where we define

$$g_y(x) = x + A^{-1}(y - f(x))$$

where $A = d_p f$ in local coords on a closed ball around p (which is a complete metric space). By the contraction mapping theorem[16] there exists f^{-1}, and one easily checks that f^{-1} is continuous. Now observe the following equation holds:

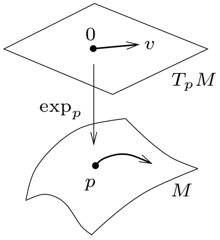
$$(df^{-1})(y) = [df(f^{-1}(y))]^{-1}$$

so by **bootstrapping**,[17]

$$\text{smoothness of } f^{-1} = \text{ smoothness of } f. \quad \square$$

Rmk. *The proof only used completeness and linear algebra, we never used compactness or finite dimensionality.*

Example.[18]



$d_0(\exp_p) \cdot \vec{v} = \partial_t|_{t=0} \exp_p(t\vec{v}) = \vec{v}$, *so* $d_0 \exp_p = I$,
$\Rightarrow \exp_p$ *is a local diffeo near* 0,
$\Rightarrow \exists \varepsilon > 0$ *so that* $\exp_p : D_\varepsilon T_pM \to M$ *is a chart!*
$\Rightarrow$ *This proves the uniqueness of geodesics joining two close enough points.*

1.2. **Regular maps.**[19]

Def. $d_p f$ *surjective* $\Rightarrow f$ *is* regular *at* p (submersion), *so* $m \geq n$.
$d_p f$ *not surjective* $\Rightarrow p$ *is a* critical point, $f(p)$ *is a* critical value.

Def. $\{\text{regular values}\} = N \setminus \{\text{critical values}\}$. Note f may not attain these values![20]

Example. $f : M \to \mathbb{R}$ regular at $p \Leftrightarrow d_p f \neq 0$, so $\mathrm{Crit}(f) = \{p \in M : d_p f = 0\}$.

Thm (Implicit Function Theorem)**.**
f regular at $p \Rightarrow \exists$ local coords near $p, f(p)$ in which f is a projection:

$$f(x_1, \ldots, x_n, \ldots, x_m) = (x_1, \ldots, x_n)$$

Pf. Locally $f : \mathbb{R}^m \to \mathbb{R}^n$ defined near $p = 0$, $f(p) = 0$.
By linear algebra (row/col operations):

$$B \circ d_p f \circ A = (I \quad 0)\,,$$

---

[15] For a detailed proof, see for example Lang, *Undergraduate analysis*.

[16] *Contraction Mapping Theorem*: a contraction mapping on a complete metric space has a unique fixed point. *Contraction mapping* $f : M \to M$ means there is a constant $0 < k < 1$ for which $\mathrm{dist}(fx, fy) \leq k \cdot \mathrm{dist}(x, y)$, for all $x, y \in M$. *Fixed point* means a point x with $f(x) = x$.

[17] f^{-1} is cts $\Rightarrow df^{-1}$ cts (because of the equation) $\Rightarrow f^{-1}$ is once ctsly diffble $\Rightarrow$ can differentiate equation $\Rightarrow$ repeat. Note that we used that inversion is C^∞ on invertible matrices.

[18] by definition $u(t) = \exp_p(t\vec{v})$ satisfies $u'(t) = \vec{v}$.

[19] Motivation: what if $m \neq n$? Hwk 3 is about $m \leq n$, immersive maps. Here we do $m \geq n$.

[20] $r \in N$ is a *regular value* iff $d_p f$ is surjective for all $p \in f^{-1}(r)$. So this holds if $f^{-1}(r) = \emptyset$!

some $A \in GL(m), B \in GL(n)$. Replace f by $B \circ f \circ A$ (a linear change of coords), so $d_p f = (I \quad 0)$. Define

$$\begin{aligned} &F : \mathbb{R}^m \to \mathbb{R}^{n+(m-n)} = \mathbb{R}^m \text{ defined near } p \text{ by} \\ &F(x) = (f(x), x_{n+1}, \dots, x_m) \end{aligned}$$

So $d_p F = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. So F local diffeo near p. The claim follows by the diagram:

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\text{old chart}"] \arrow[dd, bend right, "\text{new chart}"'] & N \arrow[dd, "\text{old chart}"] \\
\mathbb{R}^m \arrow[d, "F"] & \\
\mathbb{R}^m \arrow[r, "\text{projection}"] & \mathbb{R}^n
\end{tikzcd}$$

☐

Rmk. *diffeo/regularity are* open conditions*: if true at p then true near p.*

Rmk. *classically (algebraic geometry) you apply the theorem to*

$$F : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n \times \mathbb{R}^{m-n}, \quad F(X, Y) = (f(X, Y), Y).$$

If $X \mapsto f(X, b)$ has non-singular derivative at $X = a$, and $f(a, b) = 0$, then near (a, b) the vanishing set

$$V(f) = f^{-1}(0) \textit{ is parametrized by } Y \mapsto (0, Y) \overset{F^{-1}}{\mapsto} (g(Y), Y)$$

so $f(g(Y), Y) = 0$, and g is called implicit function.

Cor (Implicit Function Theorem).[21]

$$\boxed{q \in N \textit{ regular value of } f \Rightarrow \left\{ \begin{array}{l} f^{-1}(q) \subset M \textit{ submfd of } \operatorname{codim} = n \\ T_p f^{-1}(q) = \ker d_p f \quad \forall p \in f^{-1}(q) \end{array} \right.}$$

Pf. Locally

$$\begin{aligned} f(x_1, \dots, x_n, \dots, x_m) &= (x_1, \dots, x_n) \\ d_x f \cdot (\vec{x}_1, \dots, \vec{x}_n, \dots, \vec{x}_m) &= (\vec{x}_1, \dots, \vec{x}_n). \end{aligned}$$

$\Rightarrow f^{-1}(0) = \{(0, \dots, 0, x_{n+1}, \dots, x_m)\}$, so those x's form local coords for chart.
$\Rightarrow T f^{-1}(0) = \ker df$. ☐

---

[21] Codimension $f^{-1}(q) = n$ means $\dim f^{-1}(q) = m - n$. You lose n dimensions because you impose n conditions by asking $f = q$. The tangent space is also intuitive: if a curve moves along $f = q$, then f does not change, so $d_p f = 0$, and now recall that we related curves and vectors.

LECTURE 3.

PART III, MORSE HOMOLOGY, 2011

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1.3. Sard's theorem.

Fact.[1] *For smooth $f : M \to N$,*

$$\boxed{\text{Almost every point of } N \text{ is a regular value of } f}$$

This means: $\{$critical values$\} = f(\{$critical points$\})$ is a set of *measure zero*[2] in N. Equivalently: $\{$regular values$\} \subset N$ has *full measure*, so these points are "generic".

Cor. *$\{$regular values$\} \subset N$ is dense.*

Pf. Non-empty open sets in $\mathbb{R}^n$ have measure > 0. □

Rmk. *M, N need not be compact. The result only uses that M is second countable.*[3]

Fact. *For C^k-maps*[4] *$f : M^m \to N^n$, the above fact holds provided $k > m - n$. (Here M, N need not be smooth, just need C^k-mfds: the transition maps are C^k.)*

Examples.

(1) $f : \mathbb{R}^m \to \mathbb{R}$, $x \mapsto \sum x_i^2 - 1$
0 regular value, so $f^{-1}(0) = S^{m-1}$ mfd of dim $= m - 1$.

(2) $f :$ *Matrices*$_{n\times n} \to$ *Symmetric Matrices*$_{n\times n}$, $A \mapsto A^T A$
I regular value, so $f^{-1}(0) = O(n)$ mfd of dim $= n^2 - \frac{n(n+1)}{2}$.

(3) *Hwk.*[5] *Sard $\Rightarrow$ homotopy groups $\pi_i(S^n) = 0$ for $i < n$.*

1.4. Transversality.

Motivation:

$$\begin{array}{rcl}
q \in N \text{ regular value} & \Rightarrow & f^{-1}(q) \subset M \text{ submfd} \\
\text{① submfd } Q \subset N \text{ satisfying} \ldots ? & \Rightarrow & f^{-1}(Q) \subset M \text{ submfd} \\
\text{② submfds } Q_1, Q_2 \subset N \text{ satisfying} \ldots ? & \Rightarrow & Q_1 \cap Q_2 \subset N \text{ submfd}
\end{array}$$

① Pretend N/Q made sense ☺

$\Rightarrow F : M \xrightarrow{f} N \to N/Q \ni \overline{q} = Q/Q$

Date: May 3, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1] If you are curious about its non-examinable proof, see Milnor's *Topology from the Differentiable Viewpoint*, or Guillemin & Pollack, *Differential Topology.*

[2] A subset S of $\mathbb{R}^n$ has *measure zero* if $\forall \varepsilon > 0$, $\exists$countable covering of S by cubes C_i, with $\sum \mathrm{vol}(C_i) < \varepsilon$. A subset S of a mfd N has *measure zero* if for any chart $\varphi : U \to \mathbb{R}^n$, $\varphi(S \cap U)$ has measure 0 (it's enough to require this for a covering $\varphi_i : U_i \to \mathbb{R}^n$). Example: $\mathbb{Q} \subset \mathbb{R}$. Useful facts: countable unions of measure 0 sets have measure 0; C^1-maps between subsets of $\mathbb{R}^n$ always map measure 0 sets to measure 0 sets.

[3] *Second countable*= there is a countable covering by charts. This is always part of the definition of manifold. Consequence: any covering has a countable subcover.

[4] k-times continuously differentiable maps, with $k \geq 1$ so "regular/critical points" are defined.

[5] *Non-examinable:* the proof essentially shows Sard implies the cellular approximation theorem.

$\Rightarrow f^{-1}(Q) = F^{-1}(\overline{q})$

$\Rightarrow f^{-1}(Q)$ is mfd if $\overline{q}$ regular value of F

if d_pF surjective $\forall p \in F^{-1}(\overline{q})$

if $d_pF(T_pM) = T_{\overline{q}}(N/Q)$

if $\boxed{d_pf(T_pM) + T_qQ = T_qN \qquad \forall p \in f^{-1}(q), \forall q \in Q}$

Def. *$f : M \to N$ is transverse to Q if the above box holds. Write $f \pitchfork Q$.*

Thm.

$$\boxed{f \pitchfork Q \Rightarrow \begin{cases} f^{-1}(Q) \subset M \textit{ submfd of codim} = \operatorname{codim} Q \\ T_pf^{-1}(Q) = \ker(\,T_pM \xrightarrow{d_pf} TN \to TN/TQ) = \ker(D_pf : T_pM \to \nu_{Q,q}) \end{cases}}$$

Pf. Locally $Q \subset N$ is[6] $\mathbb{R}^a \subset \mathbb{R}^n$, so "$N/Q$" is well-defined locally: $\mathbb{R}^n/\mathbb{R}^a$. □

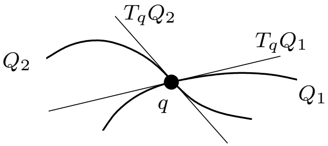
Explanation: $\nu_Q = TN/TQ =$*normal bundle* to $Q \subset N$, fibre $\nu_{Q,q} = T_qN/T_qQ$. D_pf is abuse of notation:[7] $\boxed{Df_p \cdot X = \text{vertical projection of } d_pf \cdot X \text{ at } q = f(p) \in Q}$

② For $f : Q_1 \xhookrightarrow[\text{inclusion}]{} N$ and $Q = Q_2 \subset N$,

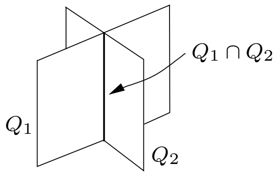
$$f^{-1}(Q) = Q_1 \cap Q_2 \subset N.$$

Def. *Q_1, Q_2 are* transverse *submfds of N, written $Q_1 \pitchfork Q_2$, if*

$$\boxed{T_qQ_1 + T_qQ_2 = T_qN \qquad \forall q \in Q_1 \cap Q_2}$$



Examples. $N \pitchfork$ any submfd! Two vector subspaces $\subset \mathbb{R}^n$ are $\pitchfork$ if they span $\mathbb{R}^n$.



Cor.

$$\boxed{Q_1 \pitchfork Q_2 \Rightarrow \begin{cases} Q_1 \cap Q_2 \subset N \textit{ submfd} \\ \textit{of codim} = \operatorname{codim} Q_1 + \operatorname{codim} Q_2 \\ T_q(Q_1 \cap Q_2) = T_qQ_1 \cap T_qQ_2 \end{cases}}$$

Rmk.

1. $\dim Q_1 + \dim Q_2 < \dim N$ *then* $Q_1 \pitchfork Q_2 \Leftrightarrow Q_1 \cap Q_2 = \emptyset$

2. $\dim Q_1 + \dim Q_2 = \dim N$ *then* $Q_1 \pitchfork Q_2 \Leftrightarrow \begin{cases} Q_1 \cap Q_2 \textit{ finite set}^{8} \\ TQ_1 \oplus TQ_2 \cong TN \textit{ at } q \in Q_1 \cap Q_2 \ (*) \end{cases}$

In case 2. you can define an intersection number

$$Q_1 \cdot Q_2 = \#(Q_1 \cap Q_2) \textit{ mod } 2 \in \mathbb{Z}/2\mathbb{Z}$$

If Q_1, Q_2, N oriented:[9]

$$Q_1 \cdot Q_2 = \#(Q_1 \cap Q_2) \in \mathbb{Z},$$

---

[6] Hwk 3: $Q \to N$ immersion $\Rightarrow$ locally has form $(x_1, \dots, x_a) \mapsto (x_1, \dots, x_a, 0, \dots, 0) \in \mathbb{R}^n$.

[7] f is not a section of ν_Q, but the construction of that vertical projection is analogous.

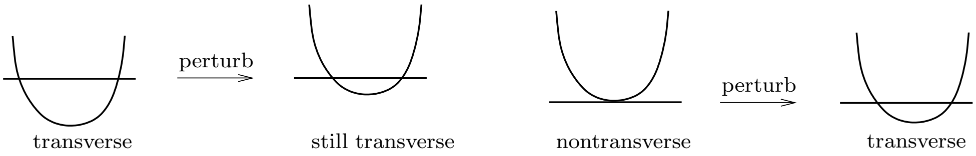
[8] assuming Q_1, Q_2 are compact submanifolds. Otherwise, replace with "discrete set".

[9] *Non-examinable:* it suffices that Q_1 is oriented, and Q_2 is co-oriented (= normal bundle $\nu_{Q_2} = TN/TQ_2$ is oriented). Assign $+1$ to $p \in Q_1 \cap Q_2$ if an oriented basis of T_pQ_1 gives rise to an oriented basis of ν_{Q_2}, and -1 else. When Q_1, Q_2, N are oriented, this sign agrees with the one above, if we orient so that $TN|_{Q_2} \cong \nu_{Q_2} \oplus TQ_2$ preserves orientation ("normals first").

where # counts with sign $+1$ *if the iso* $(*)$ *is orientation-preserving,* -1 *otherwise.*

Next time, we will deduce that one can always achieve $Q_1 \pitchfork Q_2$ *after perturbing* Q_1 *(or* Q_2*), and in case* ***2.*** *the value* $Q_1 \cdot Q_2$ *is independent of the perturbation.*

Motivation for stability and genericity. Transversality is stable and generic: Stable: perturbing preserves the property, generic: it can be achieved by perturbing.



1.5. **Stability.**

Recall a (smooth) *homotopy* f_t of $f : M \to N$ means a smooth map

$$H : M \times [0,1] \to N \qquad \text{with} \begin{cases} f_t(x) = H(x,t) \\ f_0 = f \end{cases}$$

Call f_0, f_1 (smoothly) homotopic.

Def. *A "property"* P *is* stable *for a class* C *of maps* $f : M \to N$*, if*

$$\left.\begin{array}{l} f \in C \text{ satisfies } P \\ f_t \text{ homotopy} \end{array}\right\} \Rightarrow f_t \text{ satisfies } P \text{ for each } t < \varepsilon \quad (\varepsilon > 0 \text{ depending on } f, f_t)$$

Rmk.

(1) Locally stable *means* $\forall p \in M$, $\exists$ *nbhd* $U \ni p$ *such that* P *is stable for the restrictions* $\{f|_U : f \in C\}$

(2) *For compact* M*, one can often deduce stability from local stability, by covering* M *by such* U*, taking a finite subcover, taking min of* ε*'s.*

(3) *Can use more general parameters* $t \in S =$ *metric space.*

Stability Theorem. M *compact* $\Rightarrow$ *the following classes are stable:*

$\{$*local diffeos*$\}$
$\{$*regular maps*$\}$
$\{$*maps* $\pitchfork$ *to a given topologically-closed submfd* $Q \subset N\}$

Pf. The definition of these classes locally involve the non-vanishing of some (sub) determinant of some differential. Use Rmk (2) to globalize. □

Cor. *Transversality is stable and it is an open condition.*

Pf. Stability by Thm. Open: if not, find non-transverse $f_n \to f$ as[10] $n \to \infty$. Produce a homotopy H of f with $H(1/n, t) = f_n(t)$. H contradicts stability. □

Rmk. *Here is a more direct proof that transversality is an open condition:*
Claim 1. *regular points of any smooth map* f *of mfds forms an open set.*
Pf. *Locally at regular* p*,* $d_p f = (I \quad 0)$*. So for* q *close to* p*,* $d_q f = (T \quad *)$ *for some invertible* T *since invertibility is an open condition.*[11] *So* q *is regular.* □
Transversality can be expressed as a regularity condition, so it is also open.

---

[10] the convergence is in C^∞. Also C^1 is enough: we just need the derivatives to converge.

[11] If s is an operator with small norm ($\|s\| < 1$ is enough), then $(I+s)^{-1} = I - s + s^2 - s^3 + \cdots$ is a well-defined operator. If L is invertible and $\|s\| < \|L\|$ then $(L+s)^{-1} = (I + L^{-1}s)^{-1}L^{-1}$.

1.6. **Local to global examples.**

Thm. *Any compact mfd N can be embedded in some $\mathbb{R}^k$.*

Pf. Cover N by all possible charts[12] $\varphi : B(2) \to N$.
Pick finitely many φ_i for which $\varphi_i(B(1))$ cover N.
Let β = bump function[13] $B(2) \to [0,1]$, $\beta = 1$ on $B(1)$, $\beta = 0$ near $\partial B(2)$.
$\Rightarrow N \hookrightarrow \mathbb{R}^{(n+1)\cdot\#\text{charts}}$

$$p \mapsto (\, \beta(\varphi_i^{-1}(p)) \cdot \varphi_i^{-1}(p),\ \beta(\varphi_i^{-1}(p))\,)_{i=1,2,\dots} \quad \text{(zero entry for } i \text{ if } p \notin \mathrm{im}\varphi_i\text{)}.$$

Note we are keeping track of the β values to ensure global injectivity. ☐

Cultural Rmk. *Whitney proved $N^n \hookrightarrow \mathbb{R}^{2n}$. Transversality techniques from this course can easily prove $N^n \hookrightarrow \mathbb{R}^{2n+1}$ (if you're curious, see Guillemin & Pollack).*

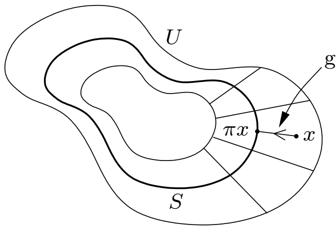
Def. *A* tubular neighbourhood *is a nbhd U of S with a regular retraction*

$$\pi : U \to S.$$

(Retraction *just means $\pi|_S = id_S$).*

Thm. *Any submanifold $S \subset M$ has a tubular nbhd $U \subset M$.*

Pf. Pick a Riemannian metric for M, use exp map. ☐



(fact: πx = closest point to x in S)

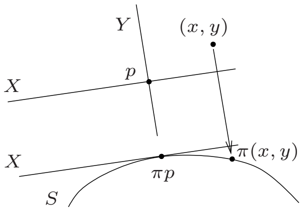
Rmk.

(1) $U \overset{exp^{-1}}{\cong}$ *nbhd of zero section of normal bundle ν_S*
$\pi \cong$ *projection*

(2) *Converse:*[14] $\boxed{\textit{A closed subset } S \subset \mathbb{R}^k \textit{ is a submfd} \Leftrightarrow S \textit{ is a smooth retract}^{15}}$

Pf. implicit function theorem for regular $\pi : U \to S$. ☐

Non-examinable details of Pf:



For $p \in U$ near S, let $X = d_p\pi(T_pU) \subset \mathbb{R}^k$ a v.subspace (secretly $T_{\pi(p)}S$). Then $\mathbb{R}^k = X \oplus Y$ some v.subspace Y. After lin change of coords,

$$d_p\pi = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] : X \oplus Y \to X \oplus Y,$$

with $p = (0,0) \in X \oplus Y = \mathbb{R}^k$. Define

$$F : X \oplus Y \to X \oplus Y, F(x,y) = \pi(x,y) + y.$$

$d_pF = I \Rightarrow$ InvFnThm $\Rightarrow F^{-1}(s) = (g(s,0),0)$ for $s \in S$ defines chart $s \mapsto g(s,0)$ at $\pi(p) \in S$.

---

[12] $B(r)$ = open ball of radius r, centre 0, in $\mathbb{R}^n$.

[13] You gain nothing from writing out explicitly a bump function you already know exists: *Non-examinable:* for $b > a > 0$, let $\alpha(x) = e^{-1/x}$ for $x > 0$, 0 for $x \le 0$; let $\gamma(x) = \alpha(x-a)\cdot\alpha(b-x)$; let $\delta(x) = \int_x^b \gamma / \int_a^b \gamma$. Then $\beta(x) = \delta(|x|)$ is 1 on $|x| \le a$, 0 on $|x| \ge b$, $\beta(x) \in (0,1)$ for $a < |x| < b$.

[14] the same proof shows this holds for C^r-mfds, π a C^r-map, $r \ge 1$ (not just $r = \infty$).

[15] Smooth retract= $\exists$open nbhd U of S, $\exists$smooth $\pi : U \to \mathbb{R}^k$ with $\pi(U) \subset S$, $\pi|_S = \mathrm{id}_S$.

LECTURE 4.

PART III, MORSE HOMOLOGY, 2011

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1.7. **Genericity.** Recall we defined: *almost every = full measure = generic.* Generic implies dense, but not conversely (e.g. $\mathbb{Q} \subset \mathbb{R}$ is dense but not generic).

Thm (Parametric Transversality)**.** *Let M, N be closed mfds, $Q \subset N$ a submfd, and S a mfd*[1] *without boundary but possibly non-compact. Suppose:*

$$F : M \times S \to N \text{ smooth map and } F \pitchfork Q$$

Then $F_s = F(\cdot, s) \pitchfork Q$ for generic $s \in S$.

Proof. Consider the projection π:

$$\begin{tikzcd} M \times S \arrow[r, "F"] \arrow[d, "\pi"] & N \\ S & \end{tikzcd} \qquad\qquad \begin{tikzcd} W = F^{-1}(Q) \arrow[r, "F|_W"] \arrow[d, "\pi"] & Q \\ S & \end{tikzcd}$$

where we used $F \pitchfork Q$ to deduce $W = F^{-1}(Q)$ is a mfd.

Claim. $s \in S$ regular for $\pi|_W \Leftrightarrow F_s \pitchfork Q$.

(So the Thm follows by Sard applied to $\pi|_W$)

Proof of Claim. Suppose $q = F(m, s) \in Q$, so $w = (m, s) \in W$. $F \pitchfork Q$ implies:

$$(*) \qquad TN = dF \cdot T(M \times S) + TQ \qquad \text{at } F(w)$$

and it implies

$$\begin{aligned} T_w W &= \ker(T(M \times S) \overset{dF}{\to} TN \to TN/TQ) \qquad \text{at } w \\ &= \{(\vec{m}, \vec{s}) \in T_w(M \times S) = T_m M \oplus T_s S : dF \cdot \vec{m} + dF \cdot \vec{s} \in TQ\} \\ &= \{(\vec{m}, \vec{s}) : dF \cdot \vec{m} = -dF \cdot \vec{s} \text{ modulo } TQ\}. \end{aligned}$$

Finally, observe that

$$\begin{aligned} s \text{ regular for } \pi|_W \;&\Rightarrow\; d\pi|_W : TW \to TS \text{ surjective at } w \\ &\Rightarrow\; \forall \vec{s},\ \vec{s} = d\pi|_W \cdot (\vec{m}, \vec{s}) \text{ some } (\vec{m}, \vec{s}) \in T_w W \\ &\Rightarrow\; \forall \vec{s},\ dF \cdot \vec{m} = -dF \cdot \vec{s} \text{ modulo } TQ \text{ some } \vec{m} \\ &\overset{(*)}{\Rightarrow}\; \forall \vec{n} \in T_q N,\ \vec{n} = dF \cdot \vec{m}_2 + dF \cdot \vec{s} + \vec{q} \text{ some } \vec{m}_2, \vec{s}, \vec{q} \\ &\Rightarrow\; \forall \vec{n} \in T_q N,\ \vec{n} = dF \cdot \vec{m}_2 - dF \cdot \vec{m} \text{ modulo } TQ \\ &\Rightarrow\; TN = dF \cdot TM + TQ \text{ at } F(w) \qquad (**) \\ &\Rightarrow\; F_s \pitchfork Q \text{ at } w. \end{aligned}$$

The proof also works by reversing the implications, which proves the converse. □

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[1] the *parameter space.*

Modern viewpoint: compute ker and coker of $d\pi|_W$ at $w=(m,s)$, $F(w)=q\in Q$:

$$\begin{aligned} \ker(d\pi|_W)_w &= \{(\vec{m},0)\in T_wW\}\\ &\cong \{\vec{m}\in T_mM : dF\cdot\vec{m}\in TQ\}\\ &= \ker\left(\begin{tikzcd} T_mM \arrow[r, "dF=dF_s"] \arrow[rr, bend right, "DF=DF_s"'] & TN \arrow[r] & TN/TQ=\nu_Q \end{tikzcd}\right)\end{aligned}$$

Therefore $\boxed{\ker(d\pi|_W)_w=\ker(DF_s:T_mM\to\nu_Q)}$ (which is $TF_s^{-1}(Q)$ if $F_s \pitchfork Q$).

Now consider $\operatorname{coker}(d\pi|_W)_w=T_sS/d\pi\cdot TW$, which you can think of as measuring how much the implication $*\Rightarrow **$ fails to hold. By linear algebra,[2]

$$d\pi:\frac{TM\oplus TS}{TM+TW}\to\frac{TS}{d\pi\cdot TW}=\operatorname{coker}(d\pi|_W)_w \qquad \text{iso at } w$$

$$\begin{aligned} F\pitchfork Q &\Rightarrow \quad TW=\ker(DF:TM\oplus TS\overset{\text{surj}}{\to}\nu_Q) \qquad \text{at } w\\ &\Rightarrow \quad \frac{TM\oplus TS}{TW}\to\nu_Q \qquad \text{iso at } w\\ &\Rightarrow \quad \frac{TM\oplus TS}{TM+TW}\to\frac{\nu_Q}{DF\cdot TM}=\frac{\nu_Q}{DF_s\cdot TM}=\operatorname{coker}DF_s \qquad \text{iso at } w\end{aligned}$$

So $\boxed{\operatorname{coker}(d\pi|_W)_w\cong\operatorname{coker}(DF_s:T_mM\to\nu_Q)}$.

These calculations only used linear algebra, so they hold also for Banach manifolds (which use a Banach space instead of $\mathbb{R}^n$ for charts, more on this in Lecture 5).

Thm (Parametric Transversality 2)**.**

$$\begin{aligned} F\pitchfork Q \quad &\Rightarrow \quad \begin{cases} \ker(d\pi|_W)_w=\ker(DF_s:T_mM\to\nu_Q)\\ \operatorname{coker}(d\pi|_W)_w\cong\operatorname{coker}(DF_s:T_mM\to\nu_Q)\end{cases}\\ &\Rightarrow \quad \begin{cases} d\pi \text{ Fredholm} \Leftrightarrow DF \text{ Fredholm}^3\\ d\pi \text{ surjective} \Leftrightarrow DF \text{ surjective}\end{cases}\end{aligned}$$

Thm (Genericity of transversality)**.** *Let $f:M\to N$ be smooth, $Q\subset N$ a submfd (M,N,Q closed mfds). Then for $S=$open nbhd of $0\in\mathbb{R}^k$, there is $F:M\times S\to N$, $F(\cdot,0)=f$, with $F\pitchfork Q$.*

Proof. Embed $N\hookrightarrow\mathbb{R}^k$. Pick tubular nbhd of N: $U\subset\mathbb{R}^k$, $\pi:U\to N$. Then

$$\begin{array}{rccccc} F: & M\times\mathbb{R}^k & \to & U & \to & N\\ & (m,s) & \mapsto & f(m)+s & \to & \pi(f(m)+s)\end{array}$$

the first map is defined for small $\|s\|$, and is clearly regular (think about it). The second map is regular by definition of U. Therefore the composite is regular. So $F\pitchfork$ anything (since dF is already surjective), in particular $F\pitchfork Q$. ☐

Cor.[4] *f is homotopic to $f_s=F(\cdot,s)\pitchfork Q$ (for generic s).*

Rmk.

---

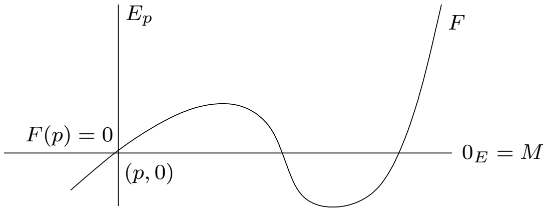
[2] $d\pi:TM\oplus TS\to TS$, $d\pi\cdot(\vec{m},\vec{s})=\vec{s}$ is surjective. Make it injective by quotienting the domain by TM. Now you want $TS/d\pi\cdot TW$ as codomain, so to make the map well-defined you quotient the domain by $TM+TW$.

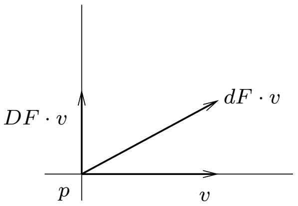
[3] Fredholm = finite dimensional kernel and cokernel, more on this in Lecture 6.

[4] Motto: You can make things transverse by perturbing!

1. *You only need to perturb f near $f^{-1}(nbhd(Q))$, indeed by replacing s by $\beta(m)s$ where $\beta : M \to [0,1]$, $\beta = 1$ near $f^{-1}(Q)$, $\beta = 0$ away from $f^{-1}(Q)$, we still get regularity of F near $F^{-1}(Q)$ so $F \pitchfork Q$.*
2. *If f is already $\pitchfork Q$ on a closed set $M_0 \subset M$ (hence near M_0 by openness of transversality), then one only needs to perturb f away from M_0: again pick $\beta : M \to \mathbb{R}$, $\beta = 0$ on M_0, $\beta = 1$ away from M_0 (ensure $0 < \beta < 1$ lies in region where $f \pitchfork Q$, so for small enough s also $f_s \pitchfork Q$ there).*
3. *Instead of using $N \subset \mathbb{R}^k$ one can also use charts $U \subset N$, $\varphi : U \to \mathbb{R}^n$, and consider $F(m,s) = \varphi \circ f(m) + \beta(\varphi(m)) \cdot s$, $\beta =$bump function supported in chart. So one can inductively perturb f on charts to make it $\pitchfork Q$.*

1.8. **Sections of a vector bundle.**





Recall $D_pF : T_pM \to E_p$ is the vertical derivative (vertical projection of dF).

Lemma. $\boxed{D_pF \textit{ surjective } \forall p \in F^{-1}(0_E) \Leftrightarrow F \pitchfork 0_E}$

Proof. $T_{(p,0)}E = T_p0_E \oplus E_p$, so $d_pF(T_pM) + T_p0_E = D_pF(T_pM) + T_p0_E$. ☐

Cor.

$$\boxed{DF \textit{ surjective along } F^{-1}(0_E) \Rightarrow \begin{cases} F^{-1}(0_E) \subset M \textit{ submfd} \\ \textit{of codim} = \textit{codim } 0_E = \textit{rank } E \\ TF^{-1}(0_E) = \ker DF \end{cases}}$$

Example. *$E = T^*M \to M$ with section $F = df$, where $f : M \to \mathbb{R}$ smooth.*

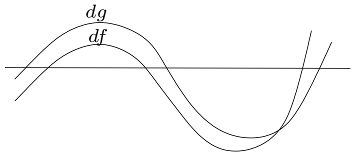
1.9. **Morse functions.**

Def. *$f : M \to \mathbb{R}$ is a Morse function if* $\boxed{df \pitchfork 0_{T^*M}}$

Consequences (for M closed):

1. $df^{-1}(0_{T^*M}) = \mathrm{Crit}(f)$ is a 0-dim submfd, so the critical points are isolated, so $\boxed{\mathrm{Crit}(f) \text{ is finite}}$.
2. $\boxed{f \text{ Morse} \Leftrightarrow \text{all critical pts are nondegenerate (Hessian is nonsingular)}}$

   Proof. Hwk 2: at $p \in \mathrm{Crit}(f)$, $\mathrm{Hess}_pf = D_p(df) = \frac{\partial^2 f(p)}{\partial x_i \partial x_j}$. ☐
3. Being Morse is stable.
4. $\boxed{\text{Being Morse is open in the } C^2\text{-topology}}$



Proof. For Morse f:
f, g C^2-close $\Rightarrow df, dg$ C^1-close
$\Rightarrow dg \pitchfork 0$ since $df \pitchfork 0$
$\Rightarrow g$ Morse. ☐

(5) $\boxed{\text{Morse functions are dense in the } C^0\text{-topology}}$

(Means: $\forall \varepsilon > 0, h : M \to \mathbb{R} \Rightarrow \exists$ Morse $f : M \to \mathbb{R}, \sup |f - h| < \varepsilon$)

Proof.[5] WLOG[6] $M \subset \mathbb{R}^k$, $h : \mathbb{R}^k \to \mathbb{R}$ (extend to $\mathbb{R}^k$ via a tubular nbhd and bump function). WLOG h smooth (since $C^\infty \subset C^0$ dense). For $q \in \mathbb{R}^n$,

$$L_q : \mathbb{R}^k \to \mathbb{R},\ L_q(x) = \langle q, x\rangle_{\mathbb{R}^k} = \sum q_i \cdot x_i$$

is called a *height function.*

Claim. $h + L_q$ is Morse for almost every q (and C^0-close to h for small q)

Proof. Consider $F(x, q) = d(h + L_q)$:

$$\begin{tikzcd} M \times \mathbb{R}^k \arrow[r, "F"] \arrow[d, "\pi"] & T^*M \\ \mathbb{R}^k & \end{tikzcd}$$

We want $F \pitchfork 0_{T^*M}$, then $d(h+L_q) \pitchfork 0_{T^*M}$ for generic q ✓. View its vertical component F^{loc} as a map $\mathbb{R}^k \times \mathbb{R}^k \to T_x^*\mathbb{R}^k$ (later restrict to $M \subset \mathbb{R}^k$):

$$F^{loc}(x, q) = \textstyle\sum_i (\frac{\partial h}{\partial x_i}(x) + q_i)\, dx_i$$
$$DF_{(x,q)} \cdot (\vec{x}, \vec{q}) = \textstyle\sum_i (\sum_j \frac{\partial^2 h}{\partial x_j \partial x_i} dx_j(\vec{x}) + dq_i(\vec{q}))\, dx_i$$

Key remark: $dq_i(\vec{q})$ is arbitrary as you vary $\vec{q} \in T_q\mathbb{R}^k$. Now restrict:

$$DF_{(x,q)} : T_xM \times T_q\mathbb{R}^k \xrightarrow{F} T_x^*\mathbb{R}^k \xrightarrow{\text{pullback}} T_x^*M$$

The first map is surjective by the Key remark (can still freely vary $\vec{q}$), the second map is surjective because $M \hookrightarrow \mathbb{R}^k$ is embedded so $T_xM \hookrightarrow T_x\mathbb{R}^k$ is injective so its dual is surjective. So $DF_{(x,q)}$ surjective, so $F \pitchfork 0$ □

Cor. *Almost any height function on $M \subset \mathbb{R}^k$ is Morse (take $h = 0$).*

(6) **Morse Lemma**

$$\boxed{f \text{ Morse} \Leftrightarrow \begin{cases} \exists \text{ local coords near each crit point } p \text{ (called \textit{Morse chart})} \\ \text{such that } f(x) = f(p) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_m^2 \end{cases}}$$

Proof. See Hwk 4. Key idea: Taylor $f(x) = f(p) + \frac{1}{2}\sum A_{ij}(x)x_ix_j$ with $A(x)$ symmetric. Diagonalize $A(x)$ smoothly in x. Then rescale coords. □

Def. *The* Morse index *of $p \in Crit(f)$ is the index i in the Morse Lemma:*

$$\boxed{|p| = ind_f(p) = i = \#(\textit{negative evalues of } Hess_p(f) \textit{ in local coords})}$$

which equals the dimension of the maximal vector subspace of T_pM on which $T_pM \otimes T_pM \to \mathbb{R}$, $(v, w) \mapsto D_p(df) \cdot (v, w)$ is negative definite.[7]

(7) $\boxed{\text{Morse functions are generic}}$ *Proof.* Hwk 6.

---

[5] A messier alternative (avoiding $M \hookrightarrow \mathbb{R}^k$, and works for noncompact M): inductively perturb f on charts by adding $\phi_j(x) \cdot L_{q_j}(x)$, where ϕ_j is a partition of unity subordinate to a countable locally finite cover by charts, q_j are generic and small chosen inductively so that f stays Morse on charts where you already perturbed.

[6] Without Loss Of Generality.

[7] $D_p(df) : T_pM \to T_p^*M$, so $D_p(df)$ eats two vectors and outputs a number.

LECTURE 5.

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Motivation. $\mathcal{M}(p,q) = \{-\nabla f$ flowlines $u : \mathbb{R} \to M\}$ is an infinite dimensional manifold. What does this mean?

2. Banach spaces and manifolds

2.1. Review of Banach spaces.

Def. *Banach space = complete*[1] *normed vector space B. Call $\|\cdot\| : B \to \mathbb{R}$ the norm.*

Fact. *The unit ball in a Banach space is compact $\Leftrightarrow$ B is finite dimensional.*

Examples.

- $C^0[0,1] = \{$continuous $[0,1] \to \mathbb{R}\}$,

$$\|f\|_\infty = \max |f| \qquad (\text{“uniform norm”})$$

- $C^k[0,1] = \{k$-times ctsly diffble $[0,1] \to \mathbb{R}\}$,

$$\|f\|_{C^k} = \sum_{0\leq j\leq k} \|D^j f\|_\infty \qquad (D = \frac{d}{dx}, D^0 f = f)$$

Arzela-Ascoli theorem.[2] *K compact metric space, $F \subset C(K) = \{cts\ K \to \mathbb{R}\}$ equibounded*[3] *and equicontinuous,*[4] *then F is precompact*[5] *using $\|\cdot\|_\infty$.*

Non-examples.

- $C_c(\mathbb{R}) = \{$compactly supported[6] cts $f : \mathbb{R} \to \mathbb{R}\}$ with $\|\cdot\|_\infty$.
- $C^0[0,1]$ with

$$\|f\|_p = \left(\int_0^1 |f(t)|^p \, dt\right)^{1/p} \qquad (1 \leq p < \infty)$$

- $C^\infty[0,1]$ with the *natural topology*[7]: the topology is generated by all $U \cap C^\infty[0,1]$ with U open in $(C^k[0,1], \|\cdot\|_{C^k})$ some $k \geq 0$. Explicitly:

$$f_n \to f \text{ in } C^\infty \Leftrightarrow \|f_n - f\|_{C^{k(n)}} \to 0 \text{ some } k(n) \to \infty.$$

Lemma. *$C^\infty[0,1]$ is a complete metric space, but not a Banach space.*

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[1] *Complete* = Cauchy sequences converge: “$\|x_n - x_m\| \to 0$ as $n, m \to \infty$” implies “$\exists\, x : x_n \to x$”.

[2] if you are curious, see Rudin, *Functional Analysis.*

[3] $\sup\{|f(x)| : f \in F\} < \infty$ for any given $x \in K$.

[4] $\forall \varepsilon > 0, x \in K,$ there is a nbhd U of x such that $|f(y) - f(x)| < \varepsilon$ for all $y \in U, f \in F$.

[5] the closure of F in $C(K)$ is compact, explicitly: any sequence $f_n \in F$ has a uniformly convergent subsequence (but the limit may not be in F).

[6] $f = 0$ outside a compact. Equivalently: the support $\mathrm{supp}(f) = \overline{\{x : f(x) \neq 0\}}$ is compact.

[7] **Motivation:** we want $\{f : \|f\|_{C^k} < \varepsilon\}$ to be open for all $k \geq 0, \varepsilon > 0$. Asking that f, g are close in C^∞ iff *all* their derivatives are within ε is too harsh as there would be very few C^∞-functions considered “close to” 0: essentially only Gaussian functions like $\varepsilon(1 - \exp(-1/x^2))$.

Proof.

- Metric: $d(f,g) = \max\limits_k \frac{2^{-k}\|f-g\|_{C^k}}{1+\|f-g\|_{C^k}}$
- Cauchy $\Rightarrow$ uniform convergence of any derivative (since $[0,1]$ compact)
  $\Rightarrow C^\infty$-convergence
  $\Rightarrow$ completeness.
- *Claim.* closed & bounded subsets F are compact

Pf. bounded means $\|f\|_{C^k} < K_k$ some constants K_k.
  $\Rightarrow D^{k-1}f$ equicts (by mean value theorem)
  $\Rightarrow$ a sequence in F has subseq f_n with $D^{k-1}f_n$ cgt in C^0 (by Arzela-Ascoli)
  $\Rightarrow$ a sequence in F has subseq f_n with $D^k f_n$ cgt in C^0, $\forall k$ (diagonal argument)
  $\Rightarrow$ a sequence in F has subseq f_n cgt in C^∞. ✓

- Suppose C^∞ is Banach ($= \exists$ norm inducing above topology). Then the closed unit ball is compact (by Claim) so C^∞ is finite dimensional (by Fact). Contradiction! $\square$

Completion. From a normed space $(V, \|\cdot\|_V)$ we can produce a Banach v.s. B:

$$B = \{\text{Cauchy sequences } x_n \in V\} / \begin{smallmatrix}(x_n)\sim(y_n) \text{ iff} \\ \|x_n - y_n\|_V \to 0\end{smallmatrix}$$

Define $\|x_n\|_B = \lim\limits_{n\to\infty} \|x_n\|_V$. Get dense isometric inclusion $V \to B, x \mapsto (x_n = x)$.

Examples.

- $L^p[0,1]$ = completion of $(C^0[0,1], \|\cdot\|_p)$ $\quad (1 \le p < \infty)$
- $W^{k,p}[0,1]$ = completion of $(C^k[0,1], \|\cdot\|_{k,p})$,

$$\|f\|_{k,p} = \sum_{0\le j\le k} \|D^j f\|_p.$$

Fact. *C^∞ is dense in C^0, C^k, so can replace C^0, C^k by C^∞ above.*

Rmk. *$p=2$ is particularly useful because it is Hilbert*[8] *using $\langle f,g\rangle_{L^2} = \int_0^1 f(x)g(x)\,dx$ on $L^2[0,1]$. Having a notion of perpendicularity goes a long way.*

Rmk. *Since C^∞ is not Banach, you must pass to $C^k, L^p, W^{k,p}$ to use big theorems (inverse fn thm, etc.). Big issue: how to recover C^∞ results e.g. from $C^k\,\forall k$?*

Trick: passing from $C^k, \forall k$ to C^∞: Suppose $\mathcal{M}_k \subset C^k = C^k[0,1]$ dense & open $\forall k$, and $\mathcal{M}_{k+1} = \mathcal{M}_k \cap C^{k+1}$. *Claim.* $\mathcal{M}_\infty = \cap \mathcal{M}_k \subset C^\infty$ dense & open.

Proof. • $f \in C^\infty \Rightarrow f \in C^k \Rightarrow \exists f_k \in \mathcal{M}_k : \|f_k - f\|_{C^k} < \varepsilon/2$
- $C^\infty \subset C^k$ dense $\Rightarrow \exists g_k \in C^\infty : \|f_k - g_k\|_{C^k} < \varepsilon/2$
- $\mathcal{M}_k \subset C^k$ open $\Rightarrow$ can ensure $g_k \in \mathcal{M}_k$, so $g_k \in \mathcal{M}_k \cap C^\infty = \mathcal{M}_\infty$

Conclusion: $\mathcal{M}_\infty \subset C^\infty$ is C^k-dense.
- $f \in C^\infty \Rightarrow \exists f_k \in \mathcal{M}_\infty : \|f - f_k\|_{C^k} < 1/k \Rightarrow f_k \to f$ in C^∞

Conclusion: $\mathcal{M}_\infty \subset C^\infty$ is C^∞-dense.
- $\mathcal{M}_\infty \subset C^\infty$ open since $\mathcal{M}_\infty = \mathcal{M}_k \cap C^\infty$ and $\mathcal{M}_k \subset C^k$ is C^k-open. $\square$

2.2. **Banach manifolds.**

Def. *A (smooth)* Banach mfd *X modeled on the Banach space B is a Hausdorff, second-countable*[9] *topological space together with:*

---

[8] Banach space with inner product such that $\|b\|^2 = \langle b, b\rangle$.

[9] *Second-countable*= there exists a countable basis for the topology. Consequence 1: any cover has a countable subcover. Consequence 2: it is *separable* = there is a countable dense subset.

- *open covering $X = \cup U_i$;*
- *charts: homeomorphisms $\varphi_i : U_i \to$ open $\subset B$;*
- *transitions: $\varphi_j \circ \varphi_i^{-1}$ are C^k-differentiable $\forall k$.*

If the transitions are only C^k, then it's a C^k-Banach mfd.

Rmk. *$\psi : B \to B$ is* differentiable *at x if $\exists$* ***bounded***[10] *linear $L : B \to B$ with $\psi(x+y) = \psi(x) + L \cdot y + o(\|y\|)$, where $o(t)$ is an error term satisfying* $\lim_{t\to 0} \frac{\mathcal{O}(t)}{t} = 0$.

Warning: (smooth) partitions of unity do not exist in general.

Non-examinable Example.
Claim. M, N closed mfds $\Rightarrow X = C^k(M,N)$ is a C^∞-Banach mfd.

- Avoiding charts: embed $N \subset \mathbb{R}^k$, pick a smooth retraction $\pi : U \to \mathbb{R}^k$ of an open tubular nbhd U of N. Observe: $X \subset C^k(M,U)$ is closed, $C^k(M,U) \subset C^k(M,\mathbb{R}^k)$ is open, $C^k(M,\mathbb{R}^k)$ is a Banach space. Since
$$\pi\circ : C^k(M,U) \to C^k(M,N) = X$$
is a smooth retraction, 1.6 Remark (2) $\Rightarrow X$ is a Banach submfd of $C^k(M,\mathbb{R}^k)$.

- Charts: we obtain nearby maps by the geodesic flow. Details: fix a Riemannian metric on N, fix a *smooth* $f : M \to N$. Then C^k-maps close to f are parametrized by[11] $C^k(D_\varepsilon f^*TN)$. Indeed the chart is $C^k(D_\varepsilon f^*TN) \to C^k(M,N), v \mapsto f_v$ where
$$f_v(p) = \exp_{f(p)} v(p).$$

One can[12] check $C^k(f^*TN)$ is a Banach space for the norm $\|v\|_{C^k} = \sum_{j\leq k} \sup |\nabla^j v(p)|$. This gives a local chart[13] near f. They cover X since smooth f are dense in X.

- Transitions: $f, g \in X$ smooth & close $\Rightarrow C^k(f^*TN) \to C^k(g^*TN)$ on overlap[14] is
$$T : v \mapsto \exp_{f(p)} v(p) \mapsto \exp_{g(p)}^{-1}(\exp_{f(p)} v(p))$$

Key observation: differentiating in v means $d_vT \cdot \vec{v} = \partial_s|_{s=0} T(v + s\vec{v})$, for $\vec{v} \in C^k(f^*TN)$. This involves derivatives of exp, g, f: these are all smooth![15]

- Second countable: any $f \in C^k(M,N)$ can be approximated by a smooth map, which can be extended to a smooth map $\mathbb{R}^l \to \mathbb{R}^k$ after embedding $M \subset \mathbb{R}^l, N \subset \mathbb{R}^k$. Now approximate that by a map $\mathbb{R}^l \to \mathbb{R}^k$ whose coordinates are polynomials in l variables.[16] Approximate those polynomials by polynomials with $\mathbb{Q}$ coefficients. Hence X can be covered by charts as above constructed for restrictions to M of such rational polynomial maps. That is a countable basis for the topology.

---

[10] *Bounded* means $\|L\| = \sup_{y\neq 0\in B} \frac{\|Ly\|}{\|y\|} < \infty$. Easy fact: L continuous $\Leftrightarrow$ L bounded.

[11] The pull-back f^*TN is just the bundle over M with $(f^*TN)_p = T_{f(p)}N$ at $p \in M$. So $v \in C^k(D_\varepsilon f^*TN)$ means $v(p) \in T_{f(p)}N$, $|v(p)| \leq \varepsilon$, and v is C^k.

[12] We'll see this norm in detail in future lectures. For now, think "manifold version" of $\|\cdot\|_{C^k}$.

[13] *Technical Rmk:* $B = C^k(f^*TN)$ is the same Banach space up to iso if you vary f by a smooth homotopy. If you change the homotopy class of f then B can change drastically. Indeed $C^k(M,N)$ is a disjoint union of connected components, each component is a Banach mfd consisting of homotopic maps. We are just using a different local model B for each component.

[14] If we wanted to use the same model $B = C^k(f^*TN)$, then we could always compose with the isomorphism induced by the diffeo $g^*TN \cong f^*TN$ obtained from the exponential map.

[15] Do not confuse this with the unrelated local derivatives $\partial_i v \in C^{k-1}$, which never arise!

[16] This can be done by the Stone-Weierstrass theorem.

Upshot: can now redo differential geometry: tangent vectors, derivative map, etc.

General principle: Results involving linear algebra or inverse function theorem generalize to Banach manifolds. But anything which involved compactness may fail unless you first reduce the problem to finite dimensional submfds/subspaces.

What works:

- Inverse function theorem (IFT) ✓
- Implicit function theorem ✓ provided $\ker d_p f$ has a *closed* complement. $(*)$

(locally $f : B_1 \to B_2$, want to apply IFT to $F : B_1 = \ker d_p f \oplus C \to B_2 \oplus C$, $F(x,y) = f(x,y) \oplus y$, so need $B_2 \oplus C$ Banach, so need C closed)

- Transversality definitions, all results which did not use Sard, assuming $(*)$ ✓

What fails:

- Sard's theorem: but this has a chance at working if the map is an isomorphism up to finite dimensional errors.

Def. *A map $f : M \to N$ between Banach mfds is* Fredholm *if $d_p f : T_p M \to T_{f(p)} N$ is a Fredholm operator, meaning $d_p f$ has finite dimensional kernel and cokernel.*

Basic example of Fredholm operators.

$B =$ all sequences of reals $(r_1, r_2, r_3, \ldots)$ with finite norm $\|r\| = \sum |r_i|$ (or $(\sum |r_i|^p)^{1/p}$)

$$\text{Right shift } R : (r_1, r_2, \ldots) \mapsto (0, r_1, r_2, \ldots).$$
$$\text{Left shift } L : (r_1, r_2, \ldots) \mapsto (r_2, r_3, \ldots).$$

These are Fredholm: $\ker R = 0$, $\operatorname{coker} R \cong \mathbb{R}$, $\ker L \cong \mathbb{R}$, $\operatorname{coker} L = 0$.

LECTURE 6.

PART III, MORSE HOMOLOGY, 2011

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2.3. **Fredholm theory.**

Def. *A bounded linear map* $L : A \to B$ *between Banach spaces is a* Fredholm operator *if* $\ker L$ *and* $\operatorname{coker} L$ *are finite dimensional.*

Def. *A map* $f : M \to N$ *between Banach mfds is a* Fredholm map *if* $d_p f : T_p M \to T_{f(p)} N$ *is a Fredholm operator.*

Basic Facts about Fredholm operators

1. $K = \ker L$ *has a closed complement* $A_0 \subset A$.
   (so the implicit function theorem applies to Fredholm maps).
   Pf. pick basis $v_1, \ldots, v_k$ of K, pick dual[1] $v_1^*, \ldots, v_k^* \in A^*$. $A_0 = \cap \ker v_i^*$. $\square$
2. $\operatorname{im}(L) = $ *image*$(L) \subset B$ *is closed.*
   (so $\operatorname{coker} L = B/\operatorname{im}(L)$ is Banach)
   Pf. pick complement C to $\operatorname{im}(L)$. C is finite dim'l, so closed, so Banach.
   $$\Rightarrow \mathcal{L} : A/K \oplus C \to B, L(\overline{a}, c) = La + c$$
   is a bounded linear bijection, hence an iso (open mapping theorem). So $\mathcal{L}(A/K) = \operatorname{Im}(L)$ is closed. $\square$
3. $A = A_0 \oplus K$, $B = B_0 \oplus C$ where $B_0 = \operatorname{im}(L)$, $C = $ complement ($\cong \operatorname{coker} L$).
   $$\Rightarrow L = \left[\begin{smallmatrix} \text{iso} & 0 \\ 0 & 0 \end{smallmatrix}\right] : A_0 \oplus K \to B_0 \oplus C$$
   Def. index$(L) = \dim \ker L - \dim \operatorname{coker} L$.
4. *Perturbing* L *preserves the Fredholm condition and the index:*
   Claim.[2] $s : A \to B$ bdd linear with small norm $\Rightarrow \exists$ "change of basis" isos
   $$\begin{matrix} i : A \cong B_0 \oplus K \\ j : B \cong B_0 \oplus C \end{matrix} \text{ such that } j \circ (L+s) \circ i = \left[\begin{smallmatrix} I & 0 \\ 0 & \ell \end{smallmatrix}\right]$$
   for some linear map $\ell : K \to C$. Note: $\dim \ker$ drops by $\operatorname{rank}(\ell)$, but also $\dim \operatorname{coker}$ drops by $\operatorname{rank}(\ell)$. So $\operatorname{index}(L) = \operatorname{index}(L+s)$.
   Proof. $s = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $L = \left[\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right]$ (where T is an iso). So:

$$\begin{aligned} \begin{bmatrix} I & 0 \\ -c(T+a)^{-1} & I \end{bmatrix} \cdot \left[\begin{smallmatrix} T+a & b \\ c & d \end{smallmatrix}\right] \cdot \begin{bmatrix} I & -(T+a)^{-1}b \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ -c(T+a)^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} T+a & 0 \\ c & -c(T+a)^{-1}b+d \end{bmatrix} \\ &= \begin{bmatrix} T+a & 0 \\ 0 & -c(T+a)^{-1}b+d \end{bmatrix} \end{aligned}$$

where we use that $(T+a)^{-1}$ is defined for small $\|s\|$:

$$(T+a)^{-1} = [T(I + T^{-1}a)]^{-1} = (I - T^{-1}a + (T^{-1}a)^2 - (T^{-1}a)^3 + \cdots)T^{-1}$$

that power series converges provided $\|T^{-1}a\| < 1$, which we guarantee by: $\|T^{-1}a\| < 1 \Leftarrow \|T^{-1}\| < \|a\|^{-1} \Leftarrow \|T^{-1}\| < \|s\|^{-1} \Leftarrow \|s\| < \|T^{-1}\|^{-1}$ (since $\|s\| \geq \|a\|$).

Date: May 3, 2011,

[1] $v_i^*(v_j) = \delta_{ij}$, the v_i^* exist by the Hahn-Banach theorem.

[2] Claim implies $\dim \ker L$ is upper semicontinuous: $\dim \ker(L+s) \leq \dim \ker L$, for small $\|s\|$.

Cor. *M connected, f Fred map $\Rightarrow$ index$(f) = \operatorname{index} d_p f$ is indep of $p \in M$.*

2.4. Sard-Smale Theorem.

$\boxed{f : M \to N \text{ smooth Fred map} \Rightarrow \{\textit{regular values of } f\} \subset N \text{ is a } \textit{Baire set.}}$

Baire set is a geometer's analogue of "full measure" or "generic" for Banach mfds.

Def. *$S \subset N$ is a* Baire set[3] *if S contains a countable intersection of open dense sets.*

Baire category thm. A Baire set in a complete metric space[4] is dense.

Proof of Sard-Smale.

Claim 1. $\exists$ charts $\substack{M \supset U \hookrightarrow A \cong B_0 \oplus K \\ N \supset V \hookrightarrow B \cong B_0 \oplus C}$ such that locally $f(b,k) = \left[\begin{smallmatrix} I & 0 \\ 0 & \ell(b,k) \end{smallmatrix}\right]$, for some nonlinear $\ell : B_0 \oplus K \to C$.

Pf. Centre the charts around $p \in U$, $f(p) \in V$. Take $K = \ker d_p f$, $C \cong \operatorname{coker} d_p f$. So $f : B_0 \oplus K \to B_0 \oplus C$, $f(0,0) = (0,0)$, $d_{(0,0)} f = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Implicit fn thm[5] $\Rightarrow$ (after a change of charts) $\boxed{f(b,k) = (b, \ell(b,k))}$ ✓

Claim 2. f is locally closed[6] (indeed closed in the above charts).

Pf. Suppose $f(b_n, k_n) \to (b,c)$, $(b_n, k_n) \subset$ bounded open $\subset B_0 \oplus K$. By Claim 1, $b_n \to b$. Now k_n bdd, K finite dim'l $\Rightarrow \exists$ cgt subseq $k_n \to k$. So $f(b,k) = (b,c)$ ✓

Claim 3. We can reduce to Sard's theorem:

From a cover by charts as above, pick[7] a countable subcover of M, so reduce to $f|_U : U \to V$. *Claim.* $V_{\text{reg}} = \{$*regular values of* $f|_U\} \subset V$ *is open and dense.*[8]

Pf. (critical points of $f|_U$) $\subset U$ is closed,[9] so by Claim 2, V_{reg} is open ✓

$$d_{(b,k)} f = \begin{bmatrix} I & 0 \\ * & d_{(b,k)}\ell|_K \end{bmatrix} \text{ surjective } \Leftrightarrow d_{(b,k)}\ell|_K \text{ surjective}$$

Note: $d_{(b,k)}\ell|_K = d_k(\ell_b)$ for $\ell_b : K \to C, k \mapsto \ell(b,k) \leftarrow$ map of finite dim'l spaces!
$\Rightarrow V_{\text{reg}} \cap (\{b\} \oplus C) = (\text{reg. val's of } \ell_b) \subset \{b\} \oplus C \leftarrow$ dense inclusion by Sard!
$\Rightarrow V_{\text{reg}} \subset V$ dense ✓ ☐

---

[3] or *generic set*, or *residual set*. We often produce $S =$ countable intersection of dense opens.

[4] Banach mfds are (complete) metric spaces. *Non-examinable proof:* Urysohn's metrization theorem says every second-countable regular space is metrizable. Banach mfds are by definition second-countable. *Regular space* means given a point p not contained in a closed subset C, there exist disjoint open nbhds of p and C (for Banach mfds, take a chart centred at p, then consider the ε-radius open ball centre p and the complement of the 2ε-radius closed ball centre p).

[5] $f(b,k) = (\alpha(b,k), \beta(b,k))$. Inverse fn thm $\Rightarrow \exists$ local inverse to $h : B_0 \oplus K \to B_0 \oplus K$, $h(b,k) = (\alpha(b,k), k)$ near $(0,0)$. Hence $f \circ h^{-1}(b,k) = (b, \ell(b,k))$. ☐

[6] locally, closed sets map to closed sets.

[7] Banach mfds are defined to be second-countable, hence Lindelöf (covers have ctble subcovers). *Non-examinable remark:* I want Banach mfds to be metric spaces (see footnote 4). For metric spaces: second-countable $\Leftrightarrow$ separable $\Leftrightarrow$ Lindelöf. As far as I know, if I replace second-countable by separable, then it's not clear Banach mfds are metric, so it's not clear Baire category applies.

[8] So the regular values of f is the intersection of the regular values of all $f|U$'s, so it's a countable intersection of open dense sets, as required.

[9] Regular points of any smooth map of Banach mfds form an open set: at regular p, $d_p f = [I\ 0]$ (after change of basis), so for q close to p, $d_q f = [T\ *]$ for some invertible T since invertibility is an open condition (which is proved by the power series argument as in (4) of 2.3).

Thm. *If $f : M \to N$ is a Fredholm C^k-map of C^k-Banach mfds, then Sard-Smale holds provided $k >$ index(f).*

Proof. $l_b : K \to C$, $\text{index}(f) = \dim K - \dim C$, now use C^k-Sard (see 1.3). ☐

Cor. *$F : M \times S \to N$ smooth map of Banach mfds, $Q \subset N$ submfd, $F \pitchfork Q$, such that $D_m F_s : T_m M \to \nu_{Q,F_s(m)}$ is Fredholm. Then $F_s \pitchfork Q$ for generic $s \in S$.*

Proof. Parametric transversality 1 & 2 (using Sard-Smale and Hwk 6). ☐

2.5. Zero sets of Fredholm sections.

Def. Banach vector bundle $\pi : E \to B$ *with fibre V, is defined analogously to finite dimensional vector bundles after replacing E, B by Banach mfds, and V by a Banach space.*

Thm. *For a Banach vector bundle $E \to M \times S$ and a smooth section $F : M \times S \to E$, assume for all (m, s) with $F(m, s) = 0$ that*

(1) $D_{(m,s)}F : T_{(m,s)}(M \times S) \xrightarrow{dF} T_{(m,s,0)}E \to E_{(m,s)}$ *is surjective*
(2) $D_m F_s : T_m M \to E_{(m,s)}$ *Fredholm*

Then, for generic $s \in S$,

$$\begin{cases} F_s^{-1}(0_E) \subset M \textit{ submfd of } \dim = \text{index}\,(D_m F_s) \quad (\textit{near } m) \\ T_m F_s^{-1}(0_E) = \ker(D_m F_s : T_m M \xrightarrow{\text{surj}} E_{(m,s)}) \end{cases}$$

Proof. This is a direct consequence of the Corollary, but since it's important:
(1) $\Rightarrow F \pitchfork 0_E \Rightarrow W = F^{-1}(0)$ mfd (implicit fn thm[10]).
Write $\pi : M \times S \to S$ for the projection, recall parametric transversality 2:

$$\ker d\pi|_W \cong \ker DF_s \qquad \text{coker } d\pi|_W \cong \text{coker } DF_s.$$

(2) $\Rightarrow d\pi|_W$ Fredholm of index = index DF_s.
Sard-Smale $\Rightarrow$ for generic s, $d\pi|_W$ is surjective along

$$\pi|_W^{-1}(s) = W \cap \pi^{-1}(s) = F_s^{-1}(0).$$

Hence DF_s is surjective (by the iso of cokernels above). So $F_s^{-1}(0)$ mfd and

$$TF_s^{-1}(0) \cong \ker DF_s$$

with $\dim T_m F_s^{-1}(0) = \dim \ker D_m F_s =$ index $D_m F_s$ (since $\text{coker}\, D_m F_s = 0$). ☐

Thm. *Thm also holds for C^k-maps of C^k-Banach mfds when $k >$ index DF_s.*

Rmk. *The dimension of $F_s^{-1}(0_E)$ can vary depending on the connected component, since index$(D_m F_s)$ depends on the connected component of m. That is why we wrote "near m" in the Thm.*

---

[10] Hwk 6 checks the closed complement condition. You should check that Cor 1.2 (implicit function theorem) works also for Banach mfds.

LECTURE 7.

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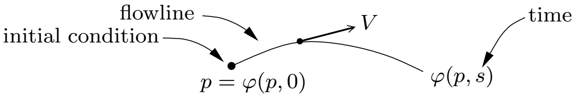
Motivation. Morse theory = studying the space of $\{-\nabla f$ flowlines $\}$.

3. Flowlines and Topology

3.1. **Flowlines.** M closed mfd, V smooth vector field.



Thm.
There exists a unique solution $\varphi : M \times \mathbb{R} \to M$ *of*

$$\frac{\partial \varphi}{\partial s} = V \circ \varphi \qquad\qquad \varphi(\cdot, 0) = id.$$

$\varphi_s = \varphi(\cdot, s) : M \to M$ *is a diffeo, with* $\varphi_s \circ \varphi_t = \varphi_{s+t}$.

Def. φ *is the* flow *of* V, *and* $s \mapsto \varphi(p, s)$ *is the* flowline *through* p.
By uniqueness, flowlines never intersect unless they coincide (*up to* $s \mapsto s + const$).

Proof. Locally: $y : [-\varepsilon, \varepsilon] \to \mathbb{R}^m$, $y(s) = \varphi(p, s)$ solves the ODE

$$y'(s) = V(y(s)) \qquad\qquad y(0) = p.$$

ODE theory[1] $\Rightarrow$ for small $\varepsilon > 0$, $\exists$ unique solution y which depends smoothly on the initial condition p.

Globally: $\forall p \in M$, local result yields a unique smooth map[2]

$$\varphi : U_p \times [-\varepsilon_p, \varepsilon_p] \to M \qquad\qquad (*)$$

Take a finite cover of M by U_p's, and ε = smallest of the ε_p's. So:

$$\varphi : M \times [-\varepsilon, \varepsilon] \to M \qquad\qquad (**)$$

Trick: $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for small s, t, since both solve $y(0) = \varphi_t(p)$, $y'(s) = V(y(s))$.
$\Rightarrow \varphi_s$ diffeo with inverse φ_{-s}
$\Rightarrow$ extend[3] φ_s to $s \in \mathbb{R}$: $\varphi_s = \varphi_{s/k} \circ \cdots \circ \varphi_{s/k}$
(k composites, with $k \gg 0$ so that $|s/k| < \varepsilon$) □

Rmk. *ODE theory* $\Rightarrow$ *If* V *is* C^k *then* φ *is* C^k.
If M *is just a* C^k*-mfd and*[4] V *is* C^{k-1} *then* φ *is* C^{k-1}.

Rmk.

Date: May 1, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1] Lang, *Undergraduate Analysis*, or Lang, *Differential Manifolds*, prove this in great detail.

[2] pick the open nbhd U_p of p small enough, so that φ lands in the given chart.

[3] φ_s is well-defined, indeed: $(\varphi_{s/k'})^{k'} = (\varphi_{s/kk'})^{kk'} = (\varphi_{s/k})^k$.

[4] Since TM is just C^{k-1}, any higher differentiability of vector fields does not make sense.

1. *If V_s depends on s, you pass to $M \times \mathbb{R}$, $V(p,s) = V_s(p) \oplus \frac{\partial}{\partial s}$. Since the mfd is now non-compact, $(*)$ holds but $(**)$ can fail (if $\varepsilon_{(p,s)} \to 0$ as $|s| \to \infty$).* ***Fact.*** *If V is C^1-bounded then $(**)$ holds.*[5] *So Thm holds.*
2. *For non-compact mfd M or Banach mfd M, $(*)$ holds by the same proof, but for $(**)$ we need the condition:* $\exists K, R > 0$ such that $\forall p \in M$, $\exists$ chart $\varphi_p : U_p \to \mathbb{R}^m$ or B, such that V is C^1-bounded by K in chart and[6] $\varphi_p(U_p) \supset$ ball with centre $\varphi_p(p)$ radius R.

3.2. Negative gradient flowlines.

(M, g) closed Riemannian mfd. Write $|v| = g(v,v)^{1/2}$ for the norm.
$f : M \to \mathbb{R}$ smooth function.

Def. *The* gradient vector field ∇f *is defined by*

$$g(\nabla f, \cdot) = df$$

Locally:[7] $\nabla f = g^{-1}\partial f = \sum \partial_i f \cdot g^{ij} \cdot \partial_j$.

Rmk. $p \in Crit(f) \Leftrightarrow d_p f = 0 \Leftrightarrow (\nabla f)_p = 0 \Leftrightarrow |\nabla f|_p = 0$

For a $-\nabla f$ flowline $u : [a,b] \to M$ (so $u' = -\nabla f$) we care how f varies along u:

$$\partial_s(f \circ u) = df \cdot u' = df(-\nabla f) = g(\nabla f, -\nabla f) = -|\nabla f|^2$$
$$f(u(b)) - f(u(a)) = \int_a^b \partial_s(f \circ u)\, ds = -\int_a^b |(\nabla f)_{u(s)}|^2\, ds \leq 0$$

Def. *So it is natural to introduce the notion of* Energy *of a path $u : (a,b) \to M$:*

$$E(u) = \int_a^b |(\nabla f)_{u(s)}|^2\, ds \geq 0.$$

Note that $E(u) = 0$ iff u is constantly equal to a critical point.

Cor. *f decreases along $-\nabla f$ flowlines, and there is an a priori energy estimate:*[8] *for any $-\nabla f$ flowline from x to y,*

$$E(u) = f(x) - f(y).$$

In particular, $E(u)$ is a homotopy invariant relative to the ends.

Rmk (Novikov theory). *A generalization of Morse theory, called* Novikov theory, *replaces df by a closed 1-form α. This gives rise to a vector field via $g(V, \cdot) = \alpha$, and one studies $-V$ flowlines. The energy $E(u) = \int_a^b |V_{u(s)}|^2\, ds \geq 0$ is zero iff u is constantly equal to a zero of α. There is no a priori energy estimate. However, $E(u)$ is still a homotopy invariant of $-V$ flowlines relative to the ends. Indeed:*

$$E(u) = -\int_{[a,b]} u^*\alpha$$

---

[5] Why C^1? Locally it implies V is Lipschitz by the mean value theorem, which is what's needed to solve the ODE. C^1 bounds guarantee the Lipschitz constant is bounded uniformly.

[6] The C^1 bounds are calculated in a chart, but they can always be achieved by rescaling a chart. So the second condition is crucial (for example: consider $V = \frac{\partial}{\partial x}$ on $\mathbb{R} \setminus \{0\}$).

[7] g^{ij} = inverse matrix of $g_{ij} = g(\partial_i, \partial_j)$, $\partial_j = \frac{\partial}{\partial x_j}$.

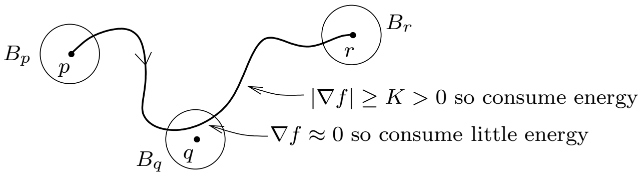
[8] *a priori* refers to the fact that the estimate only depends on boundary conditions x, y, not u.

since $u^*\alpha = \alpha \cdot u' = -\alpha(V) = -g(V,V) = -|V|^2$.
Proof: if $H : [a,b] \times [0,1] \to M$ *is a homotopy relative ends*[9]*, by Stokes's theorem*

$$\int_{[a,b]} u_0^*\alpha - \int_{[a,b]} u_1^*\alpha = \int_{[a,b]\times[0,1]} dH^*\alpha = 0$$

($dH^*\alpha = H^*d\alpha = 0$, *since* α *is closed). For* $\alpha = df$ *the energy estimate is Stokes:* $E(u) = -\int_{[a,b]} u^*df = -\int_{[a,b]} d(u^*f) = f(u(a)) - f(u(b))$ *for* $-\nabla f$ *flowlines* u.

3.3. **Energy consumption.**



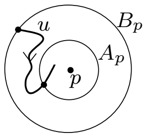
Lemma. *A* $-\nabla f$ *flowline from* x *to* y *landing in a region where* $|\nabla f| \geq K > 0$ *has*

$$E(u) \geq K \cdot \mathrm{dist}(x,y).$$

Pf.[10] Loosely:[11] $E(u) = \int |\nabla f|^2 \geq K \int |\nabla f| = K \int |u'| = K\,\mathrm{length}(u) \geq K\mathrm{dist}(x,y)$.

For example, this proof shows that: in the complement of small balls centred at the critical points of a Morse function f, any $-\nabla f$ flowline must consume at least some fixed amount $\delta > 0$ of energy to flow from one ball to another.

Notation. $A \Subset B$ (*compactly contained*) means: A, B open, and $A \subset \overline{A} \subset B$.



No escape Lemma. *Let* $p \in A_p \Subset B_p$ *with* $\overline{B_p} \cap Crit(f) = \{p\}$. *Then* $\exists \delta > 0$ *such that any* $-\nabla f$ *flowline needs* $E \geq \delta$ *to go from* ∂A_p *to* ∂B_p*, or vice-versa.*

Proof. Consider the region $\overline{B_p} \setminus A_p$, apply the Lemma. □

Energy quantum Lemma.[12] Pick disjoint nbhds $\overline{B_r}$ of each $r \in \mathrm{Crit}(f)$. $\exists \delta > 0$ such that any $-\nabla f$ flowline from B_p to B_q for $p \neq q$ consumes energy $E \geq \delta$.

Proof. Consider the region $M \setminus \cup \overline{B_r}$, apply the Lemma. □

---

[9] $H(a,\cdot) = x$, $H(b,\cdot) = y$, $H(0,\cdot) = u_0$, $H(1,\cdot) = u_1$.

[10] This proof works similarly if we use α instead of df (see previous Rmk).

[11] *Distance* $\mathrm{dist}(x,y)$ = infimum of lenth$(u) = \int |u'(s)|\, ds$ over all curves u from x to y. **Rmk.** length is parametriz'n indep: $\int |(u \circ \phi)'(s))|\, ds = \int |u'(\phi(s))|\, \phi'(s)\, ds = \int |u'(s)|\, ds$ $(\phi'(s) > 0)$.

[12] Note: you could have $f(p) = f(q)$, so the energy estimate doesn't imply result immediately.

3.4. **Convergence at the ends.**

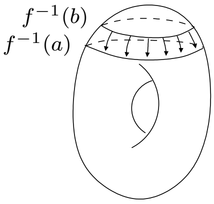
Thm. *For* $f : M \to \mathbb{R}$ *Morse,* M *closed mfd, any* $-\nabla f$ *flowline* $u : \mathbb{R} \to M$ *must converge at the ends to critical points, hence* $\exists p, q \in Crit(f)$ *with*

$$u \in W(p,q) = \{-\nabla f \text{ flowlines } \mathbb{R} \to M \text{ converging to } p,q \text{ at } -\infty,\infty\}.$$

In particular, $W(p,p) = \{\text{constant flowline at } p\}$*, since there* $E = f(p) - f(p) = 0$*.*

Proof. [13] Case $s \to +\infty$ (for $s \to -\infty$ apply proof to $-f$). No Escape Lemma: for each $p \in \mathrm{Crit}(f)$ pick A_p, B_p's (small), get $\delta > 0$. $f \circ u$ decreases in s, but f is bounded (M compact), so $f \circ u \to r \in \mathbb{R}$, so for $s \gg 0$, $f \circ u$ is within δ of r. Suppose $u \notin \cup B_p$ for some $s \gg 0$. Then u hasn't enough energy left to reach $\cup A_p$ for larger s. So $u \notin \cup A_p$ for $s \gg 0$. But $|\nabla f| \geq K > 0$ on $M \setminus \cup A_p$, so $f \circ u \to -\infty$, absurd. So $u \in \cup B_p$ for $s \gg 0$, and B_p is arbitrarily small. $\square$

3.5. **Topology of sublevel sets.** $M_a = \{x \in M : f(x) \leq a\}$ are the *sublevel sets.*



Thm. *If* $[a,b]$ *contains no critical values of* f*, then* $M_b \cong M_a$ *are diffeo.*

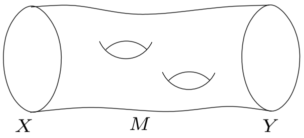
Proof. $\varphi =$ flow of $-\frac{\beta(f)}{|\nabla f|^2} \cdot \nabla f$, where $\beta : \mathbb{R} \to [0,1]$ is a bump function, $\beta = 1$ on $[a,b]$ and $\beta = 0$ on away from $[a,b]$ (in particular $\beta = 0$ at all critical values of f).

$$\partial_s(f \circ \varphi) = df \cdot \partial_s \varphi = g(\nabla f, -\tfrac{\beta(f)\nabla f}{|\nabla f|^2}) = -\beta(f)$$

which equals -1 on $[a,b]$. So $\varphi(\cdot, b-a) : M_b \to M_a$ diffeo. $\square$

Rmk. *The* $-\nabla f$ *flowlines are orthogonal to the regular level sets* $f^{-1}(a)$*.*
Pf. $g(-\nabla f, v) = -df \cdot v = 0$ *for* $v \in Tf^{-1}(a) = \ker df|_{f^{-1}(a)}$*.*

Rmk. *There is a deformation retraction*[14] *of* M_b *onto* M_a*:* $r : M_b \times [0,1] \to M_b$*,* $r(x,s) = x$ *if* $x \in M_a$*, and* $r(x,s) = \varphi(x, s(f(x)-a))$ *if* $x \in f^{-1}[a,b]$*.*



Def. *A* cobordism *between possibly-disconnected closed* X^n, Y^n *is a compact mfd* M^{n+1} *with* $\partial M = X \sqcup Y$*. Call it* h-cobordism *if in addition* X, Y *are deformation retracts of* M*.*

Fact.[15] *Equivalent definitions of h-cobordism:*

M *h-cobordism* $\Leftrightarrow$ ($X, Y \hookrightarrow M$ *hpy equivalences*) $\Leftrightarrow$ ($\pi_*(M,X) = \pi_*(M,Y) = 0$)

$\boxed{\text{For } X, Y, M \text{ simply connected: } (M \text{ h-cobordism }) \Leftrightarrow (H_*(M,X) = 0)}$

---

[13] Curiosity: $\exists$non-insightful elementary proof by contradiction, avoiding energy arguments.

[14] *Deformation retraction* $r : X \times [0,1] \to X$ of X onto A means: r cts, $r|_A =$id, $r(X,1) = A$. Note r is a hpy from id_X to a retraction $r_1 = r(\cdot,1)$ of X onto A (means $r_1(X) = A$, $r_1|_A = \mathrm{id}_A$).

[15] *Non-examinable:* By Whitehead's theorem and LES for relative hpy: inclusions $X, Y \hookrightarrow M$ are hpy equivalences $\Leftrightarrow$ they are isos on hpy gps $\Leftrightarrow \pi_*(M,X) = \pi_*(M,Y) = 0$. By hpy theory:[16] M deform retracts onto $X \Leftrightarrow \pi_*(M,X) = 0$. The 2nd fact uses Hurewicz: if $\pi_1(X) = 0$, $\pi_1(M,X) = 0$ then the first non-zero $\pi_k(M,X)$ is iso to the first non-zero $H_k(M,X)$; and it uses the Poincaré duality iso $H_*(M,X) \cong H^{m-*}(M,Y)$ (by universal coefficients, $H_*(M,Y) = 0 \Leftrightarrow H^*(M,Y) = 0$).

[16] Hilton, *An Introduction to Homotopy Theory*, Thm 1.7 p.98: if $\pi_*(Y,Y_0) = 0$, then $\forall$ subcx A of a CW cx X, any $(X,A) \to (Y,Y_0)$ can be homotoped to $X \to Y_0$ keeping it constant on A.

h-cobordism Thm[17] (Smale 1962) *If X, Y are simply connected of dim ≥ 5 then the h-cobordism M is trivial: $M \cong Y \times [0,1]$ diffeo. In particular $X \cong Y$ diffeo.*

Lemma. *If there exists a Morse function $f : M \to [a,b]$ with no critical points, $X = f^{-1}(a)$, $Y = f^{-1}(b)$, then M is a trivial h-cobordism.*

Proof. In notation of previous Rmk: $f^{-1}(b) \times [0,1] \to M, (x,s) \mapsto \varphi(x, s(b-a))$. $\square$

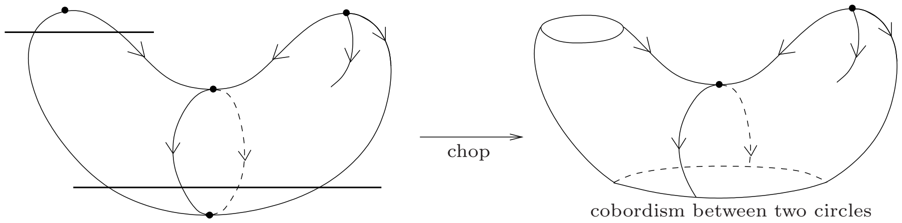
Cor. *An h-cobordism is trivial $\Leftrightarrow$ it admits a Morse function as in the Lemma.*

Idea of Pf of Thm.(hard!) Start with a Morse function on the cobordism. Systematically "cancel out" the crit points in pairs by locally modifying f and the flow, until there are no crit points left. Key: the use of ***gradient-like vector fields***: $V \in C^\infty(TM)$, $V(f) > 0$ (except at Crit(f)), such that at each $p \in$ Crit(f), $\exists$ Morse chart in which

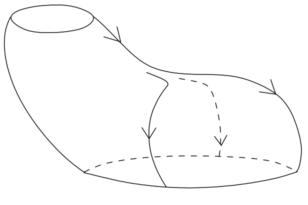
$$\begin{aligned} f(x) &= f(p) - x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_m^2 \\ V &= -x_1\partial_1 - \ldots - x_i\partial_i + x_{i+1}\partial_{i+1} + \ldots + x_m\partial_m. \end{aligned}$$

The flow you consider is the flow of $-V$, not that of $-\nabla f$. This means you know exactly what v is near critical points: up to a constant, it is the Euclidean gradient in the Morse chart (whereas $-\nabla f$ is unknown since the metric in general is not Euclidean in the Morse chart!). The $V(f) > 0$ ensures that f still decreases along $-V$ flowlines, and you get good estimates of the energy $E = \int |V|^2$.

The idea behind cancelling out critical points in pairs is as follows. Consider the sphere with two discs cut out:



You would like to push down one of the hills:



In general, you do not know what the manifold looks like globally, so you actually just modify f, v locally near the trail going up the hill:[18]

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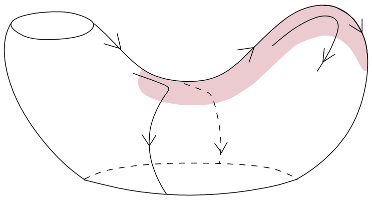
[17] Standard great reference: Milnor, *Lectures on the h-cobordism theorem.*

[18] in the figure, we modify f, v in the shaded region.



Thus cancelling out the two critical points with index difference 1.

LECTURE 8.

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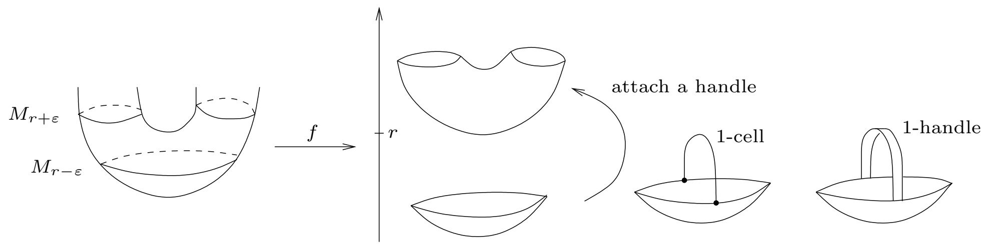
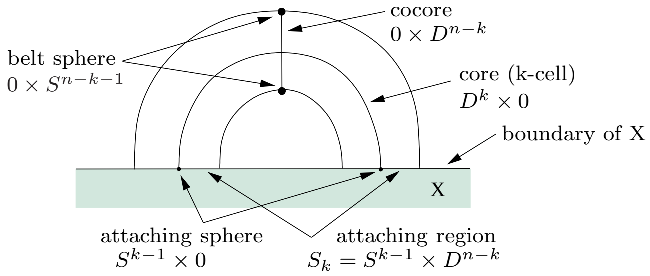


3.6. **Handle attachments.**

Def. *A k-*handle *is a "thickened up" k-cell:*[1] $D^k \times D^{n-k}$

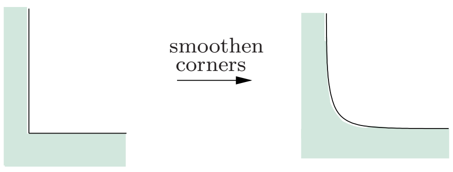


Attaching a k-handle *to X^n means:*

$$X \cup_{S_k} (D^k \times D^{n-k})$$

where the handle is attached along the attaching region $S_k = S^{k-1} \times D^{n-k}$ *via an embedding $S_k \hookrightarrow \partial X$ which needs to be specified.*

Rmk. *One can smoothen the corners of the attachment, but we omit the details.*



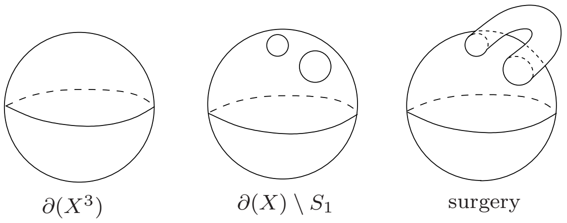
Date: May 3, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1] where D^k is the closed unit disk in $\mathbb{R}^k$.

Rmk. *The boundary becomes:*

$$(\partial X \setminus \operatorname{int} S_k) \cup_{\partial S_k} (D^k \times S^{n-k-1})$$

called a surgery on $S_k \hookrightarrow \partial X$ *(or* surgery on ∂X*). Equivalently, it is a surgery on the knot* $S^{k-1} = S^{k-1} \times 0 \overset{\varphi}{\hookrightarrow} \partial X$ *with framing*[2]*. Note that this attachment kills the homotopy class of* $S_k \hookrightarrow \partial X$*. Example: surgery on a 3-sphere with a 1-handle:*



3.7. **Topology of sublevel sets II.**

Thm (Handle-attaching theorem)**.** *M^m closed, $f : M \to \mathbb{R}$ Morse.*

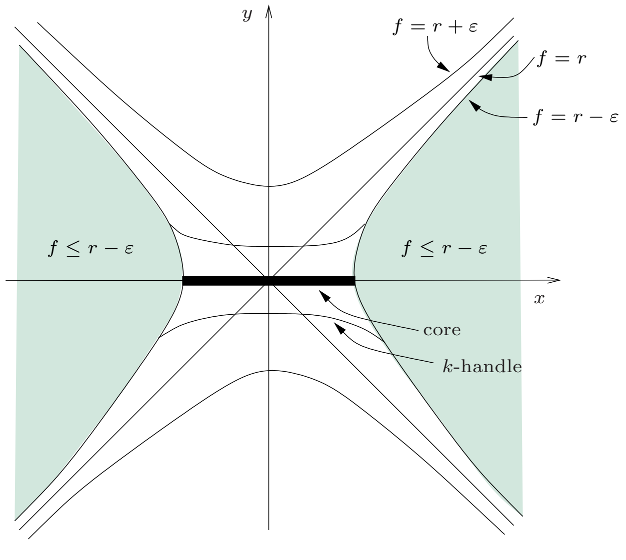
$$\begin{array}{c} f \text{ has exactly one critical point} \\ p \in f^{-1}[a,b] \text{ of index } k \end{array} \Rightarrow M_b \simeq M_a \cup k\text{-cell (hpy equivalence)}$$

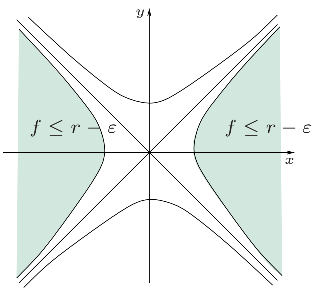
Rmk. *If there are several critical points on the same level set, then the proof still works (attach several cells).*

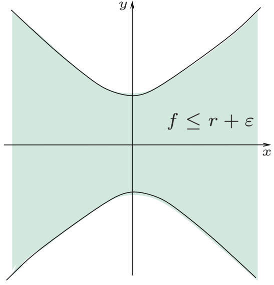
Proof. Let $r = f(p)$. By Theorem 3.5, we just need to show

$$M_{r+\varepsilon} \simeq M_{r-\varepsilon} \cup k\text{-cell} \qquad (\text{for small } \varepsilon > 0)$$

Pick Morse chart near p: $\boxed{f(x,y) = r - |x|^2 + |y|^2,\ (x,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k}}$.







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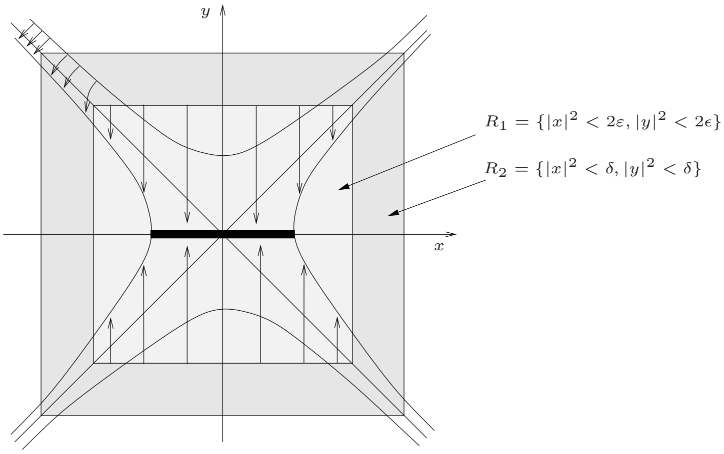
[2] framing = a given trivialization of the normal bundle $\nu_{\mathrm{im}(\varphi)} \cong S^{k-1} \times \mathbb{R}^{n-k}$. You demand the normal bundle to be trivial, since you want to extend the knot embedding to an embedding $S_k = S^{k-1} \times D^{n-k} \hookrightarrow \partial X$.

Aim: $M_{r+\varepsilon}$ deformation retracts onto $M_{r-\varepsilon} \cup$ core, where

$$\begin{aligned} \text{core} &= \{(x,0) : f(x,0) = r - |x|^2 \geq r - \varepsilon\} \\ &= \{(x,0) : |x|^2 \leq \varepsilon\} = k\text{-cell}. \end{aligned}$$

We will define two regions, R_1, R_2:



Aim: the deformation is the flow of a vector field v which is $\begin{cases} -\nabla f{=}(2x,-2y) \text{ outside } R_2 \\ \text{vertical } {=}(0,-2y) \text{ in } R_1 \end{cases}$
Pick $0 < \delta < 1$ so that $R_2 \subset$ Morse chart.
Pick a Riemannian metric on M which equals[3] the Euclidean metric on R_2.
Pick $\varepsilon < \delta/2$. Define

$$\boxed{v = \begin{cases} (2\beta(x,y)\cdot x, -2y) \text{ on } R_2 \\ -\nabla f \text{ outside } R_2 \end{cases}}$$

with $\beta : \mathbb{R}^m \to [0,1]$ equal to 0 on R_1 and 1 outside R_2. Let

$$\begin{aligned} \varphi(t,z) \quad = \quad & \text{flow of } \tfrac{v}{|v|} \text{ for time } t \text{ starting from } z \in M_{r+\varepsilon}, \text{ where} \\ & 0 \leq t \leq T(z) = \text{ time required to reach } M_{r-\varepsilon} \cup \text{ core.} \end{aligned}$$

Note: if $z \in M_{r-\varepsilon}$ then $T(z) = 0, \varphi(0,z) = z$. Moreover, $T(z)$ is finite since:[4]

$$df \cdot \frac{v}{|v|} = \begin{cases} -\|\nabla f\| \text{ outside } R_2 \\ (-2xdx + 2ydy)(2\beta x\partial_x - 2y\partial_y) = -4\beta|x|^2 - 4|y|^2 \leq -4|y|^2 \text{ in } R_2 \end{cases}$$

(so you can't stop flowing before reaching $M_{r-\varepsilon}$ or $y = 0$).

Finally, observe that $T(z)$ is continuous. Conclusion:
$(t,z) \mapsto \varphi(t \cdot T(z), z)$, for $t \in [0,1]$, deform retracts $M_{r+\varepsilon}$ onto $M_{r-\varepsilon} \cup k$-cell. □

Rmk. *The proof also shows:*

- M_b, M_r *deform retract onto* $M_a \cup k$*-cell*
- $M_b \cong M_a \cup k$*-handle are diffeomorphic*
- $f^{-1}(b) =$ *surgery on* $f^{-1}(a)$.

Def. *Recall a* CW structure *on* M *are subsets* $M_0 \subset M_1 \subset \ldots \subset M_n = M$ *where* M_0 *is a discrete set of points, and*

$$M_k = M_{k-1} \cup D_1^k \cup D_2^k \cup \cdots$$

---

[3] recall metrics can be patched together by a partition of unity argument.

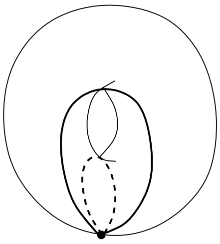
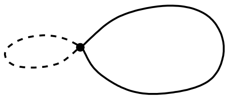
[4] abbreviating $xdx = \sum x_i dx_i$, etc.

is obtained by attaching k-cells via continuous maps $f_i^k : \partial D_i^k \to M_{k-1}$.

Cor. *M closed, $f : M \to \mathbb{R}$ Morse. Then M has a CW structure with a k-cell attached for each critical point of f of index k, and $M_0 =$ minima of f.*

Proof. For a *self-indexing Morse function*[5] the result is immediate from the Theorem (consider M_n by induction on n). So the result follows by Hwk 9: one can modify a Morse f to make it self-indexing without changing $\mathrm{Crit}(f)$ and its indices.

Alternative: use hpy theory (cellular approximation theorem[6]) to overcome the issue that the Thm is not attaching cells ordered by dimension.[7] □

Torus =  =  ∪ 2-cell

= a figure eight with a 2-cell attached.

Rmk. *Provided $M_r = (f \leq r)$ is compact, the Thm holds for non-compact M (for the Corollary you need a little care using homotopy theory*[7]*).*

3.8. **Stable and unstable manifolds.** Let φ be the flow of a vector field v on M.

Def. *For $v(p) = 0$,*

$$\begin{aligned} \textit{unstable mfd} \quad W^u(p,v) &= \{x \in M : \lim_{t\to-\infty} \varphi(x,t) = p\} \\ \textit{stable mfd} \quad W^s(p,v) &= \{x \in M : \lim_{t\to+\infty} \varphi(x,t) = p\} \end{aligned}$$

Rmk. *This of these as the points flowing "out of"/"into" p. They both contain p since $\varphi(p,t) = p$. For nasty v the W^u, W^s may not be manifolds.*

Def. *For $v = -\nabla f$ we abbreviate $W^u(p) = W^u(p, -\nabla f), W^s(p) = W^s(p, -\nabla f)$, which are often called* descending mfd $\mathcal{U}(p)$ *and* ascending mfd $\mathcal{A}(p)$ *(thinking of f as a "height" function).*

Example. In the pf of Thm 3.7, $-\nabla f = (-2x, 2y)$ near p (Euclidean metric) so

$$W^u(p) = x\text{-plane} \qquad W^s(p) = y\text{-plane} \qquad (\text{near } p)$$

so $\boxed{M_{r+\varepsilon} = M_{r-\varepsilon} \cup W^u(p)}$ and infinitesimally: $\boxed{T_pM = T_pW^u(p) \oplus T_pW^s(p)}$ $(*)$

Observe $W^u(p)$ is a mfd in a nbhd U of p, so it is globally a mfd since

$$W^u(p) = \bigcup_{t\geq 0} \varphi(W^u(p) \cap U, t),$$

so $\boxed{W^u(p) \cong \mathbb{R}^{\mathrm{index}(p)}}$ (exercise). Similarly, $W^s(p)$ is[8] a submfd $\mathbb{R}^{m-\mathrm{index}(p)} \hookrightarrow M$.

---

[5] meaning the index $|p| = f(p)$.

[6] Maps $S^n \to M$ can be homotoped to maps $S^n \to M_n \subset M$, which is an immediate consequence of Sard's theorem: first smoothen the map, then by dimensions the map cannot surject onto the higher discs D_i^m for $m > n$, so you can homotope it to avoid those discs.

[7] For details, see Milnor's *Morse theory* book.

[8] $W^u(p), W^s(p)$ are embedded submfds, but the embedding is not proper.

Thm (Hadamard-Perron & Hartman-Grobman).

$W^u(p,v), W^s(p,v)$ are "mfds" for $\left\{ \begin{array}{l} v \;=\; -\nabla f \quad (f \textit{ Morse, for any metric}) \\ v \;=\; \textit{vector field with hyperbolic fixed points}^{9} \end{array} \right.$

and $(*)$ holds.

More precisely, respectively $\left\{ \begin{array}{l} W^u(p,-\nabla f), W^s(p,-\nabla f) \textit{ are embedded planes,} \\ W^u(p,v), W^s(p,v) \textit{ are injectively immersed planes.} \end{array} \right.$

Rmk. *For general hyperbolic v a point may return arbitrarily close to itself under the flow: for example, for v you may have non-constant v-flowlines converging to p at both ends. This does not happen for $-\nabla f$ because f decreases along the flow.* Non-examinable: prove that $-\nabla f$ has hyperbolic fixed points, so the first case is a particularly well-behaved special case of the second.

Rmk. *The proof is hard. If you are* given *a metric, then you cannot hope that in a Morse chart the metric is Euclidean (e.g. consider a metric which is not flat near p). Therefore you do not know what $-\nabla f$ looks like near the critical points, so even the infinitesimal result (do $TW^u(p), TW^s(p)$ exist?) is hard.*

Two tricks to avoid the Theorem.

(1) change the metric to make it Euclidean near p via a hpy

$$g_t = (1-t)\cdot g + t\cdot((1-\beta)\cdot g + \beta \cdot g_{\text{Euclidean}})$$

using a bump function $\beta = 1$ near p (compare pf of Thm 3.7).

(2) don't use ∇f, and instead use a *gradient-like vector field* v (see Hwk 7):

$v(f) > 0$ except at $\mathrm{Crit}\, f$, $v = \nabla_{\text{Euclidean}} f$ in a Morse chart near $p \in \mathrm{Crit}\, f$.

---

[9] meaning, if $v(p) = 0$, then $d\varphi(p,t)$ has no evalues λ with $|\lambda| = 1$ $\forall t \neq 0$.

LECTURE 9.

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Motivation. Denote the (parametrized) $-\nabla f$ flowlines from p to q by:

$$\boxed{W(p,q) = \{u : \mathbb{R} \to M : u' = -\nabla f, \lim_{s\to-\infty} u(s) = p, \lim_{s\to+\infty} u(s) = q\}}$$

Identify $W(p,q) \equiv W^u(p) \cap W^s(q), u \leftrightarrow u(0)$. When is $W(p,q)$ a manifold?

3.9. **Classical approach.** $f : M \to \mathbb{R}$ Morse, and g is a Riem metric on M.

(1) Thm 3.8: Prove $W^u(p)$, $W^s(q)$ are smooth mfds of dim $= |p|, m - |q|$.

(2) ***Def.*** *$f : M \to \mathbb{R}$ is* Morse-Smale *for the Riem metric g on M if*

$$W^u(p) \pitchfork W^s(q) \quad \forall p, q \in \mathrm{Crit}(f)$$

By Cor 1.4, (f,g) Morse-Smale $\Rightarrow W(p,q) \equiv W^u(p) \cap W^s(q)$ is mfd of codim $= (m - |p|) + |q|$, hence of dimension:

$$\dim W(p,q) = |p| - |q|.$$

Rmk. $\Rightarrow W(p,p) = \{\text{constant } p\}$ since f decreases along the flow.
Rmk. $\Rightarrow W(p,q) = \emptyset$ if $|p| \le |q|$ $(p \ne q)$ since a non-constant flowline u gives a 1-dim family of flowlines $u(\cdot + \text{constant})$.

(3) **Thm [Kupka-Smale 1963]**
(M,g) closed Riem mfd $\Rightarrow$ generic $f \in C^\infty(M,\mathbb{R})$ are Morse-Smale.

Example. Hwk 1: 1(ii) Morse-Smale, but 1(i) is not by Rmk ($\exists$ 2 flowlines between the index 1 crit pts)

Rmk. The Kupka-Smale Thm is stronger: for generic smooth vector fields, the fixed pts are hyperbolic[1] and $W^u(p) \pitchfork W^s(q)$.

(4) One can describe how the $W^u(p)$'s fit together, so get a CW structure on M and can do cellular homology.

3.10. **Cellular homology for self-indexing f.**

Thm (Thom, Smale, Milnor $\sim$ 1965)**.** *For f self-indexing, and g Morse-Smale, $H_*(M) \cong$ homology of $\bigoplus_{p \in Crit(f)} \mathbb{Z}\, p$ with differential*

$$d\,p = \sum_{|q|=|p|-1} (S^u(p) \cdot S^s(q))\, q$$

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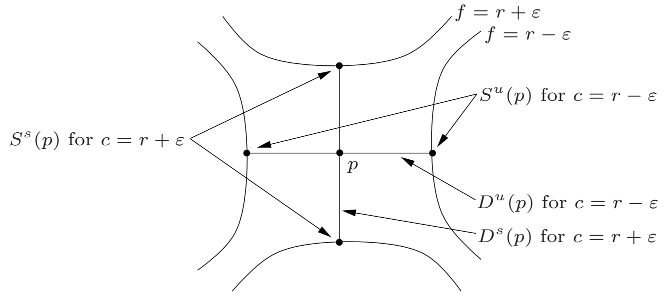
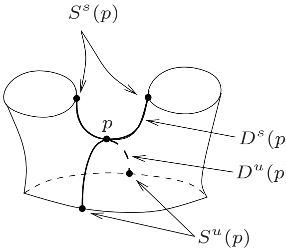
[1] so W^u, W^s are injectively immersed submfds, see 3.8.

where the intersection number $S^u(p) \cdot S^s(q)$ is calculated[2] *inside $f^{-1}(c)$, where you fix any regular value $c \in (f(q), f(p))$, and where:*[3]

$$\begin{array}{lcll} S^u(p) & = & W^u(p) \cap f^{-1}(c) & \textit{unstable sphere} \\ D^u(p) & = & W^u(p) \cap (f \geq c) & \textit{unstable disc bounded by } S^u(p) \end{array}$$





For f self-indexing Morse:

$$\begin{array}{rcl} C_k & = & H_k(M_k, M_{k-1}) \\ & \cong & H_k\,(M_k/M_{k-1} = \text{bouquet of } k\text{-spheres one for each } p \in \mathrm{Crit}(f), |p| = k) \\ & \cong & \displaystyle\bigoplus_{|p|=k} \mathbb{Z}\,p \end{array}$$

Fact.[4] $S \subset M \subset L \Rightarrow \text{LES}$[5] $H_*(M,S) \xrightarrow{\text{incl}} H_*(L,S) \xrightarrow{\text{quot}} H_*(L,M) \xrightarrow{\partial} H_{*-1}(M,S)$.
Define. $\partial : C_k \to C_{k-1}$ from LES of $M_{k-1} \subset M_k \subset M_{k+1}$.

Recall this is how cellular homology is defined (and one checks that $\partial^2 = 0$).

Thm. $H_*(M) \cong H_*(C_*, \partial)$

Proof. Use the 2 LES of triples:

$$\begin{tikzcd}
 & 0 \arrow[d] & \text{hence this is } \cong \ker\partial|_{C_k} \arrow[dl] & \text{so this is } \cong H_k(C_*,\partial) \arrow[dl] \\
C_{k+1} \arrow[r] \arrow[dr, "\partial"'] & H_k(M_k, M_{k-2}) \arrow[r] \arrow[d] & H_k(M_{k+1}, M_{k-2}) \arrow[r] & 0 \\
 & C_k \arrow[d, "\partial"] & & \\
 & C_{k-1} & &
\end{tikzcd}$$

$\Rightarrow H_k(C_*, \partial) \cong H_k(M_{k+1}, M_{k-2}) \cong H_k(M_{k+1}) \cong H_k(M)$. □

---

[2] The point is that you can't do $W^u(p) \cdot W^s(q)$ because it has dimension ≥ 1 because of the reparametrization freedom $u(\cdot + \text{constant})$.

[3] these are a sphere/disc if you choose c very close to $f(p)$, using the local model of Thm 3.7, and hence it's true for any $c \in (f(q), f(p))$ by using the flow of $-\nabla f$ (see Thm 3.5).

[4] Small, Medium, Large.

[5] Long Exact Sequence.

Intersection Pairing.
$A, B \subset X$ closed mfds, $\operatorname{codim}(A) = k$, $\dim B = k$, assume $A \pitchfork B = \{p_1, p_2, \ldots\}$ and assume A is co-oriented (the normal bundle ν_A is oriented[6])

Recall the **Thom isomorphism**:

$$\begin{array}{rcl} H_0(A) & \cong & H_k(X, X \setminus A) \\ [\mathrm{pt}] & \mapsto & \tau = \text{Thom class} = \text{fibrewise an orienation generator of } \nu_A \end{array}$$

Idea: it's a family of generators of $H_k(D^k, \partial D^k)$ where D^k is the disc in the fibre.

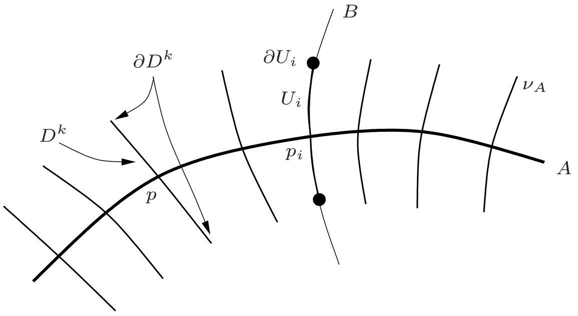
Thm 1.

$$\begin{array}{ccccc} H_k(B) & \xrightarrow{\text{incl}} & H_k(X) & \xrightarrow{\text{quot}} & H_k(X, X \setminus A) \\ [B] & \mapsto & [B] & \mapsto & (A \cdot B)\,\tau. \end{array}$$

Proof. Localize problem: pick disjoint open balls U_i around p_i in B. By naturality of the Thom iso

$$\begin{array}{rcl} H_k(U_i, U_i - p_i) & \overset{\text{incl}}{\cong} & H_k(X, X \setminus A) \\ [U_i, U_i - p_i] & \mapsto & \pm\tau \end{array}$$

where ± is the intersection number of A, B at p_i (since you can check the intersection number is the comparison of the generator $[D^k, \partial D^k]$ for $(\nu_A)_{p_i}$ with $[U_i, \partial U_i]$).



Now globalize:

$$\begin{tikzcd} H_k(B) \arrow[r] \arrow[dr] & H_k(X) \arrow[r] & H_k(X, X \setminus A) \\ & H_k(B, B \setminus A \cap B) \arrow[r, "\cong", "\text{excision}"'] & \bigoplus_i H_k(U_i, U_i - p_i) \arrow[u] \end{tikzcd}$$

□

Thm 2. $C_{k+1} \to C_k,\ p_i \mapsto \sum_j (S^s(q_j) \cdot S^u(p_i))\,[D^u(q_j)]$ (intersecting inside $f^{-1}(c)$, where one can take $c = k + \frac{1}{2}$).

---

[6] you can ignore orientations if you count intersections modulo 2.

Proof. For simplicity, do the case where C_{k+1} is generated by just p, $|p| = k+1$ and C_k by just q, $|q| = k$ (the proof easily generalizes).

$$\begin{tikzcd}
 & & & H_0(S^s(q)) \arrow[d, "\text{Thom}"', "\cong"] \\
 & & H_k(S^u(p)) \arrow[r, "(*)"] \arrow[d, "i"] & H_k(f^{-1}(c), f^{-1}(c) \setminus S^s(q)) \arrow[d, "r"] \\
C_{k+1} \arrow[r, "\text{excision}", "\sim"'] & H_{k+1}(M_{k+1} \setminus (f<c), f^{-1}(c)) \arrow[r, "\text{bdry}"] & H_k(f^{-1}(c)) & H_k(M_{k-1} \cup D^u(q), M_{k-1} \cup (D^u(q) - q)) \arrow[dd, "\text{deformation retraction}"] \\
{[D^u(p)]} \arrow[rr, mapsto] \arrow[dddrrr, bend right, "\partial"'] & & {[S^u(p)]} \arrow[d, "i"] & \\
 & & H_k(M_k) \arrow[dr, "\text{quot}"] & H_k(M_{k-1} \cup D^u(q), M_{k-1}) \arrow[d, "i"] \\
 & & & C_k
\end{tikzcd}$$

where i stands for "inclusion", r is the (restriction of the) retraction of M_k onto $M_{k-1} \cup D^u(q)$ of Thm 3.7, and the arrow "deformation retraction" pushes $D^u(q) - q$ away from q and into M_{k-1}.

Finally, one can check that this diagram commutes.[7]

By Thm 1, the arrow $(*)$ is

$$[S^u(p)] \mapsto (S^s(q) \cdot S^u(p))\,\tau$$

and the right column is $[\mathrm{pt}] \mapsto \tau \mapsto [D^u(q)] \in C_k$ (easy exercise). $\square$

3.11. Modern vs classical.

Def (Witten (1982))**.** *The* Morse complex *is* $MC_* = \bigoplus_{p \in Crit(f)} \mathbb{Z}p$. *For* $p \neq q \in Crit(f)$ *the* moduli space *of Morse trajectories*[8] *is*

$$\boxed{\mathcal{M}(p,q) = W(p,q)/\mathbb{R}}$$

where the $\mathbb{R}$ *action by time-shifting identifies* $u \sim u(\cdot + \text{constant})$.

The Morse differential *is*

$$\boxed{\partial p = \sum_{|q| = |p| - 1} \#\mathcal{M}(p,q) \cdot q}$$

where $\#\mathcal{M}(p,q)$ *counts the number of elements with orientation signs* (*exercise: this agrees with* $S^u(q) \cdot S^s(p)$).

Cor. *For a self-indexing Morse function* f *and a Morse-Smale metric* g *on a closed manifold* M, *the Morse differential satisfies* $d^2 = 0$ *and the Morse homology*[9] *is isomorphic to ordinary homology*

$$MH_*(f) \cong H_*(M).$$

---

[7] the two big columns commute at the level of topological maps if you replace $i : f^{-1}(c) \to M_k$ by r, and then since $r_* = i_*$ on homology it commutes also at the level of homology.

[8] called *instantons* or *tunneling paths* by physicists.

[9] $MH_*(f,g) = H_*(MC_*, d)$.

LECTURE 10.

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3.11. Modern vs classical (continued).

$$\begin{tikzcd}
(f,g) \arrow[r, Rightarrow, "\text{perturb } f"] & W^u(p) \pitchfork W^s(q) \arrow[r, Rightarrow] & W(p,q) \text{ mfd} \arrow[r, Rightarrow] & \mathcal{M}(p,q) = W(p,q)/\mathbb{R} \text{ mfd} \\
f \text{ Morse} \arrow[rrr, Rightarrow, bend right, "\text{modern approach}", "\text{perturb } g"'] & \forall p \neq q \in \operatorname{Crit}(f) & \dim = |p| - |q| & \dim = |p| - |q| - 1
\end{tikzcd}$$

Rmk. *From a dynamical systems perspective, the natural object to perturb is the dynamical system: f. You would not "shake the universe" to get the equations that you want. The modern approach is however reasonable: you can never measure exactly what the Riemannian metric g is, so you can always assume it is generic.*

Historical context:

1988 Floer:	$\mathcal{M}(p,q)$ = zero set of a Fredholm section of a Banach vector bundle and proved smoothness & compactness theorems
1989 Floer:	proved (an infinite dimensional generalization of) $\partial^2 = 0$, and $MH_*(f) \cong H_*(M)$ using the "Conley index"[1]
1990 Salamon:	proved $MH_*(f) \cong H_*(M)$ using the Conley index.

Importance of the modern approach:

Need to avoid 3.9 (i): in infinite dimensional generalizations of Morse theory,

(1) $W^u(p), W^s(q)$ are usually not defined[2]
(2) and even when they are defined, they are usually infinite dimensional.[3]

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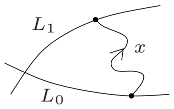
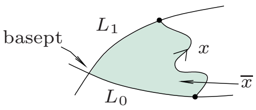
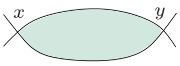
[1] which is essentially a sophisticated dynamical version of cellular homology.

[2] because flows are badly behaved in infinite dimensions.

[3] So you have a bad intersection theory. Example: take the space of sequences of real numbers. You can shift sequences by a fixed amount without losing information, so you can arbitrarily increase codimensions of subsets, so can deform any two proper subsets so that they never intersect.

3.12. **Modern generalizations.** [Non-examinable]

Homology	Manifold	Morse function f	$x \in \mathrm{Crit}(f)$	$-\nabla f$ flowline in $\mathcal{M}(x,y)$
Morse (MH)	M^n closed	$f : M \to \mathbb{R}$	$x \in M : d_x f = 0$	$u : \mathbb{R} \to M$ $\partial_s u = -\nabla f$
	Applications: count # of critical points of a function.			
∞-dim MH	$\mathcal{L}N = C^\infty(S^1, N)$ free loop space	$\mathcal{E} : \mathcal{L}N \to \mathbb{R}$ $\mathcal{E}(x) = \int_0^1 \frac{1}{2}\lvert x'\rvert^2 \, dt$	$x =$ closed geodesic of period $= 1$	$u : \mathbb{R} \times S^1 \to M$ $\partial_s u = -\nabla \mathcal{E}$
	Applications: count # of closed geodesics.			
Symplectic (SH)	$\mathcal{L}(T^*M)$ T^*M coords (q,p), $\omega = d(pdq)$ $= dp \wedge dq$ symplectic (closed 2-form ω s.t. $\omega^n = \mathrm{vol}_{T^*M}$)	$A_H(x) = -\int_x p\,dq + \int_x \frac{1}{2}\lvert p\rvert^2$ (if x contractible, it equals "area" $\int_{\overline{x}} \omega$ of disc $\overline{x}$ bounding x (Stokes))	$x =$ closed geodesic of period $= 1$ $T^*M \cong TM \ni (x, x')$	Floer's eqn $u : \mathbb{R} \times S^1 \to T^*M$ $\partial_s u = -J\partial_t u - \nabla \frac{1}{2}\lvert p\rvert^2$ (compare Cauchy-Riemann eqns: $\partial_s u = -i\partial_t u$ on $\mathbb{C}$)
	Applications: count # of closed geodesics.			
Floer (FH)	$\mathcal{L}_0 N =$ contractible loops (N^{2n}, ω) symplectic closed mfd[a]	$A_H : \mathcal{L}_0 N \to \mathbb{R}$ $A_H(x) = -\int_{\overline{x}} \omega + \int_x H$	Hamiltonian orbits $x' = X_H(x)$ $(\omega(\cdot, X_H) = dH)$	Floer's eqn $u : \mathbb{R} \times S^1 \to M$ $\partial_s u = -J\partial_t u - \nabla H$
	Applications: count # of contractible Hamiltonian orbits.			
Lagrangian FH	$\Omega(L_0, L_1) =$ paths in (N^{2n}, ω) from L_0 to L_1 (Lagrangian submfds:[b] $\omega\vert_L = 0, \dim L = n$) 	$A(x) = -\int_{\overline{x}} \omega$ 	$x \in L_0 \cap L_1$	"holomorphic" strips $u : \mathbb{R} \times S^1 \to N$ $\partial_s u = -J\partial_t u$ 
	Applications: package up LFH's ∀Lagrangians and all algebraic operations ⇒Fukaya category (Fukaya,Seidel,...) $\overset{\text{dictionary}}{\longleftrightarrow}$ *Category of coherent sheaves on a mirror manifold* *SYMPLECTIC TOPOLOGY* $\overset{\text{mirror symmetry}}{\longleftrightarrow}$ *ALGEBRAIC GEOMETRY*			
	Applications: Heegaard Floer homology "=" LFH[c] for 3-mfds (Ozsváth-Szabo,...) ⇒ *invariants of knots (detects knot genus, categorifies Alexander polynomial)*			
Instanton (IH)	$B = \{$connections x on $P\}/\sim$ ($\sim$gauge eq., $P \to Y$ principal $SU(2)$-blde, Y closed 3-mfd)	Chern-Simons functional $CS(x) = \frac{1}{8\pi^2}\int_Y \mathrm{Tr}(x \wedge dx +$ $+ \frac{2}{3} x \wedge x \wedge x)$	flat connections x	$u : \mathbb{R} \to B$ $\partial_s u = - * F_u$ ($*$ =Hodge star, F_u =curvature of $u(s)$)
Seiberg- -Witten (HM)	{spin^c-connections x with section $Y \to S$ to a reference spin bdle}	Chern-Simons-Dirac functional[d]	monopoles x	4-dim SW eqns
	Applications: invariants of 4-mfds (a 4-dim coborism gives a hom of HM^*'s of boundary 3-mfds) *Taubes (2010): Seiberg-Witten homology ≅ Heegaard Floer homology*			

---

[a] Meaning, ω is a closed 2-form with $\omega^n = \mathrm{vol}_M$. Usually you assume $\int_{\text{2-spheres}} \omega = 0$ so that $A_H : \mathcal{L}_0 N \to \mathbb{R}$ is well-defined independently of the choice of disc $\overline{x}$ bounding x. This $\omega[\pi_2(M)] = 0$ condition also prevents an issue called "bubbling": non-constant holomorphic spheres don't appear because they would have positive ω-area.

[b] Called (Lagrangian) *branes* by physicists.

[c] Loosely speaking, you mimick the construction of LFH for $N =$ the symmetric product $\mathrm{Sym}^g(\Sigma_g)$ of the surface Σ_g arising in a Heegaard splitting of the 3-mfd, with $L_0 = \prod \alpha_i$, $L_1 = \prod \beta_i$ for appropriate loops α_i, β_i which generate the H_1 of Σ_g.

[d] See Kronheimer and Mrowka, *Monopoles and Three Manifolds.*

3.13. **Key aspects of modern homology theories.**

1. ***Moduli Space:*** $\mathcal{M}(x,y)$ = zero set of some Fredholm map;
2. ***Transversality:*** perturb parameters $\Rightarrow \mathcal{M}(x,y)$ smooth mfd;
3. ***Dimension:*** index of some Fredholm operator[4] in terms of x,y gradings;
4. ***Compactness & Gluing:*** $\mathcal{M}(x,y)$ compact if allow "broken solutions";
5. ***Orientations:*** for $\mathbb{Z}$ coefficients, need to choose[5] orientations for $\mathcal{M}(x,y)$;
6. ***Invariance:*** $H_*(\oplus_{x\in \mathrm{Crit}(f)}\mathbb{Z}\,x, \partial$ counting $\mathcal{M}(x,y)$'s) indep of parameters.

4. TRANSVERSALITY THEOREM

4.0. **Transversality for $\mathcal{M}(p,q)$: outline of the proof.**
For $f: M \to \mathbb{R}$ smooth Morse function, consider

$$\begin{tikzcd} E = \text{all vector fields along paths} \arrow[d] \\ U \times G = \{\text{all paths } \mathbb{R} \to M \text{ from } p \text{ to } q\} \times \{\text{all metrics}\} \arrow[u, bend right, "F(u,g) = \partial_s u - \nabla^g f"'] \end{tikzcd}$$

where $\nabla^g f$ is ∇f for g: $g(\nabla^g f, \cdot) = df$. Then notice:

$$\boxed{\mathcal{M}^g(p,q) = (\text{ zero set of } F_g = F(\cdot, g))}$$

Aim: Parametric transversality $\Rightarrow \mathcal{M}^g(p,q)$ is smooth mfd for generic g.

What is needed: U, G, E Banach mfds, $F \pitchfork 0_E$, DF_g Fredholm.

What Banach mfd structure?

- Cannot use C^∞: not Banach
- $\boxed{G = \{C^k\text{-metrics on } M\}}$ ✓
- $\partial_s u = -\nabla^g f$ and $u \in C^1 \Rightarrow u$ is C^{k+1}:
$$(\partial_s)^{k+1} u = (\partial_s)^k(-\nabla^g f).$$
- C^1 is not so good on non-compact domains like $\mathbb{R}$. Also: want to integrate $\int_{\mathbb{R}} |u'|^2\,dt$ and $\int_{\mathbb{R}} |\nabla f_u|^2\,dt$, but $C^1 \cap L^2$ is not Banach. Try $W^{1,2}$ instead.
- $W^{1,2}$ is great since Hilbert: for smooth $u, v: \mathbb{R} \to M \subset \mathbb{R}^a$, inner product[6]
$$\langle u, v\rangle_{1,2} = \langle u, v\rangle_{L^2} + \langle \partial_s u, \partial_s v\rangle_{L^2}.$$
Recall $W^{1,2}$ is obtained by completing $C^1 \cap L^2$ for the induced norm $\|\cdot\|_{1,2}$. Hilbert spaces are great because you can tell if an operator is surjective by studying the subspace perpendicular to the image.
- First attempt:
$$\boxed{U = \{u \in W^{1,2}(\mathbb{R}, M) : u \to p, q \text{ at the ends}\}} \;\textit{(not quite correct)}$$
$$\boxed{E = \{L^2 - \text{vector fields over paths}\}}$$

However, various problems arise:

---

[4] the linearization of the Fredholm map in (1).

[5] chosen compatibly with respect to the gluing (4) of $\partial\mathcal{M}(x,y)$ onto $\mathcal{M}(x,y)$.

[6] $\langle u, v\rangle_{L^2} = \int u(s)\cdot v(s)\,ds$ using the dot product in $\mathbb{R}^a$ after embedding $M \hookrightarrow \mathbb{R}^a$.

- Does $u \to p, q$ even make sense? For smooth $u \in W^{1,2}$ get
$$\begin{aligned} |u(b) - u(a)| &= |\int_a^b (\partial_s u)\, ds| \\ &\leq \int_a^b |\partial_s u|\, ds \\ &\leq (\int_a^b 1^2)^{1/2} (\int_a^b |\partial_s u|^2)^{1/2} \\ &\leq \sqrt{b-a} \cdot \|u\|_{1,2} \end{aligned}$$
does this mean that elements in $W^{1,2}$ can be represented by a continuous function? (in which case $u \to p, q$ makes sense).
- even if u is continuous, F involves "$\partial_s u$": surely u is not C^1 in general?
- finally, $W^{1,2}(\mathbb{R}, M)$ doesn't quite make sense: a constant $\mathbb{R} \to \{x\} \in M \subset \mathbb{R}^a$ has norm $\int_{\mathbb{R}} |x|\, ds = \infty$ (for $x \neq 0$), and most other maps $\mathbb{R} \to M$ also have infinite norm. So in fact we want u to be *locally* $W^{1,2}$, and we require that u is $W^{1,2}$ at the ends with norms calculated in such a way that the critical points have norm zero. This requires some care.

Conclusion: we need to review the Sobolev spaces $W^{1,2}$ before moving on. Then we will define G, U, E properly, and prove they are Banach mfds. Finally we will prove the hypotheses required for parametric transversality. The upshot will be that the moduli spaces $\mathcal{M}(p, q)$ are smooth mfds for generic metrics g.

LECTURE 11.

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SOBOLEV SPACES

The book by Adams, *Sobolev spaces*, gives a thorough treatment of this material. We will treat Sobolev spaces with greater generality than necessary (we only use $W^{1,2}$ and L^2), since these spaces are ubiquitously used in geometry.

4.1. $W^{k,p}$ spaces on Euclidean space.

Notation: $k \geq 0$ integer, $1 \leq p < \infty$ real.

Def. *For an open set* $X \subset \mathbb{R}^n$, $W^{k,p}(X) = W^{k,p}(X, \mathbb{R})$ *is the completion*[1] *of* $C^\infty(X) = \{\text{smooth } u : X \to \mathbb{R}\}$ *with respect to the* $\|\cdot\|_{k,p}$*-norm*[2]

$$\|u\|_{k,p} = \sum_{|I| \leq k} \|\partial^I u\|_p = \sum_{|I| \leq k} \left(\int_X |\partial^I u|^p \, dx \right)^{1/p}$$

$W^{k,\infty}(X)$ *is defined analogously using* $\|u\|_{k,\infty} = \sum_{|I| \leq k} \sup |\partial^I u|$.

Def. $W^{k,p}(X, \mathbb{R}^m)$ *is the completion of* $C^\infty(X, \mathbb{R}^m)$ *using*

$$\|u\|_{k,p} = \sum_{i=1,\dots,m} \|u^i\|_{W^{k,p}(X)}$$

where u^i *are the coordinates of* u. *An equivalent norm*[3] *can be defined using the previous definition, replacing* $|\partial^I u|$ *by* $|\partial^I u|_{\mathbb{R}^n}$.

Rmks.

(1) C^∞ *is dense in* C^k *with respect to* $\|\cdot\|_{k,p}$, *so completing* C^k *is the same as completing* C^∞. ***Fact.*** *When* ∂X *smooth (or* C^1*),* $C^\infty(\overline{X}) \subset W^{k,p}(X)$ *is dense* ($\overline{X}$ = *closure of* $X \subset \mathbb{R}^n$*), so it is the same as completing* $C^\infty(\overline{X})$.

(2) $W_0^{k,p}$ *is the completion of* C_c^∞ *inside* $W^{k,p}$, *where*[4]

$$C_c^\infty(X) = \{\text{smooth compactly supported functions } \phi : X \to \mathbb{R}\}$$

These spaces typically arise in geometry when you globalize a locally defined function after multiplying by a bump function.
Example. $\phi \in C_c^\infty(X)$, $u \in W^{k,p}(X) \Rightarrow \phi \cdot u \in W_0^{k,p}(X)$.
Warning. *Usually* $W_0 \neq W$ *since the* u*'s must* $= 0$ *on* ∂X*, unlike* $C^\infty(\overline{X})$.

Date: May 3, 2011,

[1] it is always understood that we complete the subset of C^∞ of u's with *bounded* $\|u\|_{k,p}$.

[2] where $I = (i_1, i_2, \dots, i_n)$, $\partial^I = (\partial_1)^{i_1} \cdots (\partial_n)^{i_n}$, $\partial_j = \frac{\partial}{\partial x_j}$, $|I| = i_1 + \cdots + i_n$.

[3] Two norms $\|\cdot\|$, $\|\cdot\|'$ are *equivalent* if $\exists$ constants $a, b > 0$ such that $a\|x\| \leq \|x\|' \leq b\|x\|$, $\forall x$.

[4] The *support* of ϕ is $\text{supp}(\phi) = \overline{\{x \in X : \phi(x) \neq 0\}}$.

(3) $W^{k,p}_{loc}$ = locally $W^{k,p}$ maps = *completion of* C^∞_c *with respect to the topology:*[5]

$$u_n \to u \Leftrightarrow u_n|_C \to u|_C \quad \forall C \Subset X$$

Loosely think of this as saying: the restriction to any compact is $W^{k,p}$*.* ***Warning.*** *This is not a normed space, but it is a complete metric space.*

(4) *All these spaces are* separable*: there is a countable dense subset, namely the polynomials with rational coefficients.*

4.2. L^p theory.

$L^p = W^{0,p}$ with $\|u\|_p = \left(\int_X |u|^p \, dx\right)^{1/p}$, and $L^\infty = W^{0,\infty}$ with $\|u\|_\infty = \sup |u|$. Recall **Hölder's inequality**[6]

$$\int_X |u \cdot v| \, dx \leq \|u\|_p \|v\|_q \qquad \text{for } \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Lemma. $p \geq q \Rightarrow \mathrm{vol}(X)^{-1/q} \cdot \|u\|_q \leq \mathrm{vol}(X)^{-1/p} \cdot \|u\|_p$
$\Rightarrow L^p(X) \hookrightarrow L^q(X)$ *is bounded for* X *bounded.*

Proof. For $u \in C^\infty$, let $A = \int |u|^p$, then

$$\frac{\|u\|_q}{\|u\|_p} = \frac{(\int |u|^q)^{1/q}}{A^{\frac{1}{p}}} = \left(\int \left(\frac{|u|^p}{A}\right)^{q/p} \cdot 1\right)^{1/q} \leq \left[\left(\int \frac{|u|^p}{A}\right)^{q/p} \cdot (\textstyle\int 1)^{1-\frac{q}{p}}\right]^{1/q} = \mathrm{vol}(X)^{\frac{1}{q}-\frac{1}{p}}$$

using Hölder in the inequality. Therefore $(C^\infty \cap L^p, \|\cdot\|_p) \to (C^\infty \cap L^q, \|\cdot\|_q)$ is a bdd inclusion, so we can complete it:[7] $L^p \to L^q$, $[u_n] \mapsto [u_n]$. ☐

Example. $L^\infty(X) \subset L^p(X)$ is clearly true for bdd X, and clearly false for $X = \mathbb{R}$.

Cor. $p \geq q \Rightarrow W^{k,p}(X) \hookrightarrow W^{k,q}(X)$ *is bdd for* X *bdd.*

Motivation. $k > k' \Rightarrow W^{k,p} \hookrightarrow W^{k',p}$ is clearly bounded, and one might even suspect that it is compact because of a mean value thm argument. So can one combine this with the Corollary and get optimal conditions on k, p simultaneously?

Def. Recall a linear map $L : X \to Y$ is *bounded* if $\|Lx\| \leq c\|x\| \ \forall x$, and *compact* if any bounded sequence gets mapped to a sequence having a cgt subsequence.[8]

4.3. Sobolev embedding theorems.

Let[9]

$$\boxed{p^* = \begin{cases} \frac{np}{n-kp} & \text{if } kp < n \\ \infty & \text{if } kp \geq n \end{cases}}$$

From now on, assume $X \subset \mathbb{R}^n$ open, ∂X smooth (or C^1).

Thm. $\boxed{W^{k,p}(X) \overset{bdd}{\hookrightarrow} L^q(X) \text{ for } p \leq q \leq p^*}$ *(require* $q \neq \infty$ *if* $kp = n$*).*

Rmk. *For* X *bdd one can omit* $p \leq q$ *by the Lemma.*

---

[5] recall $C \Subset X$ means C, X open and $C \subset \overline{C} \subset X$.

[6] The generalization of the Cauchy-Schwarz inequality ($p = q = 2$).

[7] the inequality shows that L^p-Cauchy implies L^q-Cauchy, and the map $[u_n] \mapsto [u_n]$ is well-defined since $u_n \to 0$ in L^p implies $u_n \to 0$ in L^q, again by the inequality.

[8] equivalently: the closure of the image of the unit ball is compact.

[9] Unexpected results happen at the Sobolev borderline $kp = n$. Example: $\log\log(1 + \frac{1}{|x|})$ on the unit ball in $\mathbb{R}^n$ is $W^{1,n}$ but neither C^0 nor L^∞.

Cor. *Under the same assumptions, $W^{k+j,p}(X) \hookrightarrow W^{j,q}(X)$ is bdd.*

Proof. Idea: $u \in W^{j,q} \Leftrightarrow \partial^I u \in L^q$, $\forall |I| \leq j$. For smooth u (afterwards complete):

$$\|u\|_{j,q} = \sum_{|I|\leq j} \|\partial^I u\|_q \leq c \sum_{|I|\leq j} \|\partial^I u\|_{k,p} \leq c' \, \|u\|_{k+j,p}. \quad \square$$

Thm.[10] $\boxed{W^{k+j,p}(X) \overset{bdd}{\hookrightarrow} C^j_b(\overline{X}) \text{ for } kp > n}$

Thm (Rellich)**.** *X bdd & inequalities are strict $\Rightarrow$ above embeddings are compact.*

Example. $W^{1,2}(\mathbb{R},\mathbb{R}^m) \hookrightarrow C^0_b(\mathbb{R},\mathbb{R}^m) = \{\text{bdd cts } \mathbb{R} \to \mathbb{R}^m\}$ ($kp = 2 > n = 1$✓).
$W^{1,2}(\mathbb{R},\mathbb{R}^m) \xrightarrow{\text{restr}} W^{1,2}((0,1),\mathbb{R}^m) \hookrightarrow C^0([0,1],\mathbb{R}^m)$ is compact.

Idea of Proof of First Theorem for $kp < n$, X bdd
Note: $kp < n$, $q \leq p^* \Leftrightarrow 0 > k - \frac{n}{p} \geq -\frac{n}{q}$. By induction reduce to $k = 1$:

$$\begin{array}{lll} u \in W^{k,p} & \Rightarrow & u, \partial_j u \in W^{k-1,p} \\ & \Rightarrow & \text{(induction) } u, \partial_j u \in L^{p'} \qquad 0 > k - 1 - \frac{n}{p} = 0 - \frac{n}{p'} \\ & \Rightarrow & u \in W^{1,p'} \\ & \Rightarrow & (k=1) \;\; u \in L^q \qquad\qquad\;\; 0 > 1 - \frac{n}{p'} = k - \frac{n}{p} \geq 0 - \frac{n}{q} \end{array}$$

To prove $W^{1,p} \hookrightarrow L^q$ seek

$$\boxed{\|u\|_q \leq c\|du\|_p \qquad \forall u \in C^\infty_c(X) \qquad \text{(fails for } u = 1) \qquad (*)}$$

Sketch: You start from fund. thm of calculus "$u(x) = \int_{-\infty}^{x_i} \partial_i u(\cdots)\, dx_i$" ($p = 1$), then everything else[11] is repeated integrations and Hölder's inequalities. For general p, you just use clever exponents.

Fact. can extend $u \in W^{1,p}(X)$ to a compactly supported $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ in a way that $\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq c\|u\|_{W^{1,p}(X)}$. Then approximate by $C^\infty_c(\mathbb{R}^n)$ and use $(*)$. $\square$

Rmks.

(1) *Can replace W by W_0 (then no conditions on ∂X are needed).*
(2) *$(*)$ holds $\forall u \in W^{1,p}_0(X)$ $(0 > 1 - \frac{n}{p} > 0 - \frac{n}{q})$, a version of* Poincaré's ineq.
(3) *$kp > n$ is wonderful since $W^{k,p} \subset C^0$, so you can represent elements as continuous functions, avoiding the Cauchy rubbish.*

4.4. **Derivatives on $W^{k,p}$.**

Method 1: via completions

$$\partial_s : (C^\infty, \|\cdot\|_{k,p}) \to (C^\infty, \|\cdot\|_{k-1,p})$$
$$\partial_s : W^{k,p} \to W^{k-1,p}, [u_n] \to [\partial_s u_n]$$

Method 2: for $p = 2$, use the Fourier transform to replace ∂^I by multiplication by x^I (up to a constant factor). See Hwk 11.

Method 3: Using weak derivatives
First we want to avoid completions, and work with actual functions:

$$L^p(X) = \{\text{Lebesgue measurable } u : X \to \mathbb{R} \text{ with } \|u\|_p < \infty\} \big/ \underset{\text{almost everywhere}}{u\sim v \text{ if } u=v}$$

---

[10] using $\|u\|_{C^j} = \sum_{|I|\leq j} \sup |\partial^I u|$ on $C^j(\overline{X})$: call C^j_b the subset of u's with bdd $\|u\|_{C^j}$.

[11] If you're curious, see Evans, *Partial Differential Equations*, p.263.

For our purposes, we don't need a deep understanding of measure theory, just a vague nod: *Lebesgue measure* is a good notion of volume for certain subsets of $\mathbb{R}^n$. These subsets are called *measurable*. For example open subsets and closed subsets. The notion of volume for cubes and balls is what you think, and there are various axioms, the most important of which is: the volume of a countable disjoint union of subsets is the sum of the individual volumes. Define:

$$f \text{ is } \textit{measurable} \text{ if } f^{-1}(\text{any open set}) \text{ is measurable.}$$

Examples. Continuous functions, since $f^{-1}(\text{open})$ is open. Step functions (for example $f = 1$ on some open set, $f = 0$ outside it). Also: can add, scale, multiply, take limits of measurable fns to get measurable fns.[12]

Convention. If $u \sim$ continuous fn, then we always represent u by the cts fn!

Fact. *The above* $(L^p(X), \|\cdot\|_p)$ *is complete and* $C^\infty(X) \subset L^p(X)$ *dense, so* $L^p(X) \cong$ *completion of* $(C^\infty(X), \|\cdot\|_p)$ *(as usual, only allow smooth* u *with* $\|u\|_p < \infty$*).*

Def. $f_I \in L^p(X)$ *is an* I*-th weak derivative of* f *if* $\forall \phi \in C_c^\infty(X)$,

$$\int_X f_I \cdot \phi \, dx = (-1)^{|I|} \int_X f \cdot \partial^I \phi \, dx.$$

For smooth f *this is just integration by parts with* $f_I = \partial^I f$.

Exercise. weak derivatives are unique if they exist. So just write $f_I = \partial^I f$.

Key Fact. $\boxed{\text{if } f_I \text{ is cts, then the usual } \partial^I f \text{ exists and equals } f_I.}$

See Lieb & Loss, *Analysis*, 2nd ed. Non-examinable: If $u \in W^{k,p}_{loc}$ then $u \in W^{1,1}_{loc}$ by Lemma 4.2 (*loc* gives finite vol), hence the FTC holds (L&L p.143): $u(x+y) - u(x) = \int_0^1 y \cdot \nabla u(x+ty)\, dt$ for a.e. x, all small y. Their proof shows that this is true for all x if $u, \nabla u$ are continuous. The key fact is proved in L&L p.145. Example: Suppose $u \in W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ with cts weak $\partial_s u$, then $\frac{u(s+y)-u(s)}{y} = \int_0^1 \partial_s u(s+ty)\, dt \to \partial_s u(s)$ as $y \to 0$ by cty, so u is C^1 with deriv = weak deriv. Proof of FTC for $u \in W^{1,2}(\mathbb{R}, \mathbb{R})$: for $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$, write $\widetilde{\phi}(s) = \phi(s - ty)$.

$$\begin{aligned} \textstyle\int \phi(s)(\int_0^1 y \cdot \partial_s u(s+ty)\, dt)\, ds &= \textstyle\int_0^1 y(\int \phi(s) \cdot \partial_s u(s+ty)\, ds) dt = -\int_0^1 \int y \cdot \partial_s \widetilde{\phi}(s) \cdot u(s)\, ds\, dt \\ &= \textstyle\int \int_0^1 \partial_t \widetilde{\phi}(s)\, dt\, u(s)\, ds = \int \phi(s-y) u(s)\, ds - \int \phi(s) u(s)\, ds = \int \phi(s)(u(s+y) - u(s))\, ds. \end{aligned}$$

Hence $\int_0^1 y \cdot \partial_s u(s+ty)\, dt = u(s+y) - u(s)$ for a.e. s (all s if $u, \partial_s u$ cts (u is cts by Sobolev)).

Rmks.

(1) *Weak derivatives behave as you expect:*

$$\partial^I : W^{k,p} \to W^{k-|I|,p} \quad \textit{is linear.}$$

Also $\phi \in C_c^\infty$, $u \in W^{k,p} \Rightarrow \phi \cdot u \in W^{k,p}$ *with* $\partial^I(\phi \cdot u) =$ *Leibniz formula.*

(2) *Observe:*

$$\begin{aligned} u \in W^{k,p}(X) \quad &\Rightarrow \quad u = (\|\cdot\|_{k,p}\text{-}\textit{Cauchy sequence of smooth } u_n : X \to \mathbb{R}) \\ &\Rightarrow \quad \partial^I u_n \textit{ are } \|\cdot\|_p\text{-}\textit{Cauchy } \forall |I| \leq k \\ &\Rightarrow \quad \partial^I u_n \to u_I \textit{ in } L^p, \textit{ some } u_I \in L^p(X) \, (\textit{by completeness of } L^p). \end{aligned}$$

But $\int \partial^I u_n \cdot \phi = (-1)^I \int u_n \cdot \partial^I \phi$, *take* $n \to \infty$, *deduce* $u_I =$ *weak derivs!*

Thm. *Define*

$$W^{k,p}(X) = \{u \in L^p(X) : u \textit{ has weak derivatives } \partial^I u \in L^p(X), \forall |I| \leq k\}$$

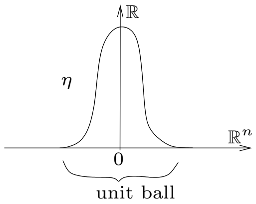
then $C^\infty(\overline{X}) \cap W^{k,p}(X) \subset W^{k,p}(X)$ *dense, so* $W^{k,p}(X) \cong$ *completion construction.*

Pf uses a standard method to smoothly approximate measurable fns: **mollifiers**.[13]

---

[12] *Non-examinable:* Any measurable fn is a limit of simple fns. Simple fns are linear combinations of characteristic fns χ_S of measurable subsets S ($\chi_S(s) = 1 \, \forall s \in S$, else $\chi_S = 0$).

[13] An explicit η is the following: $c \cdot \exp(\frac{1}{|x|^2 - 1})$ for $|x| \leq 1$, and 0 otherwise.



$\eta : \mathbb{R}^n \to \mathbb{R}$ smooth bump function, normalized so that $\int_{\mathbb{R}^n} \eta \, dx = 1$.

For $\varepsilon > 0$, define[14] $\boxed{\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \cdot \eta(\frac{x}{\varepsilon})}$

Observe: $\eta_\varepsilon \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\operatorname{supp} \eta_\varepsilon \subset \varepsilon$-ball, and $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$.

For $u : X \to \mathbb{R}$ in $L^1_{loc}(X)$, define ε-mollification as a *convolution*:

$$\begin{aligned} u_\varepsilon(x) &= (\eta_\varepsilon * u)(x) \\ &= \int_X \eta_\varepsilon(x-y)\, u(y)\, dy \\ &= \int_{\varepsilon\text{-ball}} \eta_\varepsilon(y)\, u(x-y)\, dy \end{aligned}$$

defined for $x \in X_\varepsilon = \{$points of X at distance $> \varepsilon$ from $\partial X\}$.

Fact.[15]

(1) *$u_\varepsilon(x)$ only depends on values of u near x* (*Pf. 2nd integral.*)
(2) *$u_\varepsilon \in C^\infty(X_\varepsilon)$* (*Pf. differentiate 1st integral.*)
(3) *$u_\varepsilon(x) \to u(x)$ for almost any x as $\varepsilon \to 0$*
(4) *u continuous $\Rightarrow u_\varepsilon \to u$ uniformly on compacts (hence $C^\infty \subset C^0$ is dense)*
(5) *$u \in L^p_{loc}(X) \Rightarrow u_\varepsilon \to u$ in $L^p_{loc}(X)$*

Cor. *$u \in W^{k,p}(X) \Rightarrow u_\varepsilon \to u$ in $W^{k,p}_{loc}(X)$*

Proof. Easy computation:

$$\partial^I u_\varepsilon = \eta_\varepsilon * \partial^I u \qquad (\text{in } X_\varepsilon)$$

but $\partial^I u \in L^p(X)$, so by (5), $\eta_\varepsilon * \partial^I u \to \partial^I u$ in L^p_{loc}. □

This corollary essentially implies the theorem by a clever[16] partition of unity argument (non-examinable).

4.5. Elementary proof of Sobolev/Rellich for $W^{1,2}$.

Theorem 1. *$W^{1,2}(\mathbb{R}) \overset{bdd}{\hookrightarrow} C^0_b(\mathbb{R}) = \{$bdd cts $\mathbb{R} \to \mathbb{R}\}$, and $W^{1,2}(\mathbb{R}) \overset{cpt}{\to} C^0([-S, S])$.*

Proof. For $u \in W^{1,2}$, pick $u_n \in C^0 \cap W^{1,2}$ converging to u in $W^{1,2}$ (by mollification, $C^0 \cap W^{1,2} \subset W^{1,2}$ is dense). So u_n is $W^{1,2}$-bdd and by Cauchy-Schwarz

$$|u_n(b) - u_n(a)| \leq \int_a^b |\partial_s u_n| \, ds \leq \sqrt{|b-a|} \cdot \|u_n\|_{1,2} \leq \text{const} \cdot \sqrt{|b-a|} \qquad (*)$$

so u_n is equicts. To check u_n is equibdd, suppose $u_n(a)$ is unbdd (fixed a). By $(*)$

$$\left| \min_{b \in [a-1, a+1]} u_n(b) - u_n(a) \right| \leq C$$

so that minimum is also unbdd. So u_n is L^2-unbdd, contradicting $W^{1,2}$-bdd.

---

[14] as $\varepsilon \to 0$, intuitively "$\eta_\varepsilon \to$Dirac delta".

[15] If you're curious: Evans, *Partial Differential Equations*, p.630.

[16] If you're curious: Evans, *Partial Differential Equations*, p.251-254. The Corollary gives the Thm for $C^\infty(X)$, and to get $C^\infty(\overline{X})$ one needs a little care near the boundary ∂X because the convolution requires having an ε-ball around x inside the domain. The fix is to locally (on a small open $V \subset X$) translate u: $\widetilde{u}(x) = u(x - c\varepsilon\vec{n})$ where $\vec{n}$ is the outward normal along ∂X and c is a large constant. Then $\eta_\varepsilon * \widetilde{u} \in C^\infty(\overline{V})$ cges to u in $W^{k,p}(V)$.

By Arzela-Ascoli, there is a subsequence $u_n|_{[-S,S]} \to v$ in $C^0[-S,S]$, so also in $L^2[-S,S]$, so $v = u|_{[-S,S]}$, so u is cts since S was arbitrary.

Need to check u is C^0-bounded. As in $(*)$, $|u(s+1)-u(s)| \leq \|u|_{[s,s+1]}\|_{1,2}$, so

$$|u(s+m)-u(s)| \leq \|u|_{[s,s+1]}\|_{1,2}+\cdots+\|u|_{[s+m-1,s+m]}\|_{1,2} = \|u|_{[s,s+m]}\|_{1,2} \leq \|u\|_{1,2}$$

so u is bdd at $\pm\infty$, hence bdd on $\mathbb{R}$ by cty. □

4.6. $W^{k,p}$ **for manifolds.** Let N^n be a compact mfd and M^m any mfd.

$W^{k,p}(N) = W^{k,p}(N,\mathbb{R})$ and $W^{k,p}(N,M)$ are the completion of $C^\infty(N)$ and $C^\infty(N,M)$ w.r.t. the $\|\cdot\|_{k,p}$ norm defined below. Equivalently, they are the space of measurable functions/maps[17] which are k-times weakly differentiable (in the charts below) and which have bounded $\|\cdot\|_{k,p}$-norm.

Def. *$W^{k,p}(N,\mathbb{R}^m)$ for N^n compact mfd: pick a finite cover by charts*[18]

$$\varphi_i : (\textit{ball } B_i \subset \mathbb{R}^n) \to U_i \subset N$$

For $u : N \to \mathbb{R}^m$, define $\boxed{\|u\|_{k,p} = \sum \|u\circ\varphi_i\|_{W^{k,p}(B_i,\mathbb{R}^m)}}$

$W^{k,p}(N,M)$, any mfd M^m: fix smooth embedding $j : M \hookrightarrow \mathbb{R}^a$. For $u : N \to M$ let[19]

$$\|u\|_{k,p} = \|j\circ u\|_{W^{k,p}(N,\mathbb{R}^a)}.$$

Rmk.

(1) *N compact $\Rightarrow$ get equivalent norms if change charts*

(2) *$X, Y \subset \mathbb{R}^n$ open, $k \geq 1$, call $\phi : X \to Y$ a C^k-diffeo if: ϕ is a homeomorphism, $\phi \in C^k(\overline{X},\overline{Y})$, $\phi^{-1} \in C^k(\overline{Y},\overline{X})$ and both have bdd C^k-norm.*

Fact. *$W^{k,p}(Y) \xrightarrow{\circ\phi} W^{k,p}(X)$ is bdd with bdd inverse.*

Cor. *N compact $\Rightarrow$ get equivalent norm if change φ_i, U_i.*

(3) *$\phi : X \to Y$ has bdd C^k-norm $\Rightarrow W^{k,p}(N,X) \xrightarrow{\phi\circ} W^{k,p}(N,Y)$ bdd*

Rmk. *just bound $\phi\circ u$ in terms of $\|\phi\|_{k,p}$, $\|u\|_{C^k}$. If you wanted to bound $\phi\circ u$ in terms of $\|\phi\|_{k,p}$, $\|u\|_{k,p}$, then even for smooth ϕ you need $kp > n$.*

Cor. *M compact$\Rightarrow$ choice of j does not matter (for non-cpt M it matters)*

4.7. $W^{k,p}$ **for vector bundles.** For a vector bundle $E \to N$,

$$W^{k,p}(E) = \{W^{k,p} \text{ sections } u : N \to E\}$$

In this case, you can avoid picking j:

$$B_i \times \mathbb{R}^r \overset{\varphi_i}{\cong} U_i \times \mathbb{R}^r \overset{\text{triv}}{\cong} E|_{U_i}$$

view $u\circ\varphi_i$ as a map $B_i \to \mathbb{R}^r$

$$\|u\|_{k,p} = \sum \|(\rho_i \cdot u)\circ\varphi_i\|_{W^{k,p}(B_i,\mathbb{R}^r)}$$

---

[17] $W^{k,p}(N,M) \subset W^{k,p}(N,\mathbb{R}^a)$, the $u : N \to \mathbb{R}^a$ with $u(n) \in M \subset \mathbb{R}^a$ for almost every $n \in N$.

[18] strictly speaking these are *parametrizations*: they go from $\mathbb{R}^n$ to N. If you want charts $\varphi_i : U_i \to \mathbb{R}^n$, then you need bump functions ρ_i subordinate to the U_i: $\sum \|(\rho_i\cdot u)\circ\varphi_i^{-1}\|_{W^{1,2}(\mathbb{R}^n,\mathbb{R})}$.

[19] Using charts on M would be a bad idea: think about why that would not work.

Alternatively, pick: a Riem metric g_N on N, a metric g_E on E (smoothly varying inner product for each fibre), and a connection ∇ on E. Then define:[20]

$$\boxed{\|u\|_{k,p} = \sum_{i \leq k} \left(\int_N |\nabla^i u|^p \operatorname{vol}_N \right)^{1/p}}$$

Lemma 2. *N compact $\Rightarrow$ those two definitions give equivalent norms.*

Proof. Choice of local trivializations doesn't matter since they change by multiplication by a smooth matrix-valued map (use Rmk 3 above).

Pick local trivializns using smooth local *orthonormal* sections. So $|\cdot|$ differs from $|\cdot|_{\mathbb{R}^a}$ only by use of g_N^* in $\Omega^i(N)$ directions. So get bounds since N is compact.

Locally $\nabla = d+A$ (A local section of $\operatorname{End}(E)$), hence can bound $u, \ldots, \nabla^{i-1}u, \nabla^i u$ in terms of $\|A\|_\infty$, u, $\partial^I u$ ($|I| \leq i$). Vice-versa can bound $\partial^I u$ in terms of $\|A\|_\infty$, $\nabla^i u$ ($i \leq |I|$) by the triangle inequality. ☐

4.8. **Sobolev theorems for manifolds.** For a compact mfd N, any mfd M:

$$\boxed{\begin{array}{lll} L^p(N) \overset{\text{bdd}}{\hookrightarrow} L^q(N) & \text{for } p \geq q \quad (\text{since } \operatorname{vol}(N) < \infty) & \\ W^{k,p}(N,M) \overset{\text{bdd}}{\hookrightarrow} W^{k',p'}(N,M) & \text{for } \begin{cases} k \geq k' \\ k - \frac{n}{p} \geq k' - \frac{n}{p'} \\ {\scriptstyle (p' \neq \infty \text{ if } kp = n)} \end{cases} & (\text{compact if strict } > \text{'s}) \\ W^{k,p}(N,M) \overset{\text{bdd \& cpt}}{\hookrightarrow} C^{k'}(N,M) & \text{for } k - \frac{n}{p} > k' & \end{array}}$$

Warning. Fails for non-compact N, unless you have control of the geometry at ∞: for example for $N = \mathbb{R}, \mathbb{R}^n$ the above still holds.

Def. $W^{k,p}_{loc}(N,M) = \{u : N \to M : \ u|_C \in W^{k,p}(C,M), \ \forall C \Subset N\}$

Warning. the $W^{k,p}_{loc}$ are not normed, but they are complete metric spaces with the topology: $u_n \to u$ in $W^{k,p}_{loc} \Leftrightarrow u_n|_C \to u|_C$ in $W^{k,p}(C,M) \, \forall C \Subset N$.

Exercise. $u \in W^{k,p}_{loc} \Leftrightarrow \exists u_n \in C^\infty_c$, $u_n \to u$ in $W^{k,p}_{loc}$. So $W^{k,p}_{loc} \cong$ completion defn.

Cor. *Sobolev embeddings hold for*[21] *$W_{loc}, L_{loc}, C_{loc}$ even for non-compact N.*

Proof. $u \in W^{k,p}_{loc}(N,M) \Rightarrow u|_C \in W^{k,p}(C,M) \Rightarrow u|_C \in W^{k',p'}(C,M) \Rightarrow u \in W^{k',p'}(N,M)$ ☐

---

[20] where $\nabla^0 u = u$, $\nabla^i : C^\infty(E) \to \Omega^i(N) \otimes C^\infty(E)$ (extending ∇ to higher forms by Leibniz: $\nabla(\omega \otimes s) = d\omega \otimes s + \omega \otimes \nabla s$), and $\operatorname{vol}_N = \sqrt{|\det g_N|}\, dx_1 \wedge \cdots \wedge dx_n$, and the norm in the integral combines the norm from g_E on E and the norm from the dual metric g_N^* on T^*N (which induces a metric on the exterior product $\Lambda^i T^*N$ - explicitly, use g_N to locally define an orthonormal frame for TN by Gram-Schmidt, declare the dual of that to be an o.n. frame for T^*N, this determines g_N^*, and taking ordered i-th wedge products you declare what an o.n. frame for $\Lambda^i T^*N$ is).

[21] C^k_{loc} just means C^k-convergence on compact subsets.

LECTURE 12.

PART III, MORSE HOMOLOGY, 2011

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4.9. **Sobolev setup for the transversality theorem.** Let M be a closed manifold, $f : M \to \mathbb{R}$ a Morse function, and fix critical points $p \neq q \in \mathrm{Crit}(f)$ and a reference metric g_M for M (all norms will refer to g_M, not the variable metric g).

Consider the bundle mentioned in 4.0:

$$\begin{tikzcd} E \arrow[d] \\ U \times M \arrow[u, bend right, "F(u,g) = \partial_s u + f'_g(u)"'] \end{tikzcd}$$

where:[1]

$G =$	$\{C^k\text{-metrics on } M\}$ (fix $k \geq 1$)
$U =$	$\{u \in W^{1,2}_{loc}(\mathbb{R}, M) : u(s) \to p, q \text{ as } s \to -\infty, +\infty$ and for large $\mathcal{S}$ we have : $u(s) = \exp_p(\xi(s))$ for $s \leq -\mathcal{S}$, some $\xi \in W^{1,2}((-\infty, -\mathcal{S}), T_pM)$ $u(s) = \exp_q(\xi(s))$ for $s \geq +\mathcal{S}$, some $\xi \in W^{1,2}((+\mathcal{S}, \infty), T_qM)\}$
$E =$	$\{L^2\text{-vector fields along the paths } u \in U\}$

By Sobolev, $W^{1,2}_{loc} \subset C^0_{loc}(\mathbb{R}, M)$, so the $u \in U$ are continuous, and requiring convergence to p, q at $\pm\infty$ makes sense.

E is a vector bundle over $U \times G$ with fibre $L^2(u^*TM)$, the L^2-sections of the pull-back bundle $u^*TM \to \mathbb{R}$ whose fibre is $(u^*TM)_s = T_{u(s)}M$ over $s \in \mathbb{R}$.

Lemma. *G is a smooth Banach manifold.*

Proof. $G \subset C^k(Sym^2(T^*M))$ is an open subset of the space[2] of symmetric 2-forms on TM, since positive definiteness is an open condition. We have a regular retraction $\pi : U \to M$ of a tubular nbhd of $j : M \hookrightarrow \mathbb{R}^a$ (so $\pi \circ j = id$). Recall by 1.6, that for a Banach space B and a closed subset $S \subset B$,

$$S \text{ smooth retract of an open nbhd of } S \subset B \Rightarrow S \subset B \text{ is a submfd.}$$

Also recall that for any map $\varphi : A \to B$ of mfds, g metric on B, the pull-back metric is defined by $(\varphi^* g)_a(v, w) = g_{\varphi a}(d\varphi \cdot v, d\varphi \cdot w)$ for $v, w \in T_aA$.

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[1] **Rmk.** *We cannot just say $U \subset W^{1,2}(\mathbb{R}, M)$: Sobolev spaces don't make sense for non-compact domains (unless you are considering sections of a bundle). Example: the constant $u : \mathbb{R} \to \{p\} \in M$ has $\int_{\mathbb{R}} |u(s)|^2\, ds = |p|^2 \cdot \int_{\mathbb{R}} 1\, ds = \infty$ for $|p| \neq 0$ the norm of $p \in M \subset \mathbb{R}^a$. Similarly, continuous u converging to p, q have infinite L^2-norm. So our Sobolev spaces would be empty! We want each $p \in Crit(f)$ to be considered to have zero norm, for that reason we chose (canonical, using* exp*) charts around the critical points and require u to be $W^{1,2}$ in that chart.*

[2] Example: $g = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ is the symmetric form $dx^{\otimes 2} + 2dx \otimes dy + 2dy \otimes dx + 3dy^{\otimes 2}$, so $g(\partial_x, \partial_y) = 2$.

First note that

$$S = C^k(Sym^2(T^*M)) \hookrightarrow C^k(Sym^2(T^*U)), g \mapsto \pi^* g$$

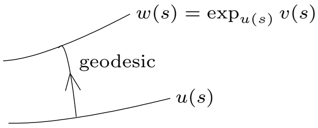
is a closed subset (injective since $j^*\pi^*g = (\pi j)^*g = g$). Secondly

$$C^k(Sym^2(T^*U)) \to S, g \mapsto \pi^* j^* g$$

is a retraction (since $\pi^*j^*\pi^*j^*g = \pi^*(\pi j)^*j^*g = \pi^*j^*g$). Finally, both these maps are smooth since linear in g. Thus, by the above result, S is a smooth manifold, hence also the open subset G is a smooth Banach manifold. ☐

Lemma. *U is a smooth Banach manifold modeled on $W_0^{1,2}(\mathbb{R}, \mathbb{R}^m)$.*

Proof. The reference metric defines and exp map, and by Cor 0.7: any C^0-close continuous paths u, w are homotopic through geodesic arcs joining $u(s), w(s)$.

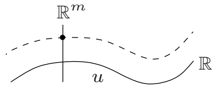


Let $\varepsilon > 0$ be as in that Corollary. For each *smooth* $u \in U$, define

$$\boxed{W = \{\exp_{u(s)} v(s) : v \in W_0^{1,2}(u^* D_\varepsilon TM)\}}$$

(so in particular, $v(s) \to 0$ as $s \to \pm\infty$). You can easily check that $W \subset U$, by construction (this crucially uses the fact that u, exp are smooth maps, so $\exp_u v$ is as smooth as v is). By Cor 0.7, and the density of C^∞ maps inside $W^{1,2}$ maps, any $w \in U$ is within ε-distance of some smooth $u \in U$, therefore these W's cover U.

Let ∇ be the Levi-Civita connection for g_M, then by parallel translation



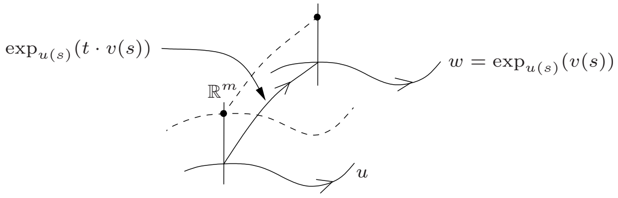
$$\boxed{u^*TM \cong \mathbb{R} \times \mathbb{R}^m}$$

so $W_0^{1,2}(u^*TM) \cong W_0^{1,2}(\mathbb{R}, \mathbb{R}^m)$, which is a Banach space. Thus the v are local coordinates for U near u. You can easily check that transition maps on overlaps are smooth, since they involve *smooth* maps u_0, u_1, exp. ☐

Rmk. *For a Banach manifold M modeled on B, the tangent space is $T_mM = B$ (compare with $T_x\mathbb{R}^n \equiv \mathbb{R}^n$). For a chart $\varphi : M \supset U \to B$ define $TM|_U = \varphi(U) \times B$. For another chart $\psi : U' \to B$, the transition $\varphi(U \cap U') \times B \to \psi(U' \cap U) \times B$ is $(x, b) \mapsto (x, d_x(\psi \circ \varphi^{-1}) \cdot b)$. In our example:* $\boxed{T_v W_0^{1,2}(u^* D_\varepsilon TM) \equiv W_0^{1,2}(u^*TM)}$

Lemma. *E is a smooth Banach vector bundle with fibre $L^2(\mathbb{R}, \mathbb{R}^m)$.*

Proof. Consider parallel transport along geodesics:



$$P_v : u^*TM \xrightarrow{\sim} (\exp_u v)^*TM = w^*TM$$

Recall this map is smooth (since linear) and is an isometry. Now, consider its dependence on v:

$$P : W_0^{1,2}(u^*D_\varepsilon TM) \times u^*TM \xrightarrow{\sim} \bigcup_{w \in W} w^*TM.$$

This is again smooth.[3]

Finally, consider parallel transporting L^2-vector fields over u:

$$P : W_0^{1,2}(u^*D_\varepsilon TM) \times L^2(u^*TM) \xrightarrow{\sim} L^2(\bigcup_{w \in W} w^*TM) = E|_W.$$

This is well-defined since parallel transport is an isometry (so L^2 sections map to L^2 sections), and is invertible by doing parallel transport backwards. It is smooth for the same reasons as before. As above, we can trivialize: $L^2(u^*TM) \cong L^2(\mathbb{R}, \mathbb{R}^m)$. Thus we have obtained a trivialization of $E|_W$:

$$E|_W \cong W \times L^2(\mathbb{R}, \mathbb{R}^m)$$

and two trivializations differ by smooth maps since $u_0, u_1, \exp$ are smooth. □

---

[3] *Non-examinable:* The ODE you solve is $\partial_t \vec{x}(t) = -A_{\exp_{u(s)}(tv(s))}((d\exp_{u(s)})_{tv(s)} \cdot v(s)) \cdot \vec{x}(t)$, $\vec{x}(0) \in T_{u(s)}M$ (where $s \in \mathbb{R}$ is fixed, $t \in [0,1]$ varies). Change $v(s)$ to $v(s) + \lambda \vec{v}(s)$: observe that $\vec{x}(1)$ is smooth in $\lambda \in \mathbb{R}$ because solutions of ODEs depend smoothly on initial conditions.

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4.9. Sobolev setup for transversality theorem (continued).

$$\begin{tikzcd} E \arrow[d] \\ U \times M \arrow[u, bend right, "{F(u,g) = \partial_s u + f'_g(u)}"'] \end{tikzcd}$$

where f'_g is ∇f calculated for g: $g(f'_g, \cdot) = df$.

Lemma. *F is a well-defined section.*

Proof. The weak derivative $\partial_s : W^{1,2} \to L^2$ is well-defined so only f'_g may be an issue. Since f'_g is C^k, it is L^2 on compacts, so we just need to check that $f'_g(u)$ is L^2 near the ends. Locally near a critical point $p = (x = 0)$:

$$|f'_g(x)| \leq c \cdot |x| \text{ by Taylor, since } f'_g(p) = 0.$$

Hence $|f'_g(u)| \leq c \cdot |u|$, so $f'_g(u)$ is L^2 near the ends since u is L^2. ☐

Lemma. *F is C^k.*

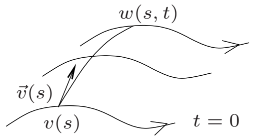
Proof. Differentiating in the g direction: only f'_g contributes, and

$$df = g(f'_g, \cdot)$$

so locally $f'_g = g^{-1} \cdot \partial f$. This is linear in g^{-1}, hence smooth in g^{-1}. Finally, inversion is smooth on non-singular matrices. So f'_g is smooth in g.

Differentiating in the direction $\vec{v} \in W^{1,2}_0(u^*TM) \equiv T_v W^{1,2}_0(u^* D_\varepsilon TM)$ (where u is smooth):

$$\begin{array}{rcll} D_v F \cdot \vec{v} & = & \nabla_t|_{t=0}(\partial_s w - f'_g(w)) & \text{(see Hwk 12)} \\ & & \text{where } w : \mathbb{R} \times [0,1] \to M & w(s,t) = \exp_{v(s)}(t\vec{v}(s)) \end{array}$$



$$\begin{array}{rcll} & = & (\nabla_s \partial_t w - \nabla_t f'_g(w))_{t=0} & \text{(using } \nabla \text{ torsion-free (see Hwk 12))} \\ & = & \nabla_s \vec{v} - (\nabla_{\vec{v}} f'_g)_v & \end{array}$$

Note: $\nabla_s = \partial_s + A(\partial_s v)\cdot$ is linear in v so C^∞ in v, $(\nabla_{\vec{v}} f'_g)_v = (d(f'_g) \bullet + A(\bullet) \cdot f'_g)V$: the first term is C^{k-1}, $A(\bullet)$ is C^∞, f'_g is C^k, so $(\nabla_{\vec{v}} f'_g)_v$ is[1] C^{k-1} in v. ☐

Date: May 3, 2011,

[1] The key point is that we never differentiate v in s.

4.10. **Transversality Theorem.**

Lemma. $F(u,g)=0 \Rightarrow u \in C^{k+1}$ *(motto: u is "as smooth as" g)*

Proof. The weak derivative $\partial_s u = f_g'(u)$ is continuous since u is cts.[2] Recall that if the weak derivative is cts then it equals the usual derivative. Now bootstrap.[3] $\square$

Conditions of Thm 2.5

$\forall\, F(u,g)=0$:

i) $D_{(u,g)}F : T_{(u,g)}(U \times G) \to E_{(u,g)}$ surjective;

ii) $D_u F_g : T_u U \to E_{(u,g)}$ Fredholm of index $< k$.

Claim 1. (ii) holds and index $= |p|-|q|$ (so pick $k > |p|-|q|$)
Claim 2. (i) holds

We will prove the Claims later. First we mention the consequence:

Transversality Theorem
For generic C^k-metrics g,

$$W^k(p,q) = \{C^k\text{-flowlines } u \text{ of } -\nabla f \text{ from } p \text{ to } q\} = F_g^{-1}(0)$$

is a C^k-submanifold of U with

$$\dim W^k(p,q) = \operatorname{index} F_g = |p|-|q|$$

and tangent space

$$T_u W^k(p,q) = \ker(D_u F_g : T_u U \overset{\text{surj}}{\to} E_u)$$

Proof. Theorem 2.5 and Claims 1 & 2 above. $\square$

Cor. *Can take $k=\infty$*

Proof. Pick C^k metric g satisfying $\pitchfork$,
$\Rightarrow \exists C^\infty$ g' close to g
$\Rightarrow g'$ satisfies $\pitchfork$ since transversality is an open condition (regularity is open).
Hwk 13: C^∞-metrics g satisfying $\pitchfork$ are generic. $\square$

$$\Rightarrow \boxed{\begin{array}{l}\text{for a generic smooth metric,}\\ \qquad W(p,q)=W^\infty(p,q) \text{ is a smooth mfd of dim} = |p|-|q| \\ \qquad \mathcal{M}(p,q)=W(p,q)/\mathbb{R} \text{ is a smooth mfd of dim} = |p|-|q|-1\end{array}}$$

Rmk. *Hwk 13 proves that the quotient $\mathcal{M}(p,q)$ really is a smooth mfd.*

Rmk. *Hwk 19 proves that the transversality condition for $W(p,q)$, that is the surjectivity of the linearization $D_u F_g$ at zeros of F_g, is equivalent to the condition that g is Morse-Smale for f. So* $\boxed{\textit{“generic } g\textit{” is equivalent to saying “}g\textit{ is Morse-Smale”}}$

---

[2] *Non-examinable:* this is an example of the elliptic regularity theorem: ∂_s is an elliptic operator of order 1, and so weak solutions of $\partial_s u = -f_g'(u)$ are as regular as f_g' plus order, so u is C^{k+1}.

[3] We proved u is C^1, so $\partial_s u = f_g'(u)$ is C^1 (since f_g' is C^k and u is C^1) so u is C^2 etc.

4.11. **Hilbert spaces tricks.** $L : A \to B$ bounded linear, A Banach, B Hilbert.

Lemma 1 *If* $\operatorname{im}(L)$ *is closed, then*

$$\operatorname{coker} L \cong (\operatorname{im} L)^{\perp} = \{b \in B : \langle La, b\rangle = 0 \ \forall a \in A\}.$$

Proof. In general, if $V \subset B$ closed subspace, then $B = V \oplus V^{\perp}$, so $V^{\perp} \cong B/V$. □

Warning. $C^{\infty} \subset L^2$ dense (non-closed) subspace: $(C^{\infty})^{\perp} = 0$, but $C^{\infty} \neq L^2$.

Def. *Define the* adjoint $L^* : B \to A$ *of* $L : A \to B$, *where* A, B *Hilbert, by*

$$\langle La, b\rangle_B = \langle a, L^* b\rangle_A \qquad \forall a, b$$

(easy exercise: $\exists$ *unique bounded linear such* L^**).*

Lemma 2 $(\operatorname{im} L)^{\perp} \cong \ker L^*$.

Proof. $b \perp \operatorname{im} L \Leftrightarrow \langle La, b\rangle = 0 = \langle a, L^* b\rangle \ \forall a \Leftrightarrow L^* b = 0$. □

$L : A = A_1 \times A_2 \to B$, A_1 Hilbert, A_2 Banach, B Hilbert.
$L(a_1, a_2) = D(a_1) + P(a_2)$
$D : A_1 \to B$, $P : A_2 \to B$ bounded linear (think of P as "perturbation").

Lemma 3 $\operatorname{im} L$ *closed* $\Rightarrow \operatorname{coker} L \subset \ker(D^* : B \to A_1) \cap (\operatorname{im} P)^{\perp}$.

Proof. $\operatorname{im} D \subset \operatorname{im} L$
$\Rightarrow \ker D^* \overset{\text{by 2}}{\cong} (\operatorname{im} D)^{\perp} \supset (\operatorname{im} L)^{\perp} \overset{\text{by 1}}{\cong} \operatorname{coker} L$
$\Rightarrow \operatorname{coker} L \perp \operatorname{im} L \supset \operatorname{im} P$. □

Lemma 4 D Fredholm $\Rightarrow \operatorname{im} L$ closed.

Proof. $B = \operatorname{im} D \oplus C$, $C =$ complement (finite dimensional).
$\operatorname{im} L = \operatorname{im} D \oplus (C \cap \operatorname{im} L)$ (equality holds since $\operatorname{im} D \subset \operatorname{im} L$)
Finally: $\operatorname{im} D$ closed, and $C \cap \operatorname{im} L$ is finite dimensional hence closed. □

4.12. **Claim 1** $\Rightarrow$ **Claim 2.** We will apply Lemma 3 to:

$$\underbrace{(D_{(u,g)}F)}_{L} \cdot (\vec{u}, \vec{g}) \quad = \quad \underbrace{D_u F_g}_{D} \cdot \vec{u} - \underbrace{D_{(u,g)} f'_g}_{P} \cdot \vec{g}$$

$A_1 = W_0^{1,2}(u^*TM) \cong W_0^{1,2}(\mathbb{R}, \mathbb{R}^m) \quad \langle a, a'\rangle_{1,2} = \int_{\mathbb{R}} g_M(a, a')\, ds + \int_{\mathbb{R}} g_M(\partial_s a, \partial_s a')\, ds$
$A_2 = T_g G$
$B = E_u = L^2(u^*TM) \cong L^2(\mathbb{R}, \mathbb{R}^m) \quad \langle b, b'\rangle_{L^2} = \int_{\mathbb{R}} g_M(b, b')\, ds.$

Where D is Fredholm by Claim 1.

Rmk. $F(u, g) = 0$ *so* u *is* C^{k+1}, *and we are using the charts defined by trivializing* TM *over* u *since we only need* C^k*-Banach mfd structures (*F *is only* C^k *anyway). If you want to use the* C^{∞}*-Banach space structures, then trivialize over a* smooth u, *and study* $F(v, g) = 0$ *(where* v *is an abbreviation for* $\exp_{u(s)} v(s)$, $v \in W_0^{1,2}(u^* D_{\varepsilon} TM)$*) so replace* u*'s by* v*'s except in the defns of* A_1, B.

Cor. $\operatorname{coker} D_{(u,g)}F \subset \ker(D_u F_g)^* \cap (\operatorname{im} P)^{\perp}$.

Claim 2 $D_{(u,g)}F$ is surjective.

Proof. If not, then $\exists b \neq 0 \in E_u$:

(1) $(D_u F_g)^* b = 0$
(2) $\langle Df'_g \cdot \vec{g}, b \rangle_{L^2} = 0 \ \ \forall \vec{g}$.

Key trick 1: (1) $\Rightarrow$ b continuous. (Proof in next Lecture)
Key trick 2: $b(s_0) \neq 0$ for some $s_0 \in \mathbb{R}$. Claim: it suffices to define $\vec{g}$ at $u(s_0)$ with

$$g_M((Df'_g)_{u(s)} \cdot (\vec{g})_{u(s)}, b(s)) > 0 \qquad \text{at } s = s_0 \qquad (*)$$

Proof of Claim:
pick any C^k-extension of $(\vec{g})_{u(s_0)}$ to $\vec{g}_x$ defined for $x \in M$ close to $u(s_0)$
$\Rightarrow$ by continuity $(*)$ holds near s_0
globalize $\vec{g}$: multiply $\vec{g}$ by a bump function ($= 0$ away from $u(s_0)$, $= 1$ at $u(s_0)$)
$\Rightarrow$ $(*)$ holds with "$\geq$" for all s, and with "$>$" near s_0
$\Rightarrow$ (2) fails. Contradiction. $\square$

Construction of $\vec{g}$ as in the Claim:
Locally $f'_g = g^{-1}\partial f$, so

$$Df'_g \cdot \vec{g} = (\partial_t|_{t=0}(g + t\vec{g})^{-1})\,\partial f.$$

Now use the usual series trick:

$$\begin{aligned} (g + t\vec{g})^{-1} &= [g \cdot (1 + tg^{-1}\vec{g})]^{-1} \\ & (1 + tg^{-1}\vec{g})^{-1} \cdot g^{-1} \qquad \text{(careful with order of matrices!)} \\ & (1 - tg^{-1}\vec{g} + \text{order}\, t^2) \cdot g^{-1} \end{aligned}$$

Hence $\boxed{Df'_g \cdot \vec{g} = -g^{-1} \cdot \vec{g} \cdot g^{-1} \cdot \partial f}$

Now $\partial f \neq 0$ since[4] $u(s_0) \notin \mathrm{Crit}(f)$.

Moreover, $g^{-1} \cdot \vec{g} \cdot g^{-1}$ is an arbitrary[5] symmetric matrix at s_0 by letting $\vec{g}$ vary: indeed to get the symmetric matrix S take $\vec{g} = gSg$.

$\Rightarrow$ $Df'_g \cdot \vec{g}$ is arbitrary at s_0. So in our case, we pick $\vec{g}$ so that $Df'_g \cdot \vec{g} = b(s_0)$. $\square$

---

[4] u is a $-\nabla f$ trajectory from p to $q \neq p$, so it is non-constant: the unique $-\nabla f$ trajectory passing through a critical point is the constant trajectory at the critical point.

[5] $G \subset \mathrm{Sym}^2(T^*M)$ is an open subset, so $T_g G = T_g \mathrm{Sym}^2(T^*M)$.

LECTURE 14.

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4.13. **Claim 1.**

Aim: $F(u,g)=0 \Rightarrow D_uF_g = \nabla_s \bullet - \nabla_\bullet f_g'(u) : T_uU \to E_{(u,g)}$ is Fred, index $|p|-|q|$.

Rmk. $F(u,g)=0 \Rightarrow u$ is C^k (Lemma 4.10). We will prove the Aim more generally for any C^k-path $u \in U$ (we will not use $F(u,g)=0$ anymore).

$u \in C^k \Rightarrow$ can trivialize u^*TM by parallel transport:
Pick basis $e_1, \ldots, e_m$ of u^*TM with $\nabla_s e_i = 0$

$V = \sum V^i e_i$
$\nabla_s V = \sum \partial_s V^i e_i$
$\nabla_V f_g'(u) = \sum V^j \nabla_{e_j} f_g'(u) = \sum (A_s)^i_j V^j e_i$ (this defines a matrix A_s)

Cor. *D_uF_g in the basis e_i is:*

$$\boxed{\begin{array}{ccc} W_0^{1,2}(\mathbb{R},\mathbb{R}^m) & \to & L^2(\mathbb{R},\mathbb{R}^m) \\ \begin{pmatrix} V^1 \\ \vdots \\ V^m \end{pmatrix} & \mapsto & (\partial_s + A_s)\cdot \begin{pmatrix} V^1 \\ \vdots \\ V^m \end{pmatrix} \end{array}}$$

Define the adjoint A_s^* by $\boxed{g_M(A_s^* x, y) = g_M(x, A_s y) \ \ \forall x,y \in \mathbb{R}^m}$.

Lemma. *$(\partial_s + A_s)^* b = 0$ for $b \in L^2 \Rightarrow b \in W^{1,2} \overset{Sobolev}{\Rightarrow} b \in C^0$.*

Proof. $0 = \langle (\partial_s + A_s)^* b, \phi \rangle_{1,2} = \langle b, (\partial_s + A_s)\phi \rangle_{L^2} = 0 \ \ \forall \phi \in L^2$
$\Rightarrow \langle b, \partial_s \phi \rangle_{L^2} = -\langle A_s^* b, \phi \rangle_{L^2} \ \forall \phi \in C_c^\infty$
$\Rightarrow$ weak $\partial_s b = A_s^* b \in L^2$
$\Rightarrow b \in W^{1,2}$. □

Cor. *4.12 **Key trick 1** holds, since $(D_{(u,g)}F_g)^* = (\partial_s + A_s)^*$ in the trivialization.*

Rmk. *In general, such Lemmas are proved by the elliptic regularity theorem.*[1]

Rmk. $\boxed{W_0^{1,2}(\mathbb{R},\mathbb{R}^m) = W^{1,2}(\mathbb{R},\mathbb{R}^m)}$ *and similarly for L^2.*

Proof. Given $b \in W^{1,2}$, $\exists b_n \in C^\infty$ with $b_n \to b$ in $W^{1,2}$. Pick bump $\beta_n : \mathbb{R} \to [0,1]$, $\beta_n = 1$ on $[-n+1, n-1]$ and $\beta_n = 0$ outside $[-n,n]$, and $|\partial_s \beta_n| \leq 2$. Then $\beta_n b_n \to b$ in L^2 since $\int_{\mathbb{R}\setminus[-n,n]} |b|^2 \, ds \to 0$ as $n \to \infty$. Also, $\partial_s(\beta_n b_n) = (\partial_s \beta_n)\cdot b_n + \beta_n \cdot \partial_s b_n$: the second term $\to \partial_s b$ in L^2 by the same argument, and the first term is supported outside $[-n,n]$ and bounded by $2\int_{\mathbb{R}\setminus[-n,n]} |b|^2 \, ds \to 0$ as $n \to \infty$. □

Date: May 2, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1] This is a general bootstrapping result for $W^{k,2}$ spaces. *Non-examinable:* for an elliptic operator L (such as ∂_s or the Laplacian) the weak solutions u of $Lu = f$ with $f \in W^{k,2}$ (and $u = 0$ on ∂U, with smooth ∂U) lie in $u \in W^{k+\mathrm{order}(L),2}$. For $f \in C^\infty = \cap_k W^{k,2}$ get $u \in \cap_k W^{k,2} = C^\infty$.

Rmk. *The same arguments show* $W^{k,p}(\mathbb{R},\mathbb{R}^m) = W_0^{k,p}(\mathbb{R},\mathbb{R}^m)$*. For other examples, see Adams,* Sobolev spaces*, p.70.*

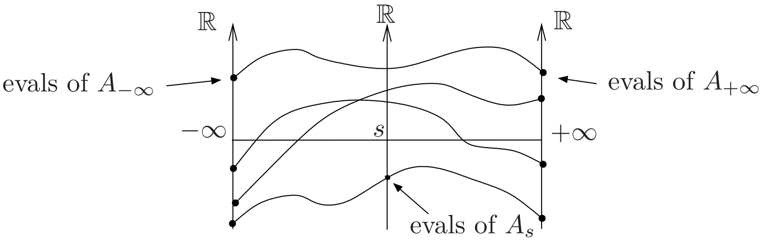
Claim 1 follows immediately from the following theorem:

Thm.

(1) $s \mapsto A_s$ *is a* C^{k-1}*-path of symmetric matrices, with* $A_s \to \mathrm{Hess}_p f$, $\mathrm{Hess}_q f$ *in* C^{k-1} *as* $s \to -\infty, +\infty$*.* Proof. Hwk 2, (1iii). $\square$

(2) $\partial_s + A_s : W^{1,2}(\mathbb{R},\mathbb{R}^m) \to L^2(\mathbb{R},\mathbb{R}^m)$ *is a Fredholm operator*

(3) $\operatorname{index}(\partial_s + A_s) =$ *"spectral flow of* A_s*"* $= |p| - |q|$.
Explanation: suppose you could[2] *diagonalize* A_s *continuously in* s*, then the "movie" of eigenvalues is:*



$$\begin{aligned}\text{spectral flow} &= \#(\textit{evalues of } A_s \textit{ moving from negative to positive}) - \\ & -\#(\textit{evalues of } A_s \textit{ moving from positive to negative}) \\ &= \operatorname{index}(\mathrm{Hess}_p f) - \operatorname{index}(\mathrm{Hess}_q f) \\ &= |p| - |q|\end{aligned}$$

In the picture: spectral flow $= 1 - 0$ *and* $|p| - |q| = 3 - 2$

Rmk. *The spectral flow is a useful "relative index" in the* ∞*-dimensional settings when the absolute indices* $|p|, |q|$ *are infinite.*

4.14. $\partial_s + A_s$ **is Fredholm.**

($*$) $L = \partial_s + A_s : W^{1,2}(\mathbb{R},\mathbb{R}^m) \to L^2(\mathbb{R},\mathbb{R}^m)$ with $A_s \to A_{\pm\infty}$ symm nondeg
$K = \text{restriction} : W^{1,2}(\mathbb{R},\mathbb{R}^m) \to L^2([-S,S],\mathbb{R}^m)$ compact (by Sobolev)

Aim: deduce $\partial_s - A_s$ has finite dimensional kernel and closed image from Lemma:

Closed range Lemma

A, B, C Banach spaces
$L : A \to B$ bounded linear
$K : A \to C$ compact[3] bounded linear

$$\boxed{\begin{array}{l}\text{If } \|a\|_A \le \text{constant}\cdot(\|La\|_B + \|Ka\|_C) \quad \forall a \in A \\ \text{then } \begin{cases}\ker L \text{ finite dimensional} \\ \operatorname{im} L \text{ closed}\end{cases}\end{array}}$$

Proof. WLOG constant $= 1$ (rescale L, K)
$\ker L \subset \{a \in A : \|a\|_A \le \|Ka\|_C\}$ closed
Suppose $a_n \in \ker L : \|a_n\| = 1$
$\Rightarrow$ subseq $Ka_n \to c$ (K compact)

---

[2] This is presumably true generically over $\mathbb{C}$, but we will take a different approach.

[3] This means: if a_n is bdd, then Ka_n has a cgt subseq.

$\Rightarrow \|a_n - a_m\|_A \leq \|Ka_n - Ka_m\|_C \to \|c - c\|_C = 0$

$\Rightarrow a_n$ Cauchy, hence cgt

$\Rightarrow$ unit sphere in $\ker L$ is compact

$\Rightarrow \boxed{\ker L \text{ finite dimensional}}$

$\Rightarrow \exists$ closed complement A_0 of $\ker L$, so $A = \ker L \oplus A_0$ (see 2.3(1))

$\Rightarrow L : A' \to B$ injective.

Suppose $La_n \to b$. WLOG $a_n \in A_0$ (without changing La_n).

Suppose by contradiction that $\|a_n\|$ is unbounded.

$\Rightarrow \exists$ subsequence $\|a_n\| \to \infty$

$\Rightarrow \widetilde{a_n} = \frac{a_n}{\|a_n\|}$ has $L\widetilde{a_n}$ cgt (since $La_n \to b$)

$\Rightarrow \|\widetilde{a_n} - \widetilde{a_m}\|_A \leq \|L\widetilde{a_n} - L\widetilde{a_m}\|_B + \|K\widetilde{a_n} - K\widetilde{a_m}\|_C \to 0$

$\Rightarrow$ Cauchy, hence $\widetilde{a_n} \to \widetilde{a} \in A_0$ (A_0 closed)

$\Rightarrow L\widetilde{a} \to 0 = L\widetilde{a}$

$\Rightarrow \widetilde{a} = 0$ (L injective on A_0)

$\Rightarrow$ contradiction ($\|\widetilde{a_n}\| = 1 \to \|\widetilde{a}\| = 1$ not 0)

$\Rightarrow \boxed{\|a_n\| \text{ bounded}}$.

$\Rightarrow$ subsequence Ka_n cgt

$\Rightarrow \|a_n - a_m\|_A \leq \|La_n - La_m\|_B + \|Ka_n - Ka_m\|_C \to 0$

$\Rightarrow a_n$ Cauchy, so cgt, so $a_n \to a, La_n \to b = La$

$\Rightarrow \operatorname{im} L$ closed. □

Thm. *Inequality of Lemma holds for operators in* $(*)$ *for* $S \gg 0$.

Proof.

Step (1). For $A_s \equiv A_{-\infty}$ constant (symmetric nondegenerate):

Aim: Solve $(\partial_s + A)V = W \in L^2(\mathbb{R}, \mathbb{R}^m)$ then bound V in terms of W $(= LV)$.

$\mathbb{R}^m = E^- \oplus E^+$ where $E^- = \oplus$eigenspaces of $A_{-\infty}$ for evals < 0, $E^+ = \oplus \cdots > 0$

WLOG $\mathbb{R}^m = E^-$ by solving equation separately on E^-, E^+. Solution:[4]

$$V(s) = \textstyle\int_{-\infty}^s e^{-A(s-t)} W(t)\, dt$$

<u>Check:</u> $\partial_s V = e^{-A(s-t)}W|_{t=s} - \int_{-\infty}^s Ae^{-A(s-t)}W\, dt = W(s) - AV(s)$ ✓

<u>Unique solution?</u> suppose $(\partial_s + A)V = 0$

$\Rightarrow$ change coords so that A diagonal

$\Rightarrow \partial_s V^i = -\lambda_i V^i \qquad \lambda_i$ evalue

$\Rightarrow V^i(s) = \text{constant} \cdot e^{-\lambda_i \cdot s}$

$\Rightarrow$ not in L^2 unless $V \equiv 0$. ✓

<u>Is V in L^2?</u>

$$\begin{aligned} V(s) &= \textstyle\int_{-\infty}^{\infty} \phi(s-t)W(t)\, dt \qquad\qquad \phi(s) = \begin{cases} e^{-As} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases} \\ &= (\phi * W)(s) \end{aligned}$$

---

[4] *Non-examinable:* In general, to solve the ODE $\partial_s V = BV + W$, you first solve for the fundamental solution $\phi : \mathbb{R} \to \operatorname{End}(\mathbb{R}^m)$ of $\partial_s \phi = B\phi$, $\phi(s_0) = I$ (so that $V(s) = \phi(s)V_0$ solves $\partial_s V = BV$ with $V(s_0) = V_0$), then our ODE has solution

$$V(s) = \phi(s)V_0 + \phi(s) \textstyle\int_{s_0}^s \phi(t)^{-1} W(t)\, dt = \phi(s)V_0 + \int_{s_0}^s \phi(s-t)W(t)\, dt$$

as can be checked by differentiating. The first piece is the homogeneous solution, the second is the particular solution. In our particular example, $\phi(s) = e^{-A \cdot s}$, and "$V(-\infty)$" $= 0$.

$$\begin{aligned}
\|\phi * W\|_2^2 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \phi(s-t)W(t)\, dt \right|^2 ds \\
&\leq \int \left(\int |\phi(s-t)|^{1/2} \cdot |\phi(s-t)|^{1/2} \cdot |W(t)|\, dt\right)^2 ds \\
&\leq \int \left(\int |\phi(s-t)|\, dt \cdot \int |\phi(s-t)| \cdot |W(t)|^2\, dt\right) ds && \text{(Cauchy-Schwarz)} \\
&= \|\phi\|_1 \iint |\phi(s-t)| \cdot |W(t)|^2\, ds\, dt && \text{(Fubini)} \\
&= \|\phi\|_1 \cdot \|\phi\|_1 \cdot \|W\|_2^2
\end{aligned}$$

$\Rightarrow \boxed{\|V\|_2 = \|\phi * W\|_2 \leq \|\phi\|_1 \cdot \|W\|_2}$ (a version of Young's inequality)

<u>Is V in $W^{1,2}$?</u>

$\partial_s V = -AV + W$

$$\begin{aligned}
\|\partial_s V\|_2 &\leq \|AV\|_2 + \|W\|_2 \\
&\leq \|A\| \cdot (\|\phi\|_1 + 1) \cdot \|W\|_2
\end{aligned}$$

where $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. Thus $\|V\|_2 + \|\partial_s V\|_2 \leq \text{constant} \cdot \|W\|_2$, equivalently:

$$\boxed{\|V\|_{1,2} \leq \text{constant} \cdot \|(\partial_s + A_{-\infty})V\|_2}$$

Step (2). For $V = 0$ away from $-\infty$:

$\|A_s - A_{-\infty}\| \leq \varepsilon$ for $s < -S + 1$ (for large S depending on ε)

Assume $V = 0$ for $s \geq -S + 1$:

$$\begin{aligned}
\Rightarrow \|V\|_{1,2} &\leq c \cdot \|(\partial_s + A_{-\infty})V\|_2 && \text{(by (1))} \\
&\leq c \cdot (\|(\partial_s + A_s)V\|_2 + \|(A_s - A_{-\infty})V\|_2) && \text{(triangle ineq)} \\
&\leq c \cdot (\|LV\|_2 + \varepsilon\|V\|_2) && (L = \partial_s + A_s) \\
\Rightarrow \|V\|_{1,2} &\leq \frac{c}{1-c\varepsilon}\|LV\|_2 && (\text{pick } \varepsilon > 0 \text{ small so } c\varepsilon < 1)
\end{aligned}$$

Step (3). Similarly do (1) for $A_{+\infty}$, and (2) for $V = 0$ for $s \leq S - 1$.

Step (4). General case:

Pick a bump function $\beta \in C^\infty(\mathbb{R}, [0,1])$ with $\beta = \begin{cases} 1 & \text{for } |s| \leq S - 1 \\ 0 & \text{for } |s| > S \end{cases}$

$$\begin{aligned}
\|(1-\beta)V\|_{1,2} &\leq \text{constant} \cdot \|L(1-\beta)V\|_2 && \text{(by (2), (3))} \\
\|\beta V\|_{1,2} &\equiv \|\beta V\|_2 + \|\partial_s(\beta V)\|_2 \\
&\leq \|\beta V\|_2 + \|L\beta V\|_2 + (\sup_s \|A_s\|) \cdot \|\beta V\|_2 \\
&\leq \text{constant} \cdot (\|\beta V\|_2 + \|L\beta V\|_2)
\end{aligned}$$

$$\begin{aligned}
\|V\|_{1,2} &\leq \|(1-\beta)V\|_{1,2} + \|\beta V\|_{1,2} && \text{(triangle ineq)} \\
&\leq \text{constant} \cdot (\|\beta V\|_2 + \|L\beta V\|_2 + \|L(1-\beta)V\|_2) && \text{(above ineq's)} \\
&\leq \text{constant} \cdot (\|KV\|_2 + \|LV\|_2) && \text{(see below)}
\end{aligned}$$

The last inequality used that $\beta, \partial_s\beta$ are bounded and vanish outside $[-S, S]$, so

$$\|(\partial_s\beta)V\|_2 \leq \text{constant} \cdot \|V|_{[-S,S]}\|_2 \equiv \text{constant} \cdot \|KV\|_2,$$

and it used that $\|\beta LV\|_2 \leq \text{constant} \cdot \|LV\|_2$ and similarly for $\|(1-\beta)LV\|_2$. $\square$

Cor. *$L = \partial_s + A_s$ has closed image and finite dimensional kernel.*

Aim of next lecture: the cokernel is also finite dimensional, so $\partial_s + A_s$ is Fred.

LECTURE 15.

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4.14. $\partial_s + A_s$ **Fredholm (continued).**

$$\boxed{L = \partial_s + A_s : W^{1,2}(\mathbb{R}, \mathbb{R}^m) \to L^2(\mathbb{R}, \mathbb{R}^m)}$$

Aim: L has finite dimensional cokernel.

Lemmas 4.11 $\Rightarrow \operatorname{coker} L \cong (\operatorname{im} L)^\perp \cong \ker L^*$.
Warning. $L^* : L^2 \to W^{1,2}$ is difficult to calculate (try!).[1]
Remedy: the *formal adjoint* $L^\star$, which is not L^*, but it has a similar kernel.

Def. *$X, Y \to M$ vector bundles with metrics over a (possibly non-compact) Riem mfd. Given a linear map on sections $L : C^\infty(X) \to C^\infty(Y)$, the* formal adjoint *(if it exists) is a map $L^\star : C^\infty(Y) \to C^\infty(X)$ satisfying*

$$\int_M \langle Lx, y\rangle_Y \, vol_M = \int_M \langle x, L^\star y\rangle_X \, vol_M \qquad \forall x \in C_c^\infty(X), \forall y \in C_c^\infty(Y).$$

Exercise: $L^\star$ is unique if it exists.

Lemma. *For $L = \partial_s + A_s$, get*[2] *$L^\star = -\partial_s + A_s^*$.*

Proof. $X = Y = \mathbb{R}^m$ and we trivialize u^*TM by g_M-ortonormal[3] e_i. Therefore $\langle\cdot,\cdot\rangle_X = \langle\cdot,\cdot\rangle_Y = \langle\cdot,\cdot\rangle_{\mathbb{R}^m}$. For all $x, y \in C_c^\infty(\mathbb{R}, \mathbb{R}^m)$, by definition of the weak derivative:[4]

$$\textstyle\int \langle(\partial_s + A_s)x, y\rangle \, ds = \int \langle x, -\partial_s y\rangle \, ds + \int \langle x, A_s^* y\rangle \, ds. \quad \square$$

Cor. *$L^\star$ extends to $-\partial_s + A_s^* : W^{1,2} \to L^2$, so $\langle L^\star x, y\rangle_2 = \langle x, Ly\rangle_2, \forall x, y \in W^{1,2}$.*

Proof. approximate x, y by C_c^∞. $\square$

Cor. $\operatorname{coker} L \cong \ker L^\star = \ker(-\partial_s + A_s^*)$

Proof. $\operatorname{coker} L \cong (\operatorname{im} L)^\perp \overset{4.11}{\cong} \ker L^* \subset W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ (Lemma 4.13).[5]
$\Rightarrow$ if $b \in (\operatorname{im} L)^\perp$ then $0 = \langle La, b\rangle_2 = \langle a, L^\star b\rangle_2$ for all a by the previous Cor, so $b \in \ker L^\star$, and conversely if $b \in \ker L^\star$ then $b \in (\operatorname{im} L)^\perp$ by the same equation.
$\Rightarrow \ker L^\star \cong (\operatorname{im} L)^\perp \cong \operatorname{coker} L$. $\square$

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[1] The issue is that it involves $\langle x, L^* y\rangle_{W^{1,2}}$ not just L^2 pairings.

[2] Where $g_M(A_s v, w) = g_M(v, A_s^* w), \forall v, w$. In our case A_s is symmetric, so $A_s = A_s^*$.

[3] Pick e_i orthonormal at a point, then parallel transport will preserve orthonormality:
$\partial_s\langle e_i, e_j\rangle = \langle\nabla_s e_i, e_j\rangle + \langle e_i, \nabla_s e_j\rangle = 0$ since $\nabla_s e_i = 0$.

[4] in this case, we are really just doing integration by parts.

[5] Observe that we are first checking that $b \in L^2$ really is $W^{1,2}$ before we apply $L^\star$!

4.15. Index of $\partial_s + A_s$.

Cor. $\boxed{\operatorname{Index}(L) = \dim\ker L - \dim\ker L^\star}$

Aim: find $\ker L, \ker L^\star$ to calculate dimensions.

Need to solve

$$\partial_s x = -A_s x \qquad x : \mathbb{R} \to \mathbb{R}^m \qquad x(s_0) = x_0$$

write $x(s) = \phi(s) \cdot x_0$, for $\phi : \mathbb{R} \to \operatorname{End}(\mathbb{R}^m) = m \times m$ matrices. So need to solve:

$$\partial_s \phi = -A_s \cdot \phi, \qquad \phi(s_0) = I \qquad (\text{so } x(s_0) = \phi(s_0)x_0 = x_0)$$

This ϕ is called **fundamental solution** (and later we will write $\phi(s) = \phi^s_{s_0}$ to emphasize the initial condition).
By 2.1 Rmk 2 $\Rightarrow \exists$ unique $\phi \in C^{k-1}$ since $A : \mathbb{R} \to \operatorname{End}\mathbb{R}^m$ is C^{k-1}-bounded.[6]

Consider $\boxed{E^\pm = E^\pm(s_0) = \{x_0 \in \mathbb{R}^m : \phi(s)x_0 \to 0 \text{ as } s \to \pm\infty\}}$

These are vector subspaces of $\mathbb{R}^m$ since the ODE is linear.

Recall $L = \partial_s + A_s$, $L^\star = -\partial_s + A_s^*$, where $A_{\pm\infty}$ are symmetric non-singular.

Thm.

$\ker L \cong E^- \cap E^+$,
$\ker L^\star \cong (E^-)^\perp \cap (E^+)^\perp$,
$E^- \cong E^u(-A_{-\infty}) = E^s(A_{-\infty}) \equiv \oplus$*eigenspaces for evals* $\lambda < 0$ *of* $A_{-\infty} = \operatorname{Hess}_p f$,
$E^+ \cong E^s(-A_{+\infty}) = E^u(A_{+\infty}) \equiv \oplus$*eigenspaces for evals* $\lambda > 0$ *of* $A_{+\infty} = \operatorname{Hess}_q f$.

Cor.

$$\begin{aligned} \operatorname{Index} L &= \dim E^- \cap E^+ - \dim((E^-)^\perp \cap (E^+)^\perp) \\ &= \dim E^- \cap E^+ - (m - \dim(E^- + E^+)) \quad (\textit{taking } \perp \textit{ of } \cap \textit{ gives } +) \\ &= \dim E^- + \dim E^+ - m \\ &= |p| - |q|. \end{aligned}$$

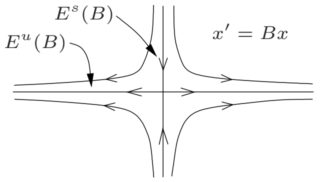
4.16. Dynamical systems and ODEs.

Aim: study $x'(t) = A(t) \cdot x(t)$, $A(t)$ matrices converging to a nonsingular matrix B as $t \to +\infty$. Want to compare solutions with the solutions of $x'(t) = B \cdot x(t)$.

Rmk. *We use a time variable $t \in \mathbb{R}$ instead of $s \in \mathbb{R}$ to emphasize that this is an abstract problem separate from previous sections.*

Example: $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Then

$$\begin{aligned} E^u(B) &= \{x_0 : \operatorname{flow}(x_0) \to 0 \text{ as } t \to -\infty\} = \mathbb{R} \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \\ E^s(B) &= \{x_0 : \operatorname{flow}(x_0) \to 0 \text{ as } t \to +\infty\} = \mathbb{R} \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \end{aligned}$$

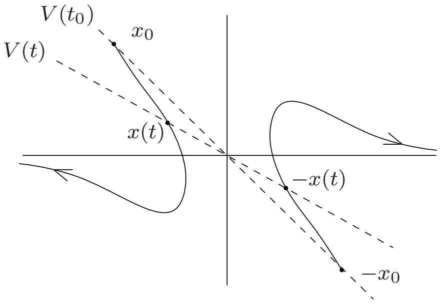


$$\begin{aligned} x &= \exp(Bt) \cdot x \\ &= \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} x_0 \end{aligned}$$

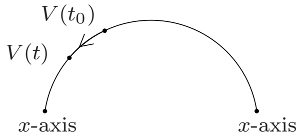
Now look at a flowline for $x'(t) = A(t)x(t)$ starting at x_0 for $t = t_0$:

---

[6] $s \mapsto A_s$ is C^{k-1}, $A_s \to A_{\pm\infty}$ in C^{k-1} as $s \to \pm\infty$, so for $s \gg 0$: $\|A_s\| \approx \|A_{\pm\infty}\|$, $\|\partial_s^j A_s\| \approx 0$.



Because the ODE is linear, we can in fact consider the flow of the lines $V(t) = \phi^t_{t_0} \cdot (\mathbb{R} \cdot x_0)$, which viewed as a flow on the Grassmannian $\mathbb{R}P^1 = \{\text{lines in } \mathbb{R}^2\}$ is:



which shows that $V(t) \to E^u(B) = x$-axis as $t \to +\infty$, so $E^u(B)$ is an attractor.

Aim of next Lecture: prove in general that $E^u(B)$ is an attractor in the Grassmannian of k-planes, $k = \dim E^u(B)$, for the flow $x'(t) = A(t)x(t)$ (assuming B is nonsingular and symmetric and $A(t) \to B$ as $t \to +\infty$).

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4.16. Dynamical systems and ODEs (continued).

Def. *B is a* hyperbolic matrix *if the real parts Re(eigenvalues)* $\neq 0$.

Rmk. *These matrices are generic. Exercise: check the details of this lecture for B symmetric and nondegenerate (so the eigenvalues are real and non-zero, so it's hyperbolic), indeed just assume WLOG that is B diagonal (by a change of basis).*

Linear Algebra Trick[1]

For $L : \mathbb{R}^n \to \mathbb{R}^n$ linear with evalues λ satisfying

$$\boxed{a < \operatorname{Re}\lambda < b}$$

there exists a basis of $\mathbb{R}^n$ inducing[2] *an inner product $(\cdot,\cdot)$ with norm $(\!(\cdot)\!)$ such that*

$$a(\!(x)\!)^2 \leq (Lx, x) \leq b(\!(x)\!)^2.$$

Key trick. If $x'(t) = Lx(t)$ then

$$\frac{d}{dt}(\!(x)\!)^2 = \frac{d}{dt}(x,x) = 2(Lx,x)$$

so $a \leq \frac{1}{2}\frac{d}{dt}\log(\!(x)\!)^2 = \frac{d}{dt}\log(\!(x)\!) \leq b$ so $\boxed{e^{at}(\!(x(0))\!) \leq (\!(x(t))\!) \leq e^{bt}(\!(x(0))\!)}$.

Example. For L restricted to[3] $\displaystyle\bigoplus_{\operatorname{Re}\lambda<0} E_\lambda(L)$ can pick $b < 0$ so $(\!(x(t))\!) \to 0$ exp fast.

Def. *Recall e^{Bt} is the flow for $x' = Bx$. The stable and unstable spaces for B:*

$$E^s(B) = \{x_0 \in \mathbb{R}^m : e^{Bt}\cdot x_0 \to 0 \text{ as } t \to +\infty\}$$
$$E^u(B) = \{x_0 \in \mathbb{R}^m : e^{Bt}\cdot x_0 \to 0 \text{ as } t \to -\infty\}$$

Rmk. $\boxed{E^s(-B) = E^u(B)}$ *since* $\displaystyle\lim_{t\to+\infty} e^{-Bt}\cdot x_0 = 0 \Leftrightarrow \lim_{t\to-\infty} e^{Bt}\cdot x_0 = 0$.

Exercise.[4] B hyperbolic $\Rightarrow$ $\boxed{\begin{array}{rcl} E^s(B) & = & \bigoplus_{\operatorname{Re}\lambda<0} E_\lambda(B) \\ E^u(B) & = & \bigoplus_{\operatorname{Re}\lambda>0} E_\lambda(B) \\ E^s(-B^*) & = & E^s(B)^\perp \end{array}}$

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[1] For a proof, see the 1974 edition of Hirsch-Smale, *Differential Equations*, p.147. For diagonal matrices this is obvious, so for symmetric matrices it's easy (take an orthonormal basis of evectors). You need to fiddle around with the Jordan Normal Form in the general case.

[2] $(x,y) = \sum x_i y_i$ where x_i, y_i are the coordinates of x, y in the given basis, and $(\!(x)\!) = (x,x)^{1/2}$. We reserve the symbols $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\perp$ for the standard $\mathbb{R}^n$ symbols.

[3] $E_\lambda(L) = \ker(L - \lambda I)^n$ is the generalized eigenspace for λ.

[4] For B symmetric, $E^s(-B^*) = E^s(-B) = E^u(B)$.

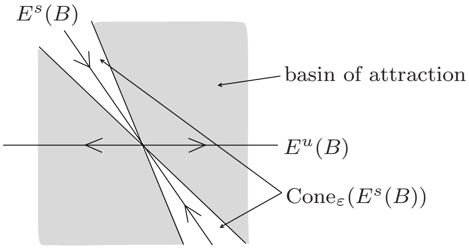
Thm. *Suppose B hyperbolic and $A(t) \to B$ as $t \to +\infty$. Then for $t \gg 0$, $E^u(B)$ is an attractor*[5] *inside*

$$\boxed{\mathrm{Gr} = \{k\text{-planes in } \mathbb{R}^m\}}$$

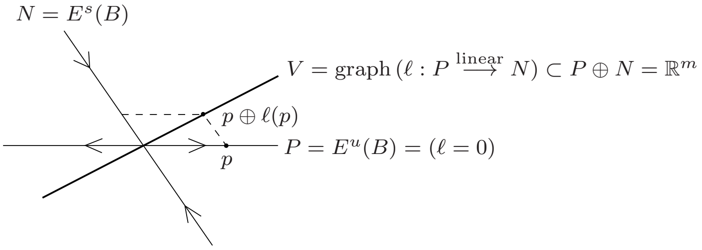
for the flow $x'(t) = A(t)x(t)$ (where $k = \dim E^u(B)$), with basin of attraction[6]

$$\mathrm{Gr} \setminus (k\text{-planes intersecting } \mathrm{Cone}_\varepsilon(E^s(B))$$

where[7] *$\varepsilon > 0$ can be chosen arbitrarily small for $t \gg 0$.*



Proof. Define local coordinates on Gr:



The $(m-k)\cdot k$ local coordinates are the entries of the matrix ℓ, and these coordinates are defined on $\mathrm{Gr} \setminus (k\text{-planes } V \text{ intersecting } N)$. We abbreviated $E^u(B), E^s(B)$ as P, N since they involve evalues with, respectively, positive and negative real parts.

$$\begin{array}{rcl} V = \ \mathrm{span}\,\{p_i \oplus \ell(p_i)\} & \overset{e^{Bt}}{\mapsto} & e^{Bt} \cdot V = \mathrm{span}\,\{e^{Bt}p_i \oplus e^{Bt} \cdot \ell(p_i)\} \\ \mathrm{graph}(\ell : p_i \mapsto \ell(p_i)) & \mapsto & \mathrm{graph}(e^{Bt} \cdot \ell \cdot e^{-Bt} : e^{Bt}p_i \mapsto e^{Bt} \cdot \ell(p_i)) \end{array}$$

So the local ODE is:

$$\ell' = \left.\frac{d}{dt}\right|_{t=0} e^{Bt}\ell e^{-Bt} = B\ell - \ell B,$$

so $\ell' = L\ell$ for the linear map $L : \ell \mapsto B\ell - \ell B$ for[8]

$$\ell \in \mathrm{Lin}(P, N) \cong \mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^{m-k}) \cong \mathbb{R}^{m-k} \otimes (\mathbb{R}^k)^*.$$

---

[5] meaning: nearby solutions converge to $E^u(B)$ as $t \to +\infty$.

[6] the subset which specifies what "nearby" means in the previous footnote.

[7] Recall $\mathbb{R}^m = E^s(B) \oplus E^s(-B^*)$ (orthogonal subspaces), so we uniquely decompose $x = x^s + x^u$ in the respective subspaces with $x^s \perp x^u$. Easy check: $\|x\|^2 = \|x^s\|^2 + \|x^u\|^2$. Define

$$\boxed{\mathrm{Cone}_\varepsilon(E^s(B)) = \{x \in \mathbb{R}^m : \|x^u\|^2 < \varepsilon \|x^s\|^2\}}$$

[8] The last isomorphism works as follows.
Identify $\mathbb{R}^k \cong (\mathbb{R}^k)^* \equiv \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R})$ by $v \mapsto (v^T : x \mapsto v^T x = \langle v, x \rangle \in \mathbb{R})$. The linear map $w \otimes v^T$ is $\mathbb{R}^k \to \mathbb{R}^{m-k}, x \mapsto w \cdot (v^T x)$. If p_i are an orthonormal basis, then $\ell = \sum \ell(p_i) \otimes p_i^T$.

We want to find the eigenvalues of L. Observe

$$L(w \otimes v^T) = Bw \otimes v^T - w \otimes v^T B = Bw \otimes v^T - w \otimes (B^T v)^T.$$

So if $w \in N$ evector of B for evalue λ_w, and $v \in P$ evector for B^T for evalue μ_v,

$$L(w \otimes v^T) = (\lambda_w - \mu_v)\ \ w \otimes v^T,$$

so $w \otimes v^T$ is an evector of L for evalue $\lambda_w - \mu_v$.

Exercise. the multiplicities (dim of gen. espaces) are what you expect.

B preserves P, N, and evalues of $B|_P$ = evalues of $B^T|_P$, so

$$\operatorname{Re} \lambda_w < 0,\ \operatorname{Re} \mu_v > 0$$

so $\operatorname{Re}(\lambda_w - \mu_v) < 0$. So by the Key trick applied to L,

$$(L\ell, \ell) \leq -\delta (\!(\ell)\!)^2 < 0 \qquad\qquad (*)$$

for any $-\delta <$ smallest Real part of all such differences of eigenvalues $\lambda_w - \mu_v$.

Now consider $x' = A(t)x$. Let $\phi_{t_0}^t$ denote its flow,[9] so

$$\ell(t) = \phi_{t_0}^t(\ell(t_0)) = \text{flow of } \ell(t_0)$$

Warning: the flow can be unbounded since the flow of V may intersect $E^s(B)$, so $\ell \to \infty$, i.e. you exit the chart for Gr!

For $t_0 \gg 0$, and away from a cone around E^s:

$$\left.\frac{d}{dt}\right|_{t=t_0} \phi_{t_0}^t(\ell(t_0)) \approx L \cdot \ell(t_0) \qquad\qquad (**)$$

since $A(t_0) \to B$ as $t_0 \to +\infty$.

$$\begin{aligned} \Rightarrow \tfrac{d}{dt}(\ell(t), \ell(t)) &= 2(\ell'(t), \ell(t)) \\ &= 2(\left.\tfrac{d}{ds}\right|_{s=t} \ell(s), \ell(t)) \\ &= 2(\left.\tfrac{d}{ds}\right|_{s=t} \phi_t^s(\ell(t)), \ell(t)) \\ &\leq 2(-\tfrac{\delta}{2}(\!(\ell)\!)^2) \end{aligned}$$

where in the last line we combined $(**)$ and $(*)$. So arguing as in the Key trick,

$$(\!(\ell(t))\!) \leq e^{-\frac{\delta}{2}(t-t_0)} (\!(\ell(t_0))\!) \to 0 \ \ (\text{“} = E^u(B)\text{”}) \text{ exponentially fast.} \quad \square$$

Aim: $E^s(t_0)$ converges to $E^s(B)$ as $t_0 \to \infty$. The idea is: solutions of $x'(t) = A(t)x(t)$ which decay to zero as $t \to \infty$ cannot get attracted to $E^u(B)$ because if you are close to $E^u(B)$ then you flow off to infinity. The difficulty is that we don't know the dimension of $E^s(t_0)$, and a priori the dimension can jump at $t_0 = \infty$ since we haven't proved a continuity “at infinity” result. The key idea is: show that the adjoint problem produces a space orthogonal to $E^s(t_0)$, and use that cleverly.

Def. *Write $\phi_{t_0}^t$ for the flow for $x' = A(t)x$ starting at t_0, and $\widetilde{\phi}_{t_0}^t$ for $x' = -A(t)^* x$.*

$$\begin{aligned} E^s(t_0) &= \{x_0 \in \mathbb{R}^m : \phi_{t_0}^t(x_0) \to 0 \text{ as } t \to +\infty\} \\ \widetilde{E}^s(t_0) &= \{x_0 \in \mathbb{R}^m : \widetilde{\phi}_{t_0}^t(x_0) \to 0 \text{ as } t \to +\infty\} \end{aligned}$$

Lemma.

$$\boxed{\mathbb{R}^m = E^s(t_0) \bigoplus \widetilde{E}^s(t_0) \to E^s(B) \bigoplus E^s(-B^*) \text{ as } t_0 \to +\infty}$$

---

[9] fundamental solution with initial condition t_0, so $\partial_t \phi_{t_0}^t = A(t) \circ \phi_{t_0}^t$ and $\phi_{t_0}^{t_0} = I$.

Proof.

Claim 1: $E^s(t_0)$ gets "attracted" to $E^s(B)$, meaning: $E^s(t_0)$ lies in an arbitrarily small cone around $E^s(B)$ for large t_0 (not claiming $\dim E^s(t_0) = \dim E^s(B)$ yet).

Proof: Suppose not. Then some vector $x \in E^s(t_0)$ lies outside that small cone. So x lies in the basin of attraction of $E^u(B)$ by the Theorem.[10] But once x stays within a small cone around $E^u(B)$, the flow of the vector x will go off to infinity exponentially fast by Hwk 16, ex.1.[11] ✓

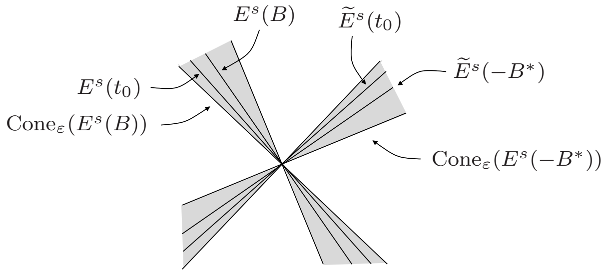
So, via this argument, the Theorem has shown that $E^s(t_0), \widetilde{E}^s(t_0)$ get "attracted" to $E^s(B), E^s(-B^*)$, but we still need to show that the dimensions agree.

Observe: any vector subspace inside $\mathrm{Cone}_\varepsilon(E^s(B))$ has dimension at most $\dim E^s(B)$, by basic linear algebra.[12] So $\boxed{\dim E^s(t_0) \leq \dim E^s(B)}$, similarly for $\widetilde{E}$.

Key observation:

$$\partial_t \langle \phi^t_{t_0}(x_0), \widetilde{\phi}^t_{t_0}(y_0)\rangle = \langle A(t)\phi^t_{t_0}(x_0), \widetilde{\phi}^t_{t_0}(y_0)\rangle + \langle \phi^t_{t_0}(x_0), -A(t)^* \widetilde{\phi}^t_{t_0}(y_0)\rangle = 0$$

so $\langle \phi^t_{t_0}(x_0), \widetilde{\phi}^t_{t_0}(y_0)\rangle$ is constant. For $x_0 \in E^s(t_0), y_0 \in \widetilde{E}^s(t_0)$ this constant is zero[13] so at $t = t_0$: $\langle x_0, y_0 \rangle = 0$. So $\boxed{E^s(t_0) \perp \widetilde{E}^s(t_0)}$.



Claim 2: $E^s(t_0) = \widetilde{E}^s(t_0)^\perp \cap \mathrm{Cone}_\varepsilon(E^s(B))$ (in particular $\dim E^s(t_0) = \dim E^s(B)$).
Proof. Suppose not, by contradiction.

$\Rightarrow \exists x_0 \in \widetilde{E}^s(t_0)^\perp \cap \mathrm{Cone}_\varepsilon(E^s(B))$ but $x_0 \notin E^s(t_0)$
$\Rightarrow x(t) = \phi^t_{t_0}(x_0) \in \widetilde{E}^s(t)^\perp = (\widetilde{\phi}^t_{t_0}\widetilde{E}^s(t_0))^\perp$ (by the Key observation)
$\widetilde{E}^s(t) \subset \mathrm{Cone}_\varepsilon(E^s(-B^*))$, $\forall t \gg 0$ (by Claim 1)
$\Rightarrow x(t) \in (\mathrm{Cone}_\varepsilon(E^s(-B^*)))^\perp = \mathrm{Cone}_\varepsilon(E^s(B))$, $\forall t \gg 0$

By Hwk 16: for $t_0 \gg 0$, $x_0 \in \mathrm{Cone}_\varepsilon(E^s(B))$ (with $\varepsilon > 0$ sufficiently small),

$$\frac{d}{dt}(\!(\phi^t_{t_0}(x_0))\!)^2 \leq -\delta(\!(x_0)\!)^2 \quad \text{ if } \ \phi^t_{t_0}(x_0) \in \mathrm{Cone}_\varepsilon(E^s(B)) \ \forall t \geq t_0.$$

So, in our case, $(\!(x(t))\!) \leq e^{-\frac{\delta}{2}(t-t_0)}(\!(x_0)\!) \to 0$, so $x_0 \in E^s(t_0)$. Contradiction.

Similarly $\widetilde{E}^s(t_0) = E^s(t_0)^\perp \cap \mathrm{Cone}_\varepsilon(E^s(-B^*))$.

---

[10] In order to work on Gr, you just turn x into a subspace of the correct dimension, k, by taking the span of x and any $k-1$ vectors from $E^u(B)$.

[11] Strictly speaking Hwk 16 ex.1 proves exponential convergence to 0 of vectors whose flow stays in a small cone around $E^s(B)$, but a similar argument proves what we mentioned here.

[12] The vector subspace in fact lies inside the graph of a linear map constructed as in the local coords construction for Gr.

[13] since both vectors in the inner product converge to zero as $t \to \infty$ by definition.

So $\mathbb{R}^m = E^s(t_0) \oplus \widetilde{E}^s(t_0)$, the summands are $\perp$ to each other, and given any small $\varepsilon > 0$, $E^s(t_0) \oplus \widetilde{E}^s(t_0) \subset \mathrm{Cone}_\varepsilon(E^s(B)) \oplus \mathrm{Cone}_\varepsilon(E^s(-B^*))$ for $t_0 \gg 0$. ☐

Cor. *$E^s(t_0) \to E^s(B)$ as $t_0 \to \infty$ and the dimensions agree.*
Similarly, if $A(t) \to C$ as $t \to -\infty$ then

$$E^u(t_0) \equiv \{x_0 \in \mathbb{R}^m : \phi^t_{t_0} x_0 \to 0 \text{ as } t \to -\infty\} \to E^u(C) \text{ as } t_0 \to -\infty.$$

Proof. The first claim is by the Lemma.

Note $x'(t) = A(t)x(t) \Rightarrow y(t) = x(-t)$ satisfies $y'(t) = -A(-t)y(t)$.
$\Rightarrow E^u_{A(t)}(t_0) = E^s_{-A(-t)}(-t_0) \to E^s(-C) = E^u(C)$ as $t_0 \to \infty$. ☐

4.17. **Proof of Theorem 4.15.** We go back to the notation of 4.15:

$$\begin{array}{lcl} Lx = 0 & \Leftrightarrow & x'(s) = -A_s x(s) \\ L^\star x = 0 & \Leftrightarrow & x'(s) = A_s^* x(s) = -(-A_s^*)x(s) \end{array}$$

Notice that we are now using $s, -A_s$ in place of the $t, A(t)$ of Section 4.16.
By Cor 4.16, $E^+ \to E^s(-A_{+\infty}) = E^u(A_{+\infty})$ as $s_0 \to +\infty \Rightarrow 4^{th}$ claim Thm 4.15.✓
By Cor 4.16, $E^- \to E^u(-A_{-\infty})$ as $s_0 \to -\infty \Rightarrow 3^{rd}$ claim Thm 4.15.✓

Lemma (1^{st} claim Thm 4.16)**.**

$$\begin{array}{rcl} \ker(\partial_s + A_s) & \cong & E^- \cap E^+ \\ (x : \mathbb{R} \to \mathbb{R}^m) & \to & x(s_0) \\ (x(s) = \phi^s_{s_0} \cdot x_0) & \leftarrow & x_0 \end{array}$$

where ϕ denotes the flow for $x'(s) = -A_s x(s)$.

Proof. Suppose $x \in \ker(\partial_s + A_s : W^{1,2} \to L^2)$. By 4.10, x is C^k and satisfies $x(s) = \phi^s_{s_0} \cdot x(s_0)$. By Sobolev, $x \to 0$ at $\pm\infty$ since $x \in (C^0, \|\cdot\|_\infty)$. So $x(s_0) \in E^-(s_0) \cap E^+(s_0)$. Conversely: for $x_0 \in E^- \cap E^+$ need $x(s) \in W^{1,2}$ so need to prove fast convergence at the ends. By Hwk 17 (*Exponential Convergence at the ends*):

$$(\partial_s + A_s)x(s) = 0,\ x \in C^1 \Rightarrow \left\{ \begin{array}{l} \text{either } |x(s)| \to \infty \text{ as } s \to +\infty \text{ or } -\infty, \text{ so } x \notin L^2 \\ \text{or } x(s) \to 0 \text{ exponentially fast as } |s| \to \infty, \text{ so } x \in L^2 \end{array} \right.$$

So $x_0 \in E^- \cap E^+ \Rightarrow x \to 0 \Rightarrow x \in L^2 \Rightarrow \partial_s x = -A_s x \in L^2 \Rightarrow x \in W^{1,2}$. ☐

Cor (2^{nd} claim Thm 4.16)**.** $\ker(\partial_s - A_s^*) \cong \widetilde{E}^- \cap \widetilde{E}^+ = (E^-)^\perp \cap (E^+)^\perp$.

4.18. **The clever way to calculate the Index.**
Aim: if you only care about the index of $\partial_s + A_s$, then one can avoid the difficult asymptotic analysis of 4.16. Trick: the index will not change if homotope A_s so that it becomes constant in s near $s = \pm\infty$.

Consider

$$A_\bullet : \mathbb{R} \to \mathrm{End}(\mathbb{R}^m) \quad C^{k-1}\text{-differentiable}$$

with $A_s \to A_\pm$ as $s \to \pm\infty$, and $A_\pm$ hyperbolic matrices. By 4.14 (and Hwk 15),

$$\partial_s + A_s : X \to Y \text{ is Fredholm} \quad (X = W^{1,2}(\mathbb{R}, \mathbb{R}^m), Y = L^2(\mathbb{R}, \mathbb{R}^m))$$

The Index of a Fred operator is constant under small-operator-norm perturbations (by 2.3). If $\widetilde{A_s}$ is a homotopy of A_s:

$$\begin{array}{rcl} \|(\partial_s + \widetilde{A_s})V - (\partial_s + A_s)V\|_2 & = & \|\widetilde{A_s}V - A_s V\|_2 \\ & \leq & \sup_{s\in\mathbb{R}} \|\widetilde{A_s} - A_s\| \cdot \|V\|_2 \\ & \leq & \sup_{s\in\mathbb{R}} \|\widetilde{A_s} - A_s\| \cdot \|V\|_{1,2} \end{array}$$

So if $\|\widetilde{A_s} - A_s\|$ is small enough for all s, the indices of $\partial_s + \widetilde{A_s}$ and $\partial_s + A_s$ agree.

This proves index invariance for small homotopies. To prove index invariance for a general homotopy $(A_{s,\lambda})_{0\leq\lambda\leq 1}$: the above shows that the index of $\partial_s + A_{s,\lambda}$ is constant on the interval $(\lambda - \delta_\lambda, \lambda + \delta_\lambda)$ (some small $\delta_\lambda > 0$). Now cover $[0,1]$ by finitely many such intervals. Hence, the index is constant for all $\lambda \in [0,1]$.

Upshot: By homotopying A_s, we may assume:

$$A_s = \begin{cases} A_+ & \text{for } s \geq S \\ A_- & \text{for } s \leq -S \end{cases}$$

Then $E^+(s_0) \equiv E^s(-A_+)$ for $s_0 \geq S$ since both definitions involve the same equation $x'(s) = -A_+ x(s)$. Similarly for E^-.

So you avoid the asymptotic analysis of 4.16! (but you still need 4.17).[14]

Rmk. *The same arguments show that the index of $\partial_s + A_s$ is invariant under homotopying the*[15] *$W^{1,2}$-path u with fixed ends p, q. Just compare the two operators by parallel transport for small hpies, and repeat the above arguments.*

---

[14] Note: $\phi^s_{s_0} : E^+(s_0) \to E^+(s)$ is an isomorphism. So all we care about is $E^+(s)$ for large s.

[15] recall we trivialized our section F on u^*TM to obtain $\partial_s + A_s$.

LECTURE 17.

PART III, MORSE HOMOLOGY, 2011

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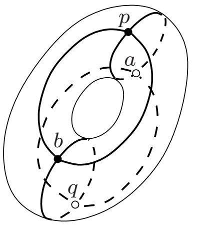
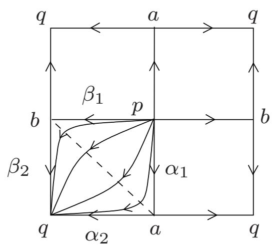
5. Compactness Theorem

5.1. **Motivating example.**

Consider $M = T^2$ tilted $\subset \mathbb{R}^3$ (see Hwk 1), $f =$ height function.

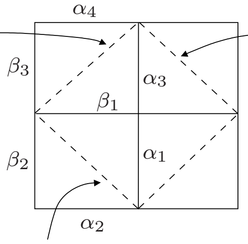




where we view the torus as a square with parallel sides identified. Observe:

$$\begin{array}{ll} \dim W(p,q) = |p| - |q| = 2 & \text{(parametrized flowlines)} \\ \dim \mathcal{M}(p,q) = 1 & \text{(unparametrized flowlines = trajectories)} \end{array}$$

Consider the lower left subsquare: the transverse intersection of the flowlines with the dashed subdiagonal parametrizes the space of trajectories $\mathcal{M}(p,q)$ (since you fix the parametrization by declaring that $u(0)$ is the point of intersection). Considering all squares, the four open subdiagonals parametrize $\mathcal{M}(p,q)$:



∂(this interval) $= \{\beta_1 \# \beta_3, \alpha_3 \# \alpha_4\}$

$\mathcal{M}(p,q) \cong$ 4-open intervals

∂(this interval) $= \{\alpha_1 \# \alpha_2, \beta_1 \# \beta_2\}$

The boundaries of the open intervals involve broken trajectories, such as $\alpha_1 \# \alpha_2$ which means the trajectory α_1 followed by α_2. Thus:

$$\begin{array}{rcll} \partial \mathcal{M}(p,q) & \cong & \mathcal{M}(p,a) \times \mathcal{M}(a,q) \sqcup \mathcal{M}(p,b) \times \mathcal{M}(b,q) & \text{(0-dimensional)} \\ & = & \text{8 broken trajectories} & \end{array}$$

Date: May 3, 2011,

and the obvious way to compactify this is to compactify each interval separately:[1]

$$\begin{aligned}\overline{\mathcal{M}}(p,q) &= \mathcal{M}(p,q)\cup\partial\mathcal{M}(p,q) = \text{disjoint union of 4 closed intervals}\\ &= \text{compact smooth 1-manifold.}\end{aligned}$$

Key idea: to compactify $\mathbb{R}$ ($\cong$ open interval)

(1) identify which sequences fail to have a convergent subsequence:
$x_n \subset \mathbb{R}$ with $|x_n| \to \infty$

(2) artificially add the limit points to your set:
$\overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,+\infty\}$

(3) declare new open sets[2] to ensure those new points are indeed limit points:
$[-\infty, r), (r,+\infty]$ for all $r \in \mathbb{R}$

Upshot: $\overline{\mathbb{R}}$ is compact, whose *open part*[3] $\mathbb{R}\subset\overline{\mathbb{R}}$ has the original topology, and it has *boundary* $\partial\overline{\mathbb{R}} = \{\text{new points}\}$.

5.2. **The topology of $\mathcal{M}(p,q) = W(p,q)/\mathbb{R}$.** Recall that

$$W(p,q) \subset U \subset W^{1,2}_{loc}(\mathbb{R}, M) \subset C^0_{loc}(\mathbb{R}, M)$$

and $W(p,q)$ is a submanifold of U. So it comes with a topology induced from U which in particular implies C^0-convergence on compact sets.

Call $\pi : W(p,q) \to \mathcal{M}(p,q)/\mathbb{R}, u \mapsto \pi(u) = [u]$ the quotient map. The $\mathbb{R}$-action is just the reparametrization action given by shifting the s-variable by a constant:

$$v = [u] = [u(\cdot + \text{constant})].$$

We call such a $u = \widetilde{v} \in W(p,q)$ a ***lift*** of $v \in \mathcal{M}(p,q)$. The quotient topology is: $\mathcal{O} \subset \mathcal{M}(p,q)$ is open $\Leftrightarrow \pi^{-1}\mathcal{O} \subset W(p,q)$ is open. More usefully:

$$v_n \to v \in \mathcal{M}(p,q) \Leftrightarrow \exists \widetilde{v}_n \to \widetilde{v} \text{ in } W(p,q)$$

or even more explicitly:

$$\boxed{[u_n] \to [u] \text{ in } \mathcal{M}(p,q) \Leftrightarrow u_n(\cdot + s_n) \to u(\cdot) \text{ in } W(p,q) \text{ for some } s_n \in \mathbb{R}}$$

Example. $u \in W(p,q) \Rightarrow u_n(s) = u(s+n)$ not cgt in $W(p,q)$, but cgt to $[u]$ (indeed constant!) in $\mathcal{M}(p,q)$.

5.3. **C^0_{loc}-convergence.** M^m closed mfd, $f : M \to \mathbb{R}$ Morse.

Lemma. *$\partial_s u_n = -\nabla f(u_n) \Rightarrow \exists$ subseq $u_n \to u$ in C^0_{loc} ($= C^0$-cgce on compacts).*

Proof. $u_n|_{[-S,S]} : [-S,S] \to M \subset \mathbb{R}^a$ are equibounded (since M is compact) and equicontinuous by the mean value theorem using $|\partial_s u_n| \le \sup_{x\in M} |\nabla f(x)|$. Now apply Arzela-Ascoli (see 2.1). □

Cor. $\boxed{W = \bigcup_{p\neq q\in Crit(f)} W(p,q) \text{ is } C^0_{loc}\text{-compact.}}$

---

[1] You do not want to identify the ends of the intervals pairwise, giving a circle: consider two sequences of trajectories, one which converges to $\beta_1\#\beta_2$ and $\beta_1\#\beta_3$ respectively (for example). Then the two sequences are not actually approaching each other, because β_2 and β_3 are not equal. So you do not want to identify the two limiting broken trajectories.

[2] the new topology is then the one generated by the old topology and these new open sets. Example: $\{-\infty\}$ is a closed set because, for example, $\{-\infty\} = \overline{\mathbb{R}} \setminus (\,(-\infty,0)\cup(-1,+\infty]\,)$.

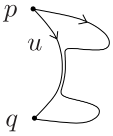
[3] $\overline{\mathbb{R}}$ is open & closed, $\mathbb{R}\subset\overline{\mathbb{R}}$ is open but no longer closed (since the complement of $\emptyset$ is now $\overline{\mathbb{R}}$).

Rmk. *Suppose* $\left\{ \substack{\partial_s u_n = -\nabla f(u_n) \\ \partial_s u = -\nabla f(u)} \right\}$. *Then* $u_n \to u$ *in* $C^0_{loc} \Leftrightarrow$ *in* $C^k_{loc}\ \forall k \Leftrightarrow$ *in* C^∞_{loc}.
Proof. Usual bootstrapping: $u_n \to u$ in $C^0_{loc} \Rightarrow -\nabla f(u_n) \to -\nabla f(u)$ in C^0_{loc} $\Rightarrow u_n \to u$ in $C^1_{loc} \Rightarrow$ etc.

Cor. *W is C^∞_{loc}-compact.*

Thm. $\boxed{C^0_{loc}\text{-convergence in } W(p,q) \Rightarrow \text{convergence in } W(p,q) \text{ (in 5.1 topology)}}$

Proof. Suppose $u_n \to u$ in C^0_{loc}, where $u_n, u \in W(p,q)$. On $[-S,S]$ we have C^1 convergence hence $W^{1,2}$ convergence. So we reduce to showing convergence at the ends. The key trick is to show that for energy reasons, the u_n, u must be uniformly[4] close to the critical points at the ends, so this does not happen:



Recall $E(u) = \int_{\mathbb{R}} |\partial_s u|^2\, ds$. Pick $S \gg 0$ such that

$$\int_{\mathbb{R}\setminus[-S,S]} |\partial_s u|^2 = \frac{1}{2}\delta \quad \text{(small)}.$$

By Rmk:

$$\begin{array}{ll} \Rightarrow & u_n|_{[-S,S]} \to u|_{[-S,S]} \text{ in } C^1 \\ \Rightarrow & E(u_n|_{[-S,S]}) \to E(u|_{[-S,S]}) \\ \overset{\text{by } 3.2}{\Rightarrow} & E(u_n|_{\mathbb{R}\setminus[-S,S]}) \leq \delta \quad (\text{ for } n \gg 0, \text{ say } n \geq N) \\ \overset{\text{by } 3.3}{\Rightarrow} & u_n(s) \in \text{small ball around } p \text{ for } s \leq -S, n \geq N \\ & u_n(s) \in \text{small ball around } q \text{ for } s \geq +S, n \geq N \end{array}$$

Where 3.2 refers to the a priori estimate $f(p) - f(q) = E(u) = E(u_n)$ and 3.3. refers to the No escape Lemma: if you have little energy left and you have to converge to p, q then you must be close to p, q.

Thus we reduce to showing $W^{1,2}$ convergence near p, q. WLOG do the case q.

Hwk 17 Exponential cgce at the ends: *Locally near* $q = 0$, $\exists c, \delta > 0$, $\exists$ *ball* $B_r(0)$ *s.t. if* $u : [S, \infty) \to B_r(0)$ *solves* $\partial_s u = -\nabla f(u)$ *then*

$$|u(s)| \leq c \cdot e^{-\delta s} \quad \text{for } s \geq S.$$

So in our case, $u_n(s) \to q$ exponentially fast as $s \to \infty$ at a rate independent of n.

$\Rightarrow \text{dist}(u_n(s), u(s)) \leq \text{dist}(u_n(s), q) + \text{dist}(q, u(s)) \to 0$ exp fast in s.

$\Rightarrow -\nabla f(u_n(s)), -\nabla f(u(s))$ exp close, indeed exp close to $-\nabla f(q) = 0$

$\Rightarrow u_n, u$ are $W^{1,2}$-close on $[S, \infty)$. □

Rmk. *In more complicated situations, when you don't have a result like 3.2 in the above argument, you would* assume *that $E(u_n)$ is bounded. So the Thm would hold for sequences having a bound on the energies.*

---

[4] independently of n.

5.4. **Convergence to broken trajectories.** Given $u_n \in W(p,q)$. By Cor 5.2, a subsequence $u_n \to u$ in W in C^0_{loc}. Two situations can arise:

Case 1: no breaking. if $u \in W(p,q)$ then $u_n \to u$ in $W(p,q)$ by Thm 5.2.

Case 2: breaking. $u \in W(p_u, q_u) \neq W(p,q)$. Observe:

$f(p) \geq f(p_u) \geq f(q_u) \geq f(q)$ since f decreases along flowlines (3.2)

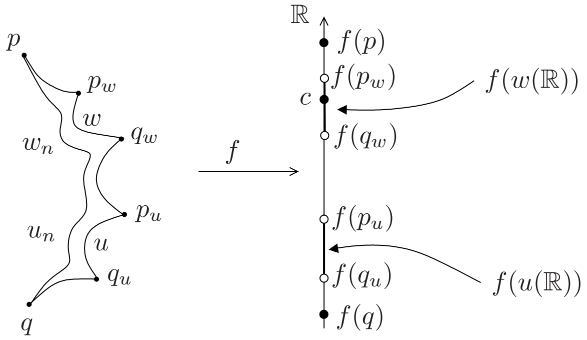
$\boxed{\text{Trick (3.3): } \exists \delta > 0 \text{ s.t. flowlines with ends near distinct crit pts consume } E \geq \delta}$

Lemma (Reparametrization Trick)**.** *Assuming we are in Case 2,* $\exists s_n \in \mathbb{R}$ *such that* $w_n = u_n(\cdot + s_n) \overset{C^0_{loc}}{\to} w$ *with* $f(w(\mathbb{R})) \cap f(u(\mathbb{R})) = \emptyset$*.*

Proof. Suppose $p_u \neq p$ (case $q_u \neq q$ is similar). Then

$$f(p) > f(p_u),$$

otherwise $u_n|_{[-a_n, -b_n]}$ has ends close to p, p_u (for b_n large, a_n very large) with energy $\approx f(p) - f(p_u) = 0$ contradicting Trick 3.3.



Fix a regular value c of f with

$$f(p) > c > f(p_u)$$

and any $s_n \in \mathbb{R}$ with

$$f(u_n(s_n)) \to c.$$

Cor 5.2 $\Rightarrow$ $w_n = u_n(\cdot + s_n) \overset{C^0_{loc}}{\to} w$ some $w \in W(p_w, q_w)$, with

$$f(p) \geq f(p_w) > c > f(q_w) \qquad (*).$$

We claim[5] that $s_n \to -\infty$. Proof:

$$\begin{array}{lll} u_n \overset{C^0_{loc}}{\to} u & \Rightarrow & c > f(p_u) > f(u(-S)) \overset{n \gg 0}{\approx} f(u_n(-S)) & \\ & \Rightarrow & c > f(u_n|_{[-S,\infty)}) & \text{(for } n \gg 0) \\ & \Rightarrow & s_n \leq -S \ \checkmark & \text{(by } (*)) \end{array}$$

It remains to check $f(q_w) \geq f(p_u)$.

Suppose by contradiction $f(p_u) > f(q_w)$. Observe:

---

[5] intuition: it takes ∞ time to reach p_u since $\nabla f \to 0$.

$$\begin{tikzcd}[column sep=small]
 & & S \gg 0 \arrow[dl] & & n \gg 0 \arrow[dl] & \\
f(q_w) & \approx & f(w(S)) & \approx & f(u_n(S+s_n)) & \\
\wedge & & & & \wedge & \begin{array}{l}\text{hence this follows. But } s_n \to -\infty \\ \text{so contradicts } f \text{ decreasing along } u_n\end{array} \arrow[l] \\
f(p_u) & \approx & f(u(-S)) & \approx & f(u_n(-S) &
\end{tikzcd}$$

□

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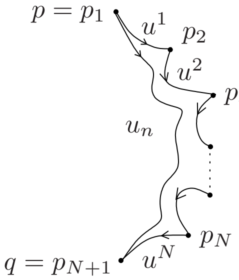
5.3. **Convergence to broken trajectories (continued).** Recall that by the reparametrization trick, for any sequence $u_n \in W(p,q)$ without a convergent subsequence, $\exists s_n \in \mathbb{R}$ with $w_n = u_n(\cdot + s_n) \to w$ in C^0_{loc} with $f(w(\mathbb{R})) \cap f(u(\mathbb{R})) = \emptyset$.

Thm. *$u_n \in W(p,q) \Rightarrow \exists$ subseq u_n such that:*

- $\exists\, s^i_n \in \mathbb{R} \qquad i = 1, \dots, N$
- $\exists\, u^i \in W(p_i, p_{i+1}) \qquad p = p_1, q = p_{N+1}$
- $f(p_1) > f(p_2) > \cdots > f(p_{N+1})$

with

$$\boxed{u^i_n = u_n(\cdot + s^i_n) \to u^i \text{ in } W(p_i, p_{i+1})}$$



Proof. Cover $[f(p), f(q)]$ by closures of disjoint intervals obtained by the reparametrization trick. This is a finite cover by Trick 3.3.[1] □

Def. *Call $(u^1, u^2, \dots, u^N) \in W(p_1,p_2) \times \cdots \times W(p_N, p_{N+1})$ a* broken flowline.

5.4. **Compactness theorem.**

Rmk. *In the Theorem, $u^i_n \in W(p,q)$ are different lifts of the same $[u_n] \in \mathcal{M}(p,q)$.*

Def. *In the Theorem, denote $v_n = [u_n] = [u^i_n] \in \mathcal{M}(p,q)$, $v^i = [u^i] \in \mathcal{M}(p_i, p_{i+1})$. Then we summarize the conclusion of the Theorem by the* broken limit *symbol*

$$v_n \rightrightarrows v^1 \# \cdots \# v^N$$

and we call $v^1 \# \cdots \# v^N \in \mathcal{M}(p, p_2) \times \cdots \mathcal{M}(p_N, q)$ an (N-times) broken trajectory.

Cor. *$v_n \in \mathcal{M}(p,q) \Rightarrow \exists$ subseq $v_n \rightrightarrows v^1 \# \cdots \# v^N$ with $v^i \in \mathcal{M}(p_i, p_{i+1})$ $(f(p) = f(p_1) > \cdots > f(p_{N+1}) = f(q),\ p = p_1,\ q = p_{N+1})$.*

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[1] you consume energy $\geq$ length of interval $\geq \delta > 0$.

Rmk. *From now on, assume transversality holds (it does for a generic metric). So*

$$|p| = |p_1| > |p_2| > \cdots > |p_{N+1}| = |q|$$

since $\mathcal{M}(p_i, p_{i+1}) = \emptyset$ *if*[2] $|p_i| \leq |p_{i+1}|$ (*note* $\dim \mathcal{M}(p_i, p_{i+1}) = |p_i| - |p_{i+1}| - 1 < 0$).

Repeat the Key idea 5.0 for the compactification of $\mathcal{M}(p,q)$:

1. sequences $u_n \in \mathcal{M}(p,q)$ which do not have a convergent subsequence:
   those with a subsequence $\rightrightarrows$ broken trajectory
2. artificially add limit points to $\mathcal{M}(p,q)$:

$$\overline{\mathcal{M}}(p,q) = \mathcal{M}(p,q) \cup \partial\mathcal{M}(p,q)$$
$$\partial\mathcal{M}(p,q) = \bigcup_{N \geq 2, |p|>|p_2|>\cdots>|q|} \mathcal{M}(p,p_2) \times \cdots \times \mathcal{M}(p_N, q)$$

3. enlarge the topology to make them limit points:
   topology of $\rightrightarrows$ convergence to broken trajectories

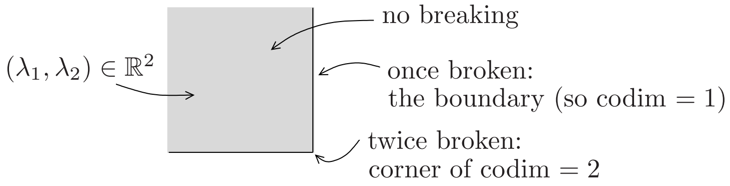
Upshot: Theorem. $\overline{M}(p,q)$ *is compact.*

Two problems:

- 5.3 $\not\Rightarrow$ $\boxed{\text{every broken flowline arises as a } \rightrightarrows \text{ limit}}$
- $\overline{\mathcal{M}}(p,q)$ smooth mfd (with corners)?

Answer: Yes, by the gluing theorem! (next section)

We will only study once-broken trajectories, so there are no corners. But, for example, you should think of a 2-dimensional moduli space as follows:



5.5. **Gluing theorem.** For once broken flowlines (for simplicity):

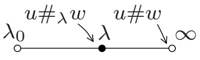
$$\dim W(p,q) = |p| - |q| = 2$$
$$\dim \mathcal{M}(p,q) = 1$$

Thm. *(Assuming transversality) For all* $a \in Crit(f)$ *with* $|p| - |a| = 1 = |a| - |q|$, *there is a* gluing map

$$\boxed{\begin{array}{rrcl} \# : & W(p,a) \times W(a,q) \times (\lambda_0, \infty) & \to & W(p,q) \\ & (u, w, \lambda) & \mapsto & u\#_\lambda w \end{array}}$$

1. $\#$ *induces a smooth embedding on* $\mathcal{M}(\cdot,\cdot)$ *spaces*
2. $u\#_\lambda w \rightrightarrows u\#w$ *as* $\lambda \to \infty$
3. *if* $v_n \rightrightarrows u\#w$ *then for* $n \gg 0$, $v_n = [u\#_{\lambda_n} w] \in \mathcal{M}(p,q)$, *for some* $\lambda_n \to \infty$

Cor. $\dim \mathcal{M}(p,q) = 1 \Rightarrow \overline{\mathcal{M}(p,q)}$ *smooth compact* 1*-mfd with bdry* $\partial\mathcal{M}(p,q)$.

Proof. Thm $\Rightarrow \exists$ (collar nbhd of $u\#w$) $\cong$  $\subset \mathbb{R}$. ☐

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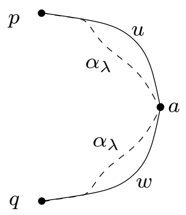
[2] $p_i \neq p_{i+1}$ since $f(p_i) > f(p_{i+1})$.

Sketch of Proof of Theorem[3]



Step 1. construct a smooth *approximate solution* of $F(u) = 0$:

$$\alpha_\lambda(s) = \begin{cases} u(s+2\lambda) & \text{for } s \leq -\lambda \\ a & \text{for } s \in [-\lambda+1, \lambda-1] \\ w(s-2\lambda) & \text{for } s \geq \lambda \end{cases}$$

and we use $\exp_a(\cdot)$ to interpolate this data.[4]

Then:

- $F(\alpha_\lambda(s)) \neq 0$ since you would need[5] ∞ time s to reach the crit pt a
- $(*)$ $F(\alpha_\lambda(s)) \to 0$ as $\lambda \to \infty$ since

$$\begin{aligned} &F(u(\cdot+2\lambda)) = 0 = F(w(\cdot-2\lambda)) \\ &F(s \mapsto a) = -\nabla f_a = 0 \\ &F(\text{interpolation}) \approx -\nabla f_a = 0 \end{aligned}$$

Step 2. $(*) \Rightarrow \exists$ "unique" actual solution $u\#_\lambda w$ close to α_λ,

$$F(u\#_\lambda w) = 0.$$

This "$\Rightarrow$" is proved using the contraction mapping theorem and the implicit function theorem. "Unique" is imprecise: one can construct a cts bijection $\alpha_\lambda \to u\#_\lambda w$.

Step 3. $\alpha_\lambda(s) \rightrightarrows u\#w$, indeed make s-shifts by -2λ and $+2\lambda$ when you lift α_λ.

Ideas used in Step 2. $L_u = D_uF$, $L_w = D_wF$, $L_\lambda = D_{\alpha_\lambda}F$

Rmk. *D_uF, D_wF, L_λ are Fredholm (Thm 4.14*[6]*)*

Technical Fact:

$$\begin{array}{c} L_u, L_w \text{ surjective} \\ \text{(by transversality)} \end{array} \Rightarrow \begin{cases} \text{①}\ L_\lambda \text{ surjective for } \lambda \gg 0 \\ \text{②}\ \exists c > 0 \text{ s.t. for } \lambda \gg 0: \\ \boxed{\|L_\lambda^\star V\|_{1,2} \leq c \cdot \|L_\lambda L_\lambda^\star V\|_2} \quad \forall V \in W^{1,2}(\mathbb{R}, \alpha_\lambda^* TM) \end{cases}$$

① One can patch[7] together elements in $\ker L_u, \ker L_w$ to obtain approximate solutions to $L_\lambda V = 0$, and one proves that for $\lambda \gg 0$ this defines an isomorphism:

$$\boxed{\begin{array}{rcl} \ker L_u \oplus \ker L_w & \overset{\sim}{\to} & \ker L_\lambda \\ V_u \oplus V_w & \mapsto & (\text{orthogonal projection}) \cdot (V_u \#_\lambda V_w) \end{array}}$$

where $\#$ is the patching. This we call ***linear gluing***. It is quite simple to prove because it just involves linear subspaces. This linear gluing map arises as the differential of the gluing map, and this isomorphism is used to prove the embedding property in (2).

---

[3] This would take too many Lectures to prove in detail, and the details are not enlightening.

[4] *Non-examinable:* $\exp_a(\beta(-s-\lambda+1)\cdot u(s+2\lambda))$ for $s \in [-\lambda, -\lambda+1]$; $\exp_a(\beta(s-\lambda+1)\cdot w(s-2\lambda))$ for $s \in [\lambda-1, \lambda]$, where $\beta : \mathbb{R} \to [0,1]$ is increasing with $\beta = 0$ on $s \leq 0$, $\beta = 1$ on $s \geq 1$.

[5] Hwk 22. ex. 2

[6] recall the theorem only used that the path was C^k, not that $F(\text{path}) = 0$.

[7] *Non-examinable:* For operators L, K which are asymptotically constant at $+\infty, -\infty$ respectively, then for $\lambda \gg 0$ we can glue $L(\cdot+2\lambda)\#K(\cdot-2\lambda) = L\#_\lambda K$, then $\ker L \oplus \ker K \overset{\sim}{\to} \ker(L\#_\lambda K)$ is the orthog projection of the patching $V\#_\lambda W = V(\cdot+2\lambda) + W(\cdot-2\lambda)$ (for fixed s this is small for $\lambda \gg 0$ since the solutions V, W decay to zero fast at the ends). This map is an iso for $\lambda \gg 0$.

By invariance of the Fredholm index under homotopying paths (indeed we know it is the difference of the Morse indices of the ends):[8]

$$\boxed{\text{index}\,(L_u) + \text{index}\,(L_w) = \text{index}\,(L_\lambda) \quad (\lambda \gg 0)}$$

so $\dim\operatorname{coker} L_\lambda = \dim\operatorname{coker} L_u + \dim\operatorname{coker} L_w = 0$, so L_λ is surjective. ✓

② Why that inequality? For A, B Hilbert,

$$\begin{tikzcd} L : A \to B \text{ Fredholm and surjective} \Rightarrow A = K \oplus A_0 \arrow[r] & B \arrow[l, bend right, "L^*"'] \arrow[l, bend left, "R = (L|_{A_0})^{-1}"] \end{tikzcd}$$

where[9] $\boxed{A_0 = \operatorname{im} L^*}$ and "R" stands for right-inverse since $LR = I$.

Cor. *$L : A \to B$ Fred and surj $\Leftrightarrow \exists$ bdd right inverse and* $\dim\ker L < \infty$

Lemma. $\|L^*b\| \le c \cdot \|LL^*b\| \ \forall b \Leftrightarrow \|Rb\| \le c \cdot \|b\| \ \forall b$

Proof. Both are equivalent to: $\|a\| \le c \cdot \|La\| \ \forall a \in A_0$. □

Upshot: Combining inequality ② with the Lemma:[10]

$\Rightarrow$ L_λ have uniformly bounded right inverses.

$\overset{Hwk\ 19}{\Rightarrow}$ $\exists$ unique actual solution $\exp_{\alpha_\lambda}(L_\lambda^\star V)$ (some unique $V \in W^{1,2}$) and all nearby actual solutions are of form $\exp_{\alpha_\lambda}(k \oplus g(k))$ where $k \in K$ is small and $g : K \to A_0$ is a smooth implicit function, $g(0) = \exp_{\alpha_\lambda}(L_\lambda^\star V)$.

So we define $\boxed{u\#_\lambda w = \exp_{\alpha_\lambda}(L_\lambda^\star V)}$

Rmk. *The key is that L^* provides a way to obtain uniqueness. $L_\lambda^\star V$ is constrained to be inside A_0, whereas if you allow vectors in the whole of A, such as $k \oplus g(k)$, then you no longer get uniqueness.[11] This is crucial also in Hwk 19: the contraction mapping principle (Picard's method) is applied to A_0, not the whole of A.*

Hwk 19: Picard's method.

For $F : A \to B$ a C^1-map of Hilbert spaces, by Taylor:

$$F(x) = c + L \cdot x + N(x)$$

where $c = F(0)$, $L = d_0F$ linear, N non-linear. Assume L Fred & surj, so as above:

$$L : K \oplus A_0 \to B \qquad R : B \to A_0 \qquad LR = I.$$

Assume the following two estimates hold:

(1) $\|Rc\| \le \frac{\varepsilon}{2}$

(2) $\|RN(x) - RN(y)\| \le C \cdot (\|x\| + \|y\|) \cdot \|x - y\|$ for all $x, y \in \text{ball}_\varepsilon(0)$, $\varepsilon \le \frac{1}{3C}$.

then

- by the contraction mapping theorem for $P : A_0 \to B$, $P(x) = -Rc - RN(x)$, there is a unique $a_0 \in A_0 \cap \text{ball}_\varepsilon(0)$ with $F(a_0) = 0$.

---

[8] or use formal adjoints to get isos of cokernels like for linear gluing of kernels.

[9] $(\operatorname{im} L^*)^\perp = \ker L = K$, and A_0 is closed since it is the complement of a finite dim'l subspace.

[10] which works in our setup for the formal adjoint $L_\lambda^\star$ instead of L_λ^*.

[11] Unsurprisingly, since when the Morse index difference is large, there is a large dimensional family of actual solutions, so the actual solution $u\#_\lambda v$ is not isolated. Indeed, the family is parametrized by K via $\exp_{\alpha_\lambda}(k \oplus g(k))$.

- by the implicit function theorem at a_0, there is a C^1-map $g : K \to A_0$ such that $F(k \oplus g(k)) = 0$ for small $k \in K$ (with $0 \oplus g(0) = a_0$).

Application: *We apply Picard's method to* $F =$ *local expression of the vertical part of our section* $\mathcal{F} = \partial_s + \nabla f : U \to E$ *in a chart around* $\alpha_\lambda \in U$ *(so* α_λ *is* 0 *in the chart). So*

$$\begin{aligned}
&F : W^{1,2}(\mathbb{R}, \alpha_\lambda^* TM) \to L^2(\mathbb{R}, \alpha_\lambda^* TM),\\
&c = F(0) = \mathcal{F}(\alpha_\lambda),\\
&L = d_0 F = D_{\alpha_\lambda}\mathcal{F} = L_\lambda.
\end{aligned}$$

Thus g *defines a parametrization of all the actual solutions* $\mathcal{F}(u \#_{\lambda(k)} w) = 0$ *close to the approximate solution* $\mathcal{F}(\alpha_\lambda) \approx 0$*, where* $\lambda(0) = \lambda$.

LECTURE 19.

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6. Morse Homology

6.1. **Definition.** M closed mfd, $f : M \to \mathbb{R}$ Morse, g generic metric ($\Rightarrow$transversality).

The *Morse(-Smale-Witten) complex* is the $\mathbb{Z}/2$-vector space generated by the critical points of f:

$$\boxed{MC_k = \bigoplus_{p \in \mathrm{Crit}(f),\ |p|=k} \mathbb{Z}/2 \cdot p}$$

where $k \in \mathbb{Z}$ is the $\mathbb{Z}$-grading by the Morse index.

The *Morse differential* $\partial : MC_k \to MC_{k-1}$ is defined on generators p by

$$\boxed{\partial p = \sum_{\dim \mathcal{M}(p,q)=0,\ p \neq q} \#\mathcal{M}(p,q) \cdot q}$$

and extend ∂ linearly to MC_*. Note[1] $\dim \mathcal{M}(p,q) = 0$ is equivalent to $|q| = |p| - 1$.

Rmk. *The sum is well-defined because $\mathcal{M}(p,q)$ is a 0-dimensional compact manifold, hence finite, so can count[2] the number of elements $\#\mathcal{M}(p,q)$. Proof: It is a smooth manifold by transversality, and it is compact by the following argument:*[3]

$$\begin{aligned} \dim \mathcal{M}(p,q) = 0 \quad &\Rightarrow \quad \dim \partial\mathcal{M}(p,q) < 0 \\ &\Rightarrow \quad \partial\mathcal{M}(p,q) = \emptyset \\ &\Rightarrow \quad \mathcal{M}(p,q) = \overline{\mathcal{M}}(p,q) \text{ compact 0-dim mfd } \checkmark \end{aligned}$$

Thm. $\partial^2 = 0$.

Proof. $|p| = k$. Compute:

$$\begin{aligned} \partial^2 p &= \partial \sum_{|a|=k-1} \#\mathcal{M}(p,a) \cdot a \\ &= \sum_{|a|=k-1} \sum_{|q|=k-2} \#\mathcal{M}(p,a) \cdot \#\mathcal{M}(a,q) \cdot q \\ &= \sum_{|a|=k-1,\ |q|=k-2} \#(\mathcal{M}(p,a) \times \mathcal{M}(a,q)) \cdot q \\ &= \sum_{|a|=k-1,\ |q|=k-2} \#\partial\mathcal{M}(p,q) \cdot q \end{aligned}$$

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[1] *Cultural Remark:* In more general situations, $\mathcal{M}(p,q)$ may have components of different dimensions, and you only count the $u \in \mathcal{M}(p,q)$ in the 0-dimensional part $\mathcal{M}_0(p,q)$.

[2] *Non-examinable:* To work over $\mathbb{Z}$ instead of $\mathbb{Z}/2$ you must count the elements of $\mathcal{M}(p,q)$ with orientation signs ± 1. Orientations of moduli spaces are an unpleasant technical detail which we decided to omit from this course (compare with sign headaches in singular homology arguments).

[3] Exercise. Can you think of a simple argument which only involves using tranversality, the compactness thm and dimension arguments, but which does not use the gluing theorem?

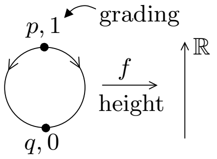
Finally observe:

$\dim \mathcal{M}(p,q) = |p| - |q| - 1 = k - (k-2) - 1 = 1$
$\Rightarrow \overline{\mathcal{M}}(p,q)$ compact 1-mfd with boundary
$\Rightarrow \overline{\mathcal{M}}(p,q)$ is a disjoint union of finitely many circles and closed intervals
$\Rightarrow \#\partial\mathcal{M}(p,q)$ even, so 0 modulo 2
$\Rightarrow \partial^2 p = 0$
$\Rightarrow \partial^2 = 0$ by linearity. $\square$

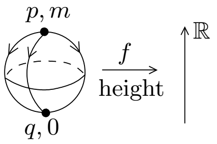
Def. $\boxed{MH_*(M,f,g) = \dfrac{\ker \partial}{\operatorname{im} \partial}}$ *is the Morse homology of* (M,f,g).

Rmk. *If you are given a metric g, then a priori you need to perturb g unless you know/check that transversality holds (see Hwk 1). Key Trick: perturbing g does not affect Crit(f) and indices, this often helps.*[4]

Examples (all homologies are over $\mathbb{Z}/2$):

(1) 

$$\partial p = q+q = 0$$
$$\begin{array}{rclcll} MH_* & = & \mathbb{Z}/2 & \oplus & \mathbb{Z}/2 & \cong H_*(S^1) \\ * & = & 0 & & 1 & \leftarrow \text{ grading} \end{array}$$

(2) 

$\partial = 0$ as the indices are ≥ 2 apart (see Rmk)
$$\begin{array}{rclcll} MH_* & = & \mathbb{Z}/2 & \oplus & \mathbb{Z}/2 & \cong H_*(S^m) \\ * & = & 0 & & m & \leftarrow \text{ grading} \end{array}$$

(3) **Thm.** $MH_*(M,f,g) \cong H_*(M)$.
Proof. Next time we will prove *invariance*:

$$MH_*(M,f_1,g_1) \cong MH_*(M,f_2,g_2).$$

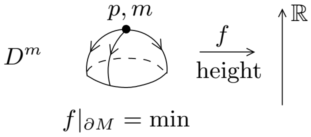
So

$$\begin{array}{rcl} MH_*(f) & \cong & MH_*(\text{self-indexing Morse function}) \\ & \cong & H_*^{\text{cellular}}(M) \quad (3.10 \ \& \text{ Hwk } 19) \\ & \cong & H_*(M). \end{array}$$

(4) M compact mfd with boundary:

Ensure $f|_{\partial M} = \text{constant min} < f|_{\text{interior}}$
$(\Rightarrow \nabla f \pitchfork \partial M \Rightarrow \text{no crit pts on } \partial M)$
$\Rightarrow \mathcal{M}(p,q)$ stay away from ∂M (f decreases along flowlines)
$\Rightarrow MH_* = \dfrac{\ker \partial}{\operatorname{im} \partial} \cong H_*(M, \partial M)$ (proved like in (3))

Example: $M = D^m$ disc.



$$\begin{array}{rcl} MH_* & = & \mathbb{Z}/2 \cdot p \quad (\text{in degree } m) \\ & \cong & H_*(D^m, \partial D^m) \\ & & (\text{have } H_0 = 0) \end{array}$$

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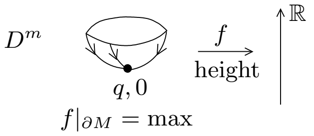
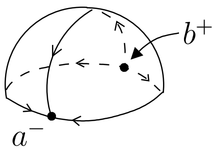
[4] Key example: if all indices are even then, after perturbing, the 0-dimensional moduli spaces are empty for index reasons. So there is no differential. So $MH_* = MC_*$, which you know.



$$\begin{aligned} MH_* &= \mathbb{Z}/2 \cdot q \quad \text{(in degree 0)} \\ &\cong H_*(D^m) \\ &\ \text{(compare Handle attaching)} \end{aligned}$$

But both are useless for LES of pair $(M, \partial M)$: cannot recover $MH_*(\partial M)$. Instead of making $\nabla f \pitchfork \partial M$ we will now try ∇f tangent to ∂M.

6.2. **Morse homology for mfds with bdry.** [*Non-examinable*]
Assume $\boxed{\nabla f \text{ is tangent to } \partial M}$ (that is: $\nabla f \in T(\partial M)$). This ensures that the flow of a point in ∂M stays in ∂M and $f|_{\partial M}$ is Morse, so hope to recover $MH_*(\partial M)$.



Write $\boxed{a = a^- \text{ if } df(\text{outward normal}) < 0 \text{ at } a}$
$\Rightarrow W^u(a) \subset \partial M$ (exercise)

Write $\boxed{b = b^+ \text{ if } df(\text{outward normal}) > 0 \text{ at } b}$
$\Rightarrow W^u(b)$ intersects interior, $\partial W^u(b) = W^u(b) \cap \partial M$

$\exists$ Similar statements for W^s reversing the roles of $+, -$ (*Proof*: switch sign of f).

$$\Rightarrow \boxed{MC_* = MC_*^0 \oplus MC_*^- \oplus MC_*^+}$$

respectively generated by $p \in \operatorname{int} M$, a^-'s, b^+'s.

Bad case:[5]

$$\begin{aligned} a^-, b^+ \quad &\Rightarrow \quad W^u(a) \subset \partial M,\ W^s(b) \subset \partial M \\ &\Rightarrow \quad \text{cannot hope } W^u(a) \pitchfork W^s(b) \text{ in } M \\ &\Rightarrow \quad \text{require } \pitchfork \text{ just in } \partial M. \end{aligned}$$

Therefore:

$$\boxed{\begin{array}{ll} \dim \mathcal{M}(a^-, b^+) = |a| - |b| & \text{bad case} \\ \dim \mathcal{M}(p, q) = |p| - |q| - 1 & \text{otherwise (as usual)} \end{array}}$$

Getting $MH_*(\partial M)$:
$p, q \in \partial M \Rightarrow \boxed{B(p,q) = \{[u] \in \mathcal{M}(p,q) : u \subset \partial M\} = \mathcal{M}(p, q, f|_{\partial M})}$
$p \in \partial M \Rightarrow \operatorname{index}_{f|_{\partial M}}(p) = \left\{ \begin{array}{ll} |p| & \text{if } p^- \\ |p| - 1 & \text{if } p^+ \end{array} \right.$
Therefore $MC_k(\partial M, f|_{\partial M}) = MC_k^- \oplus MC_{k+1}^+$, with differential

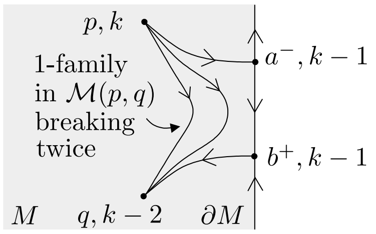
$$Bp = \sum_{\dim B(p,q) = 0,\ p \neq q} \#B(p,q) \cdot q$$

whose homology recovers $MH_*(\partial M)$.

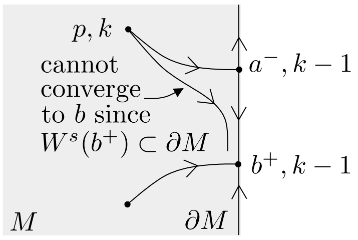
Getting $MH_*(M, \partial M)$:
There are 2 new bad phenomena:

---

[5] We are tweaking the definition of Morse-Smale to suit the situation.



① ∃ flowlines between same index bdry pts! (Bad case)



② Gluing fails for this twice broken trajectory!

Upshot: $\boxed{\partial^2 \neq 0}$

The argument in 6.1 for $\partial^2 = 0$ will yield:

$$\sum_{r\in \operatorname{int} M} \#\mathcal{M}(p,r)\cdot\#\mathcal{M}(r,q)+\sum_{|a^-|=|b^+|=k-1}\#\mathcal{M}(p,a)\cdot\#B(a,b)\cdot\#\mathcal{M}(b,q)=0 \pmod 2 \quad (*)$$

where $p, q \in \operatorname{int} M$, $|p| = k$, $|q| = k-2$.

Miracle: ① essentially fixes ②. The two problems suggest that one should not keep both MC^+ and MC^-, one should use only one of the two. Try keeping

$$\boxed{MC_*^0 \oplus MC_*^+}$$

Notation:[6] $B : MC_k^- \oplus MC_{k+1}^+ \to MC_{k-1}^- \oplus MC_k^+$,

$$B = \begin{bmatrix} B_-^- & B_-^+ \\ B_+^- & B_+^+ \end{bmatrix}$$

Similar notation for ∂. Then $(*)$ can be rewritten as:

$$\partial_0^0\partial_0^0 + \partial_0^+ B_+^- \partial_-^0 = 0$$

Define a differential d by combining ∂ with B's, so that a once-broken trajectory breaking at an a^- point is considered as if it were just one flowline.[7]

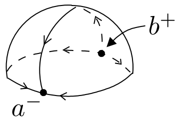
$$\boxed{d = \begin{bmatrix} \partial_0^0 & \partial_0^+ \\ B_+^-\partial_-^0 & B_+^+ + B_+^-\partial_-^+ \end{bmatrix} : MC_k^0 \oplus MC_k^+ \to MC_{k-1}^0 \oplus MC_{k-1}^+}$$

$$\Rightarrow d^2 = \begin{bmatrix} \partial_0^0\partial_0^0 + \partial_0^+ B_+^- \partial_-^0 & \bullet \\ \bullet & \bullet \end{bmatrix}$$

By $(*)$, the first entry is 0. Similar arguments show the other entries are zero. So

$$\Rightarrow \boxed{d^2 = 0}$$

Example



$MC_*^0 = 0$
$MC_*^+ = \mathbb{Z}/2 \cdot b$ in degree 2
$\Rightarrow H_*(d) = \mathbb{Z}/2 \cdot b \cong H_*(D^2, \partial D^2)$

Thm. $\boxed{H_*(d) \cong H_*(M, \partial M)}$

---

[6] The top index is "from", the bottom index is "to", so B_-^+ goes from MC^+ to MC^-.

[7] This is to fix the two problems by pretending that ① is a once-broken trajectory, and that the first breaking in ② is not a breaking.

Proof Sketch. First you show that MH_* changes by an iso if you change f. Then you construct your favourite f by the methods of Hwk 7: one for which

$$df(\text{outward normal}) \leq 0$$

near ∂M (so all critical $a \in \partial M$ are of type a^-, and no trajectory from the interior will get arbitrarily close to ∂M unless it ends there). The claim then follows by examples (3) & (4). $\square$

If you instead try just keeping

$$\boxed{MC^0_* \oplus MC^-_*}$$

then the appropriate differential is

$$\boxed{\delta = \left[\begin{array}{cc} \partial^0_0 & \partial^+_0 B^-_+ \\ \partial^0_- & B^-_- + \partial^+_- B^-_+ \end{array} \right]}$$

In the above Example, $H_*(\delta)$ is generated by a in degree $|a| = 0$.

Thm. $\boxed{H_*(\delta) \cong H_*(M)}$

Proof idea. Make all critical $b \in \partial M$ to be of type b^+. $\square$

Def. *The homologies of B, d, δ (also denoted $\overline{\partial}, \hat{\partial}, \check{\partial}$) are called:*

$$\begin{array}{lcll} \overline{MH} & \cong & H_*(\partial M) & \text{“}MH\text{ bar”} \\ \widehat{MH} & \cong & H_*(M, \partial M) & \text{“}MH\text{ from”} \\ \widecheck{MH} & \cong & H_*(M) & \text{“}MH\text{ to”} \end{array}$$

The hat tells you the movement of flowlines: from/to the boundary ∂M.

Thm. LES of pair $(M, \partial M)$

$$\cdots \to \overline{MH}_* \to \widecheck{MH}_* \to \widehat{MH}_* \to \overline{MH}_{*-1} \to \cdots$$

at the chain level, the maps are:

$$\begin{array}{ccccccccc} \cdots \to MC^- \oplus MC^+ & \to & MC^0 \oplus MC^- & \to & MC^0 \oplus MC^+ & \to & MC^- \oplus MC^+ \to \cdots \\ & \left[\begin{array}{cc} 0 & \partial^+_0 \\ I & \partial^+_- \end{array} \right] & & \left[\begin{array}{cc} I & 0 \\ 0 & B^-_+ \end{array} \right] & & \left[\begin{array}{cc} \partial^0_- & \partial^+_- \\ 0 & I \end{array} \right] & \end{array}$$

An excellent reference for further details is the CUP book *Monopoles and Three-Manifolds* by Kronheimer & Mrowka.

LECTURE 20.

PART III, MORSE HOMOLOGY, 2011

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6.3. **Invariance Theorem.** Let M be a closed mfd.

Thm. *$MH_*(M)$ does not depend on the auxiliary parameters you chose: given Morse $f_0, f_1 : M \to \mathbb{R}$ and generic metrics g_0, g_1, there is an isomorphism*

$$[\varphi_{10}] : MH_*(f_0, g_0) \xrightarrow{\cong} MH_*(f_1, g_1)$$

and these isomorphisms satisfy

1. $[\varphi_{21}] \circ [\varphi_{10}] = [\varphi_{20}] \qquad (\forall f_i, g_i,\ i = 0, 1, 2)$
2. $[\varphi_{00}] = \mathrm{id} \qquad (\forall f_0, g_0)$

Outline

1. Construct a *continuation map*

   $\boxed{\varphi : MC_*^- \to MC_*^+}$ $(MC_*^\pm = MC_*(f^\pm, g^\pm), f^\pm \text{ Morse}, g^\pm \text{ generic})$

   defined on generators as follows (then extend linearly):

   $$\varphi(p^-) = \sum_{\dim \mathcal{C}(p^-,q^+)=0} \#\mathcal{C}(p^-, q^+) \cdot q^+$$

   which counts the *moduli space of continuation solutions*

   $$\boxed{\begin{array}{lll} \mathcal{C}(p^-, q^+) = \{v : \mathbb{R} \to M & : & \partial_s v = -\nabla^s f_s(v), \\ & & v(s) \to p^-, q^+ \text{ as } s \to -\infty, +\infty\} \end{array}}$$

   where $p^- \in \mathrm{Crit}(f^-)$, $q^+ \in \mathrm{Crit}(f^+)$, $g_s(\nabla^s f_s, \cdot) = df_s$. This moduli space depends on a choice of smooth homotopy f_s, g_s,

   $$s \mapsto (f_s : M \to \mathbb{R}), \quad s \mapsto g_s$$

   where the functions f_s need not be Morse and the metrics g_s need not be Morse-Smale for f_s. The key requirement[1] is that for some S:

   $$\boxed{f_s = \begin{cases} f^- & \text{for } s \le -S \\ f^+ & \text{for } s \ge S \end{cases} \qquad g_s = \begin{cases} g^- & \text{for } s \le -S \\ g^+ & \text{for } s \ge S \end{cases}} \quad (*)$$

   Rmk. *Transversality for $\mathcal{C}(p^-, q^+)$ is achieved for generic paths g_s.*

   Rmk. *Note that we do not*[2] *quotient $\mathcal{C}(p^-, q^+)$ by an $\mathbb{R}$-action by shifting s, unlike what we did for $\mathcal{M}(p, q) = W(p, q)/\mathbb{R}$.*

2. φ = identity for the constant data $f_s = f$, $g_s = g^+ = g^-$.

Date: May 3, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1] this is crucial for the energy estimate, later.

[2] indeed, cannot: $\nabla^s f_s$ is not invariant under shifting $s \mapsto s + \text{constant}$.

(3) φ is a chain map:

$$\boxed{\varphi\circ\partial^- = \partial^+\circ\varphi} \qquad (\partial^\pm : MC_*^\pm \to MC_{*-1}^\pm)$$

hence we get a map on homology:

$$[\varphi] : MH_*^- \to MH_*^+.$$

(4) $\boxed{[\varphi] \text{ does not depend on } f_s, g_s}$

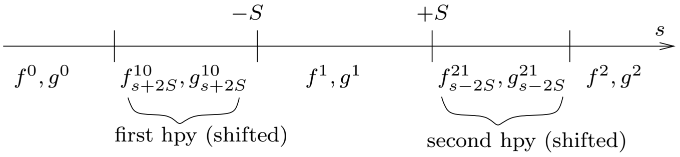
Consider a homotopy $(f_s^\lambda, g_s^\lambda)_{0\leq\lambda\leq 1}$ from f_s^0, g_s^0 to f_s^1, g_s^1, where we assume $(*)$ also for f_s^λ, g_s^λ. Denote φ^0, φ^1 the continuations for f_s^0, g_s^0 and f_s^1, g_s^1.

Claim φ^0, φ^1 are chain homotopic:

$$\boxed{\begin{array}{l} \exists\, K : MC_*^- \to MC_{*+1}^+ \\ \varphi^0 - \varphi^1 = K\circ\partial^- + \partial^+\circ K \end{array}}$$

hence[3] $[\varphi^0] - [\varphi^1] = I$. ✓

(5) Suppose f_s^{10}, g_s^{10} is a homotopy from f^0, g^0 to f^1, g^1, and f_s^{21}, g_s^{21} is a homotopy from f^1, g^1 to f^2, g^2. Glue[4] the (reparametrized) homotopies:



Claim. For $S \gg 0$, the φ obtained for this glued homotopy equals the composite $\varphi^{21}\circ\varphi^{10}$ of the continuation maps for the two homotopies.

(6) **Consequences of these properties:**

(4) and (5) $\Rightarrow [\varphi^{21}]\circ[\varphi^{10}] = [\varphi^{20}]$ (independently of choices of hpies) ✓
(2) and (4) $\Rightarrow [\varphi^{00}] =$ identity (independently of choice of hpy) ✓
$\Rightarrow [\varphi^{01}]\circ[\varphi^{10}] = [\varphi^{00}] =$ identity so $[\varphi^{10}]$ injective
$\Rightarrow [\varphi^{10}]\circ[\varphi^{01}] = [\varphi^{11}] =$ identity so $[\varphi^{10}]$ surjective
$\Rightarrow [\varphi^{10}]$ isomorphism
$\Rightarrow$ Theorem

Key ideas in the proofs:

(1) Redo the transversality proof, now using:

$$G = \{C^k\text{-paths of metrics } s\mapsto g_s \text{ with } g_s = \begin{cases} g^- & \text{for } s\leq -S \\ g^+ & \text{for } s \geq S\end{cases} \quad \}$$
$$F(u, g_s) = \partial_s u - \nabla^s f_s(u) \quad (\text{where } g_s(\nabla^s f_s, \cdot) = df_s)$$

$\Rightarrow$ Parametric transversality, Fredholm analysis, etc. like we did for $W(\cdot,\cdot)$

$$\Rightarrow \boxed{\begin{array}{l} \mathcal{C}(p^-, q^+) \text{ smooth mfd for generic smooth } g_s \\ \dim\mathcal{C}(p^-, q^+) = |p| - |q| \end{array}}$$

In (3) we explain how to compactify $\mathcal{C}(p^-, q^+)$, and it shows that $\mathcal{C}(p^-, q^+)$ is compact when $\dim\mathcal{C}(p, q) = |p| - |q| = 0$. So φ is well-defined.

---

[3] since $\partial^- = 0$ on $\ker\partial^-$, $\partial^+(K\bullet) = 0$ modulo $\operatorname{im}\partial^+$.

[4] for large S these glue correctly.

(2) For constant data $f = f_s$, $g = g_s$,

$$\mathcal{C}(p,q) = W(p,q).$$

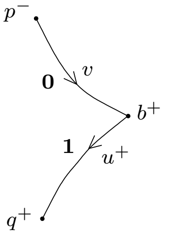
So since $g = g^- = g^+$ is generic, $W(p,q)$ is a smooth mfd, thus so is $\mathcal{C}(p,q)$. Finally we make a dimension argument:

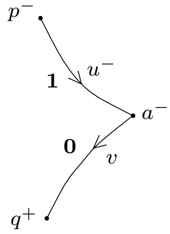
If $v \in \mathcal{C}(p,q)$ is a solution then $v(\cdot + \text{constant})$ is a solution since $\partial_s v = -\nabla f(v)$ has ∇f independent of s.
$\Rightarrow$ if v non-constant, then there is a 1-family of solutions $v(\cdot + \text{constant})$
$\Rightarrow \dim \mathcal{C}(p,q) \geq 1$ if $p \neq q$
$\Rightarrow \varphi$ only counts constant solutions $\mathcal{C}(p,p) = W(p,p) = \{\text{constant at } p\}$
$\Rightarrow \varphi(p) = p$
$\Rightarrow \varphi = \text{identity}$.

Rmk. *The key is* not *to make an s-dependent perturbation of $g_s = g^- = g^+$, but rather to perturb s-independently (in fact, since we assume $g^- = g^+$ is generic, we don't need to). This gives transversality for $W(p,q) = \mathcal{C}(p,q)$.*

(3) Study the breaking of 1-dimensional $\mathcal{C}(p^-, q^+)$, so $|p^-| - |q^+| = 1$.

The key claim is that a once-broken continuation solution does not consist of two continuation solutions, but rather consists of one continuation solution and one $f^\pm$-trajectory:[5]





Proof. Consider a sequence v_n of continuation solutions in $\mathcal{C}(p^-, q^+)$. Consider the interval $[-S,S]$ where f_s, g_s depend on s. Since $[-S,S]$ is compact, by Arzela-Ascoli a subsequence will satisfy C^0-convergence on $[-S,S]$ so there is no breaking there.

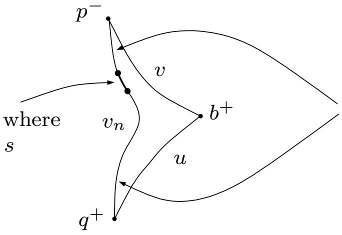


image of $[-S,S]$ where f_s, g_s depend on s

here can apply the reparametrization trick (think "stretching of an interval $[-s_n, -S]$ or $[S, s_n]$")

---

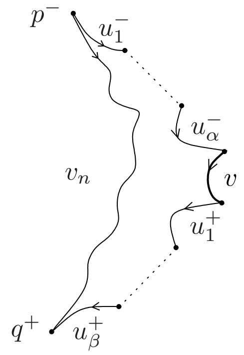
[5] In the figure, we denote by v the continuation solutions, and by $u^\pm$ the $-\nabla^\pm f^\pm$-trajectories. We write boldface numbers which indicate the Fredholm index of the linearization of the Fredholm section. So in the figure, $|b| - |q| = 1$, $|p| - |a| = 1$. Recall that

$$\dim(\mathcal{C} \text{ or } W \text{ spaces}) = \dim \text{ tangent space} = \dim \ker(\text{surj Fred operator}) = \text{index}.$$

For W spaces, dimension 1 implies that $\mathcal{M} = W/\mathbb{R}$ has dimension 0. So solutions v, u in $\mathcal{C}, W$ spaces of dim $0, 1$ respectively are called ***rigid*** (or *isolated*).

Finally, these dimension numbers add to give the correct dimension of the breaking family because of linear gluing (see 5.5 Step 2, details to ①): "gluing kernels of Fred operators is iso to kernel of glued Fred operator". So indices add correctly under gluing.

In the general case (when we do not assume $|p|-|q|=1$): mimick the proof of 5.3 and use the above observation $\Rightarrow$ general breaking for $\mathcal{C}(p^-,q^+)$ is:



Here u_i^- are $-\nabla^- f^-$-trajectories, u_j^+ are $-\nabla^+ f^+$-trajectories, and v is a continuation map for (f_s, g_s).

Details. Reviewing the proof of compactness for W spaces, observe that what we needed crucially was an a priori energy estimate. In our case it is:

$$\begin{array}{rcl} E(v) & = & \int_{-\infty}^{\infty} |\partial_s v|^2\, ds \\ & = & \int g_s(\partial_s v, \partial_s v)\, ds \\ & = & -\int df_s(\partial_s v)\, ds \quad (\text{since } \partial_s v = -\nabla^s f_s) \\ & = & -\int (\partial_s(f_s \circ v) - (\partial_s f_s)(v))\, ds \\ & \leq & f^-(p^-) - f^+(q^+) + \int |\partial_s f_s|_v\, ds \\ & \leq & f^-(p^-) - f^+(q^+) + 2S \cdot \max\limits_{x\in M} |\partial_s f_s(x)| \end{array}$$

We also needed the energy consumption trick 3.3. This can also be used in our setup in the regions $s \leq -S$, $s \geq S$ where f_s, g_s do not depend on s.

Key observation: each u_i^+, u_j^- contributes to 1 to the index difference $|p^-|-|q^+|$, since the $\mathcal{M}^\pm$ spaces are empty if the index difference of the ends is zero or negative.

Key $\Rightarrow$ for $|p^-|-|q^+|=0$ no breaking can occur $\Rightarrow$ $\mathcal{C}(p^-,q^+)$ is compact.
Key $\Rightarrow$ for $|p^-|-|q^+|=1$ only 1 breaking can occur for dimension reasons.

Hence (after reproving the gluing theorem) for $|p|-|q|=1$:

$$\partial\overline{\mathcal{C}}(p^-,q^+) = \bigsqcup_{a^-} \mathcal{M}_0^-(p,a) \times \mathcal{C}_0(a,q) \cup \bigsqcup_{b^+} \mathcal{C}_0(p,b) \times \mathcal{M}_0^+(b,q)$$

where the numbers indicate the dimension we request[6] and $\mathcal{M}^\pm$ are the $\mathcal{M}$ spaces for $(f^\pm, g^\pm)$.

$$\begin{array}{l} \Rightarrow \overline{\mathcal{C}}(p,q) \text{ compact 1-mfd} \\ \Rightarrow \#\partial\overline{\mathcal{C}}(p,q) \text{ is even} \\ \Rightarrow \varphi \circ \partial^- + \partial^+ \circ \varphi = 0. \quad \square \end{array}$$

(4) $(f_s^\lambda, g_s^\lambda)_{0\leq\lambda\leq1}$ is called *homotopy of homotopies* $(*)$

---

[6] $|p|=|b|=k$, $|a|=|q|=k-1$.

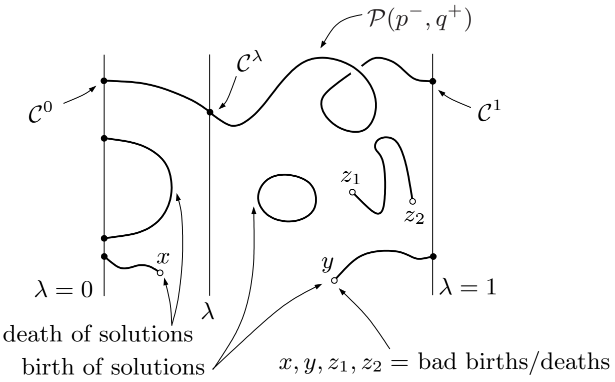
Fix p^-, q^+ with $|p^-| - |q^+| = 0$. Look at the "movie"

$$\boxed{\mathcal{C}^\lambda = \mathcal{C}^\lambda(p^-, q^+; f_s^\lambda, g_s^\lambda)}$$

as λ varies. This "movie" is called the *parametrized moduli space*

$$\mathcal{P}(p^-, q^+) = \bigsqcup_{0 \le \lambda \le 1} \mathcal{C}^\lambda$$

For generic data $(*)$, it is a smooth 1-mfd:[7]



Warning. *$\mathcal{C}^\lambda$ may not be a smooth manifold for fixed λ. Genericity of the family (in λ) does not guarantee genericity of each point of the family (fixed $\lambda = \lambda_0$). However, one can guarantee that each $\mathcal{C}^\lambda$ satisfies transversality except for finitely many values of λ.*

Breaking analysis: a subsequence (λ_n, v_n) has $\lambda_n \to 0, 1$ or $\lambda_0 \in (0, 1)$.

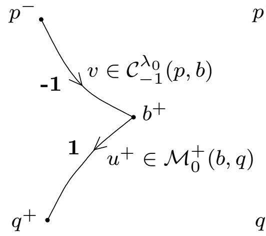
$$\Rightarrow \partial\overline{\mathcal{P}} = \mathcal{C}^0 \sqcup \mathcal{C}^1 \sqcup B$$

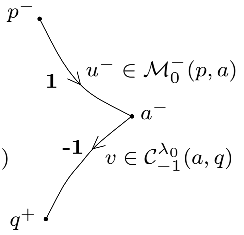
where $B = \{\text{bad births/deaths}\}$.

If $B = \emptyset$, then P is a 1-cobordism from $\mathcal{C}^0$ to $\mathcal{C}^1$,

$$\begin{aligned} &\Rightarrow \#\mathcal{C}^0 - \#\mathcal{C}^1 = \#\partial P = \text{even} = 0 \bmod 2 \\ &\Rightarrow \varphi^0 = \varphi^1 \end{aligned}$$

If $B \neq \emptyset$, let K count the bad set B:





Question. How is it possible that such v exist: the relevant moduli space $\mathcal{C}^{\lambda_0}$ is negative dimensional!

---

[7] $\dim \mathcal{P}(p, q) = |p| - |q| + 1$, where the additional 1 is because of the parameter λ.

Answer. This happens because $f_s^{\lambda_0}, g_s^{\lambda_0}$ is not generic.[8] So "-1" is the *virtual dimension*: the dimension you would get *if* transversality held true:

$$\text{virdim } \mathcal{C}^{\lambda_0}(p,q) = |p| - |q|.$$

Def. *Such* $v \in \mathcal{C}^{\lambda_0}_{-1}(\cdot,\cdot)$ *(virtual dimension* -1*) are called* rogue trajectories.

There are no rogue trajectories at $\lambda = 0, 1$ since by assumption f_s^0, g_s^0 and f_s^1, g_s^1 are generic. So define

$$\boxed{\begin{array}{rl} K \;:\; & MC_*^- \to MC_{*+1}^+ \\ & Kx^- = \displaystyle\sum_{|y^+|=|x^-|+1} \#(\text{rogue trajectories from } x \text{ to } y)\cdot y^+ \end{array}}$$

So in the above pictures, the contributions would be:

$$\begin{array}{rcl} Kp^- &=& b^+ + \cdots \\ Ka^- &=& q^+ + \cdots \\ \partial^- p^- &=& a^- + \cdots \\ \partial^+ b^+ &=& q^+ + \cdots \end{array}$$

So $\boxed{\varphi^0 - \varphi^1 = \partial^+ \circ K + K \circ \partial^-}$ comes from counting the even number of elements in:

$$\begin{aligned} \partial\overline{\mathcal{P}}(p^-, q^+) = \mathcal{C}^0 \sqcup \mathcal{C}^1 \quad &\sqcup \bigsqcup_{\lambda_0 \in (0,1), b^+ \in \text{Crit } f^+} \mathcal{C}^{\lambda_0}_{-1}(p,b) \times \mathcal{M}^+_0(b,q) \\ &\sqcup \bigsqcup_{\lambda_0 \in (0,1), a^- \in \text{Crit } f^-} \mathcal{M}^-_0(p,a) \times \mathcal{C}^{\lambda_0}_{-1}(a,q) \end{aligned}$$

(5) This is a gluing argument: you can *approximately* glue solutions, then for large S (depending on p, r, q) you can associate a "unique" actual solution. This produces a bijection:

$$\bigsqcup_{q^1 \in \text{Crit} f^1} \mathcal{C}_0(p^0, q^1; 1^{\text{st}} \text{ hpy}) \times \mathcal{C}_0(q^1, r^2; 2^{\text{nd}} \text{ hpy}) \to \mathcal{C}_0(p^0, r^2; \text{glued hpy})$$

So $\varphi^{21} \circ \varphi^{10}(p^0)$ and $\varphi^{20}(p^0)$ have the same r^2 coefficients. Therefore

$$\varphi^{21} \circ \varphi^{10} = \varphi^{20}$$

(there are only finitely many critical points, so you can pick the largest of the S's, as you vary p, q, r).[9]

---

[8] Just because the family $(*)$ is generic, does not mean that each $f_s^{\lambda_0}, g_s^{\lambda_0}$ is generic.

[9] *Non-examinable:* In more complicated situations, when there are infinitely many generators, you can still prove the equation at the level of homology: cycles involve *finite* linear combinations of generators, so only finitely many generators are involved in showing that the two expressions agree on a given cycle.

LECTURE 21.

PART III, MORSE HOMOLOGY, 2011

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7. Applications

7.1. **Poincaré duality.** Define

$$MC^* = \text{ Morse cochain complex } = \sum_{p\in \mathrm{Crit}\, f} \mathbb{Z}/2 \cdot p$$

$$\delta p = \sum_{\dim \mathcal{M}(q,p)=0,\ q\neq p} \#\mathcal{M}(q,p)\cdot q \quad \text{(note the changed order!)}$$

Cor. *$MH^*(f) = H_*(MC^*,\delta) =$ Morse cohomology $\cong H^*(M)$ (proved like 6.1 (3)).*

Thm. *$MH_*(f) \cong MH^{m-*}(-f)$*

Proof. Note that $\mathrm{ind}_{-f}(p) = m - \mathrm{ind}_f(p)$. So define on generators

$$\phi : MC_*(f) \to MC^{m-*}(-f),\ p \mapsto p$$

and extend linearly. So ϕ is clearly an isomorphism of vector spaces. Moreover,

$$\begin{array}{ccc} \mathcal{M}(p,q;f) & \cong & \mathcal{M}(q,p;-f) \\ [u(s)] & \mapsto & [u(-s)] \end{array}$$

so ϕ is a chain map:

$$(\phi\circ\partial)(p) = \sum \#\mathcal{M}(p,q;f)\cdot q = \sum \#\mathcal{M}(q,p;-f)\cdot q = (\delta\circ\phi)(p). \quad \square$$

7.2. **Künneth's theorem.**

Thm. *Over $\mathbb{Z}/2$ coefficients,*

$$\begin{tikzcd} MH_*(f_1\oplus f_2) \arrow[r, "\cong"] & MH_*(f_1)\otimes MH_*(f_2) \arrow[d, "\cong"] \\ H_*(M_1\times M_2) \arrow[u, "\cong"'] & H_*(M_1)\otimes H_*(M_2) \end{tikzcd}$$

where $f_1 : M_1 \to \mathbb{R}$, $f_2 : M_2 \to \mathbb{R}$ are Morse functions on closed mfds.

Proof.

$$\begin{array}{cccc} f_1\oplus f_2 : & M_1\times M_2 & \to & \mathbb{R} \\ & (m_1,m_2) & \mapsto & f_1(m_1)+f_2(m_2) \end{array}$$

So $d(f_1\oplus f_2) = df_1 \oplus df_2 : TM_1 \oplus TM_2 \to \mathbb{R}$.

Also f_1, f_2 Morse $\Rightarrow$ $f_1\oplus f_2$ Morse.

For g_1, g_2 generic metrics, we will use[1] $g_1\oplus g_2$ on $M_1\times M_2$. Then

$$W(p_1\times p_2, q_1\times q_2, f_1\oplus f_2) = W(p_1,q_1,f_1)\times W(p_2,q_2,f_2)$$

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[1] We will see directly that transversality holds for $(f_1\oplus f_2, g_1\oplus g_2)$.

since $\partial_s u_1 \oplus \partial_s u_2 = -\nabla f_1 \oplus -\nabla f_2$. The grading is:

$$|p_1 \times p_2| = |p_1| + |p_2|.$$

To get 1-dimensional moduli spaces $W(\cdot,\cdot)$: need u_1 or u_2 constant, otherwise have $\dim \geq 2$ by shifting s in u_1 and in u_2.

$$\Rightarrow d \text{ counts} \begin{cases} \mathcal{M}(p_1 \times p_2, q_1 \times p_2) = \mathcal{M}(p_1, q_1, f_1) \times \{p_2\} \\ \mathcal{M}(p_1 \times p_2, p_1 \times q_2) = \{p_1\} \times \mathcal{M}(p_2, q_2, f_2) \end{cases}$$

$\Rightarrow d(p_1 \times p_2) = \partial p_1 \times p_2 + p_1 \times \partial p_2$

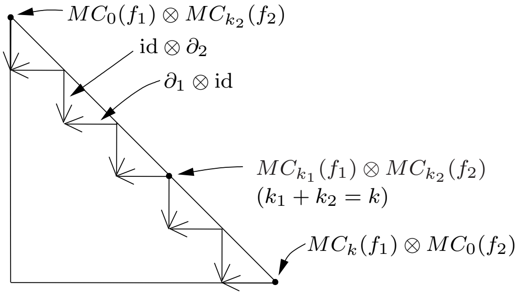
$$\Rightarrow MC_k(f_1 \oplus f_2) \cong \bigoplus_{k_1+k_2=k} MC_{k_1}(f_1) \oplus MC_{k_2}(f_2)$$

is an isomorphism of chain complexes, where the differential on the right is

$$d = \partial_1 \times \mathrm{id} + \mathrm{id} \times \partial_2$$

and this complex is denoted $(MC_*(f_1) \otimes MC_*(f_2))_{\text{grading } k}$.

Hence, by abstract algebra,[2] $MH_*(f_1 \oplus f_2) \cong MH_*(f_1) \otimes MH_*(f_2)$. ☐



7.3. **Morse-inequalities.** Abbreviate:

$$\begin{aligned} c_k &= \#(\text{crit pts of Morse } f \text{ of index } k) = \dim MC_k(f) \\ m_k &= \dim MH_k(f) \\ b_k &= \dim H_k(M) \text{ (Betti numbers)} \end{aligned}$$

computed using $\mathbb{Z}/2$ coefficients (although everything in this Section holds also if you use $\mathbb{Z}$ or $\mathbb{R}$ coefficients[3]).

Cor. $c_k \geq b_k$

Proof. $c_k = \dim MC_k \geq m_k = b_k$ (since $MH_k \cong H_k(M)$). ☐

Lemma. *If* $0 \longrightarrow C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C_n \longrightarrow 0$, $d_{i+1} \circ d_i = 0$, *where* C_i *are vector spaces over some field, then the Euler-characteristics agree:*

$$\chi(H_*) = \sum (-1)^i \dim H_i = \sum (-1)^i \dim C_i = \chi(C_*)$$

where H_* *is the homology of* (C_*, d).

---

[2] see May, *Concise Course in Algebraic Topology*, p.131: this is an abstract version of Künneth's theorem, and it uses the fact that we work over $\mathbb{Z}/2$.

[3] this involves orienting the moduli spaces $\mathcal{M}(p,q)$, and in the definition of ∂ you count the elements of $\mathcal{M}(p,q)$ with orientation signs ± 1. When using $\mathbb{Z}$, replace dim by rank.

Proof. $K_i = \ker d_i$, $J_{i+1} = \operatorname{im} d_i \subset C_{i+1}$,

$$C_i \cong K_i \oplus \text{complement} \xrightarrow{d_i} J_{i+1} \subset C_{i+1}$$

and the complement maps isomorphically onto J_{i+1}. So, using $H_i = K_i/J_i$

$$\begin{array}{rcl} \dim C_i & = & \dim K_i + \dim J_{i+1} \\ \dim H_i & = & \dim K_i - \dim J_i \end{array}$$

Writing c_i, h_i, k_i, j_i for the dimensions,

$$\sum_i (-1)^i c_i = \sum_i (-1)^i k_i + \sum_i (-1)^i j_{i+1} = \sum_i (-1)^i k_i - \sum_\ell (-1)^\ell j_\ell = \sum_\ell (-1)^\ell h_\ell$$

□

Thm (Morse inequalities)**.**

$$c_k - c_{k-1} + \cdots + (-1)^k c_0 \geq b_k - b_{k-1} + \cdots + (-1)^k b_0 \qquad \forall k$$

and for $k = m$ get equality: $\sum (-1)^i c_i = \sum (-1)^i m_i = \chi(M)$.

Proof.

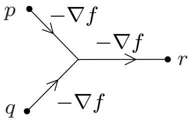
$$0 \longrightarrow MC_0 \xrightarrow{\partial_0} MC_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{k-1}} MC_k \xrightarrow{\partial_k} \operatorname{im}(\partial_k) \longrightarrow 0$$

So by the Lemma,

$$c_0 - c_1 + \cdots + (-1)^k c_k + (-1)^{k+1} \dim(\operatorname{im} \partial_k) = m_0 - m_1 + \cdots + (-1)^k m_k.$$

Now multiply by $(-1)^k$ and use $(-1)^{2k+1} \dim(\operatorname{im} \partial_k) \leq 0$. Finally use $m_i = b_i$. □

7.4. **Products.** You would like to count the following rigid configurations:



which is an abbreviation for:

$$\begin{array}{l} u_1 : (-\infty, 0] \to M,\ \partial_s u_1 = -\nabla f \\ u_2 : (-\infty, 0] \to M,\ \partial_s u_2 = -\nabla f \\ u_3 : [0, +\infty) \to M,\ \partial_s u_3 = -\nabla f \\ u_1(0) = u_2(0) = u_3(0). \end{array}$$

Note the intersection point lies in

$$W(p, q; r) = W^u(p) \cap W^u(q) \cap W^s(r). \qquad (*)$$

Problems: *Firstly, such configurations are unlikely: flowlines do not intersect unless they coincide. Secondly, we need $W(p,q;r)$ to be a manifold. To fix both these issues, one needs to perturb the function f on each edge,*[4] *so we use Morse f_1, f_2, f_3 on the 3 edges, and we pick a Morse-Smale metric g for all three edges.*[5]

So, for generic f_i, $W(p, q; r)$ is a manifold of codimension

$$\begin{array}{rcl} \operatorname{codim} W(p,q;r) & = & \sum \text{codims} \\ & = & m - |p| + m - |q| + |r| \end{array}$$

---

[4] in fact, it is enough to perturb f on only one of the first two edges, since we already have $W^u(p) \pitchfork W^s(r)$. But for more general graphs, you will have to perturb all f's.

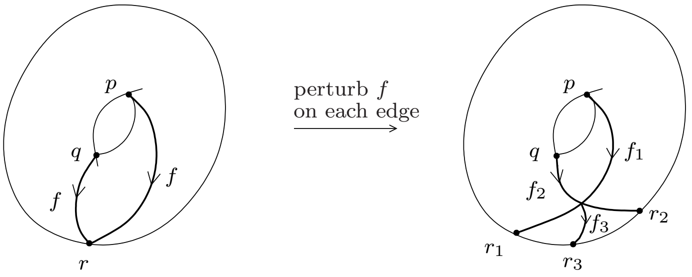
[5] a finite intersection of generic sets is generic.

we want this to be m (so we get dim = 0) so want $|r| = |p| + |q| - m$. Thus

$$MC_a \otimes MC_b \xrightarrow{\psi} MC_{a+b-m}$$
$$p \cdot q = \psi(p,q) = \sum_{\dim W(p,q;r)=0} \#W(p,q;r) \cdot r$$

on generators, and extend bilinearly.

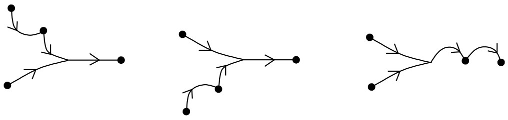
Example. *For the torus, you can check directly that $p \cdot q = r$ in the figure:*



which is Poincaré dual to the statement $H^1_{dR} \otimes H^1_{dR} \to H^2_{dR}$, $ds \wedge dt =$ (area form)*.*

Claim. *ψ is a chain map*

Proof. Breaking/gluing analysis:



$\Rightarrow \psi \circ (\partial_1 \otimes \mathrm{id} + \mathrm{id} \otimes \partial_2) = \partial_3 \circ \psi$ □

Cor.

$$\begin{tikzcd} MH_a \otimes MH_b \arrow[r, "\cdot", "intersection\ product"'] \arrow[d, "P.D."', "\cong"] & MH_{a+b-m} \arrow[d, "P.D."', "\cong"] \\ MH^{\alpha} \otimes MH^{\beta} \arrow[r, "\cup", "cup\ product"'] & MH^{\alpha+\beta} \end{tikzcd}$$

where $\alpha = m - a$, $\beta = m - b$, $\alpha + \beta = m - (a + b - m)$.

Comparison with algebraic topology:

Claim. ψ is compatible with homotopying f_1, f_2, f_3.

Consequence. WLOG f_1, f_2, f_3 are self-indexing. Then $MC_* \to C_*^{\mathrm{cell}}$, $p \mapsto W^u(p)$ is an isomorphism (see 3.10), and the cellular intersection product is:

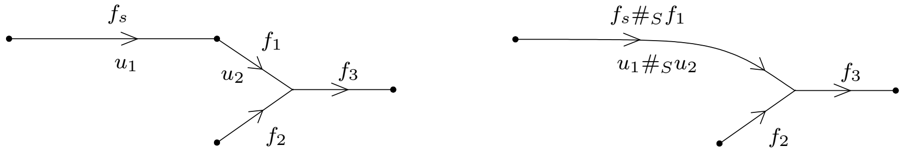
$$W^u(p) \cdot W^u(q) = \sum_r \#(W^u(p) \pitchfork \widetilde{W^u(q)} \pitchfork \widetilde{W^s(r)}) \cdot W^u(r)$$

where you homotope $W^u(q), W^s(r)$ to $\widetilde{W^u(q)}$, $\widetilde{W^s(r)}$ to ensure the intersection is transverse and that all intersections occur away from the boundaries[6] of the unstable/stable mfds. So in fact, we do not need to homotope if we choose f_1, f_2, f_3 generically. This proves you recover the classical intersection product. □

---

[6] more precisely, we mean: away from the boundary of the compactification of the image. *Non-examinable:* the notion of *pseudo-manifolds* makes these arguments very rigorous.

Sketch of Proof of the Claim. Consider $\psi \circ (\varphi \circ \mathrm{id})$, where φ is a continuation map (so we are only homotopying f_1, using a hpy f_s from f_0 to f_1). Consider the following two configurations:



On the left, are the broken solutions counted by $\psi_{f_1,f_2,f_3} \circ (\varphi \circ \mathrm{id})$ where we emphasize for which f's ψ is defined. On the right, are the glued solutions counted by $\psi_{f_s\#_S f_1,f_2,f_3}$. By a gluing argument, for a large gluing parameter S, there is a bijection between the broken solutions and the glued solutions.

Now homotopying $f_s\#_S f_1$ to f_0, produces a chain homotopy K like in 6.3 (4):

$$\psi_{f_s\#_S f_1,f_2,f_3} - \psi_{f_0,f_2,f_3} = \partial_3 \circ K + K \circ (\partial_0 \otimes \mathrm{id} + \mathrm{id} \otimes \partial_2).$$

So the maps agree on homology. So $\psi_{f_1,f_2,f_3} \circ (\varphi \circ \mathrm{id}) = \psi_{f_0,f_2,f_3}$ on homology. □

LECTURE 22 AND 23.

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7.5. **Spectral sequences.**

Spectral sequence = algebraic gadget that interlocks a bunch of exact sequences. It usually arises for a chain complex C_* with

$$d = d_0 + d_1 + d_2 + \cdots$$

where d_0 is "dominant" in some way over the higher order terms $d_1, d_2, \ldots$, and we hope to approximate $H_*(C_*, d)$ by

$$E^1 = H_*(C_*, d_0),\ E^2 = \text{“}H_*(E^1, d_1)\text{”},\ \ldots \overset{\text{cges?}}{\Rightarrow} H_*(C_*, d).$$

Def. *An* exact couple *is an exact*[1] *triangle of vector spaces of the form*

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

Given an exact couple, define

$$\boxed{d = j \circ k : E \to E}$$

Then $d^2 = jkjk = 0$ since $kj = 0$ by exactness. Thus we obtain the *derived couple*:

$$\begin{tikzcd}
& i(A) \arrow[rrrr, "i'"] & & & & i(A) \arrow[ddll, "j'"] & \\
kb & & & & & & ia \arrow[ddll, mapsto] \\
& & & H(E;d) \arrow[uull, "k'"] & & & \\
& & {[b]} \arrow[uull, mapsto] & & {[ja]} & &
\end{tikzcd}$$

Exercise. Check these maps are well-defined, and that this new triangle is exact.

Rmk. *If i = inclusion, then $k = 0$, so $d = 0$, so $E \equiv H(E, d) = A/iA$ unchanged!*

7.6. **Example: the spectral sequence for a bounded filtration.**

Suppose (C_*, d) is a $\mathbb{Z}$-graded chain complex[2] with a filtration by subcomplexes[3]

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_n = C_*$$

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[1] recall *exact* means the kernel of one arrow equals the image of the previous arrow.

[2] $d : C_k \to C_{k-1}$, $d^2 = 0$.

[3] $dF_p \subset F_p$.

$\Rightarrow 0 \to F_{p-1} \xrightarrow{i} F_p \xrightarrow{j} F_p/F_{p-1} \to 0$ exact
$\Rightarrow$ define $F_p = 0$ for $p < 0$, $F_p = C_*$ for $p \geq n$. Then define

$$E^0_{p,*-p} = \bigoplus_p (F_p/F_{p-1})_{\text{the part in } \mathbb{Z}\text{-grading } *}$$

Then the LES asssociated to the above SES:[4]

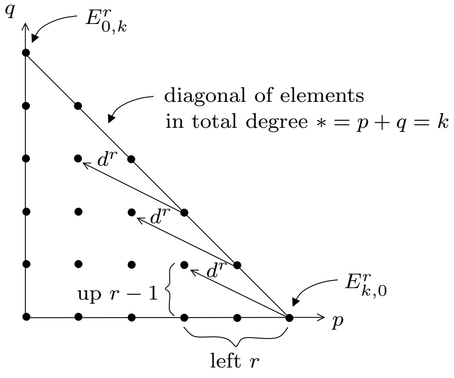
$$\begin{tikzcd}
A^1 = \bigoplus_p H_*(F_{p-1}) \arrow[rr, "i^1"] & & \bigoplus_p H_*(F_p) = A^1 \arrow[dl, "j^1"] \\
& H_*(\bigoplus_p E^0_{p,*-p}) = \bigoplus_p E^1_{p,*-p} \arrow[ul, "k^1", "/" marking] &
\end{tikzcd}$$

The dash on the arrow indicates that the grading $*$ drops by 1. Abbreviate $q = * - p$ ($* = p + q$ is called the *total degree*). Deriving the couple, we obtain:

$$d^1 = k^1 \circ j^1 : E^1_{p,q} \to E^1_{p-1,(*-1)-(p-1)} = E^1_{p-1,q}.$$

$$\begin{tikzcd}
A^2 \arrow[rr, "i^2"] & & A^2 \arrow[dl, "j^2"] \\
& H(E^1, d^1) = E^2 \arrow[ul, "k^2", "/" marking] &
\end{tikzcd}$$

and keep deriving. So obtain $\boxed{E^r,\ d^r = j^r \circ k^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}$



$$A^1 = \text{sum up } (0 \to H(F_0) \xrightarrow{i} H(F_1) \xrightarrow{i} \cdots \xrightarrow{i} H(F_n) \xrightarrow{=} H \xrightarrow{=} \cdots)$$

where $H = H(C_*, d) = H(F_p)$ for $p \geq n$. The image under i^n becomes:

$$\begin{aligned} A^{n+1} &= \text{sum up } (0 \to i^n H(F_0) \subset i^n H(F_1) \subset \cdots \subset i^n H(F_n) = H = \cdots) \\ &= \text{sum up } (G_0 \subset G_1 \subset \cdots \subset G_n = H = \cdots) \text{ where } G_p = \operatorname{im}(H(F_p) \xrightarrow{i^n} H). \end{aligned}$$

$\overset{\text{Rmk}}{\Rightarrow} E^r_{p,*-p} = \oplus (G_p/G_{p-1})_{\text{the part in } \mathbb{Z}\text{-grading } *} = E^\infty_{p,*-p}$ constant for $r \gg 0$.

But now $H \cong G_0 \oplus (G_1/G_0) \oplus (G_2/G_1) \oplus \cdots \oplus (G_n/G_{n-1})$, so rewriting:

$$\Rightarrow \boxed{H_*(C_*, d) \cong \bigoplus_p E^\infty_{p,*-p}}$$

---

[4] recall that every short exact sequence gives rise to a long exact sequence. E.g. see Hatcher's *Algebraic Topology.*

One abbreviates this result by writing

$$\boxed{E^1_{p,q} \Rightarrow H_*(C_*, d)}$$

(read "⇒" as "converges to") and one says the *spectral sequence* $E^1_{p,q}$ *converges.*[5]

Warning: the last two isomorphisms are not canonical, because you are recovering the group from certain successive quotients.

7.7. **Leray-Serre spectral sequence.**

Thm.

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

Let E be a fibre bundle with simply connected base B, and with fibre F, where B, E, F are closed mfds. Then there is a spectral sequence

$$E^2_{p,q} = MH_p(B) \otimes MH_q(F) \Rightarrow MH_*(E).$$

Example. *Künneth's theorem: $E = B \times F$, then $E^2_{p,q} = E^\infty_{p,q}$.*

Proof. Fix Morse-Smale data:

$$\begin{array}{ll} (b : B \to \mathbb{R}, g_B) & (f : F \to \mathbb{R}, g_F) \\ \operatorname{Crit} b = \{b_1, b_2, \ldots, b_n\} & \operatorname{Crit} f = \{y_1, \ldots, y_m\}. \end{array}$$

Pick disjoint opens B_i around $b_i \in B$ with trivializations

$$\begin{tikzcd} E|_{B_i} \arrow[r, "\cong"] \arrow[d, "f_i"] & B_i \times F \arrow[d, "f"] \\ \mathbb{R} \arrow[r, equal] & \mathbb{R} \end{tikzcd}$$

Fix bump functions $\rho_i : B \to [0,1]$, $\rho_i = \begin{cases} 0 & \text{outside } B_i \\ 1 & \text{near } b_i \end{cases}$

Then we obtain a function on E:

$$h = b + \varepsilon \sum \rho_i f_i : E \to \mathbb{R}$$

where we abusively write b but mean $b \circ \pi : E \to B \to \mathbb{R}$.

Claim. *h is Morse for $0 < \varepsilon \ll 1$.*

Proof. $h = b \oplus \varepsilon f$ on $(\rho_i = 1) \subset B_i \times F$ is Morse (compare Künneth proof) ✓

Outside $\cup_i(\rho_i = 1)$: $|db| > \delta > 0$, so for $\varepsilon \ll \delta$ get $|db| > \frac{1}{2}\delta > 0$ (ρ_i, f_i are C^1-bdd since F compact). ☐

This also proves that

$$\operatorname{Crit} h = \operatorname{Crit} b \times \operatorname{Crit} f \quad \text{(in the trivializations).} \qquad (*)$$

Now want to build a metric on E such that in the above trivializations we are in the Künneth setup:

$$\boxed{\begin{array}{rcl} g_E & = & g_B \oplus g_F \text{ on } B_i \times F \\ \Rightarrow \nabla h & = & \nabla b \oplus \varepsilon \nabla f_i \text{ on } \rho_i = 1. \end{array}}$$

Problem: *∇h is useless: outside $\rho_i = 1$ you get $\nabla \rho_i$ terms and also you will need to perturb g_E to get transversality*

---

[5] In our case, one also says the spectral sequence *degenerates* at sheet $n+1$ because $d^r = 0$ for $r \geq n+1$, so we may identify $E^{n+1} = E^{n+2} = \cdots$.

$\Rightarrow$ *you have no idea what* $d\pi(\nabla h)$ *is.*
$\Rightarrow$ *no idea what* $\pi \circ (-\nabla h$ *trajectory) is.*

Trick: we will construct a gradient-like vector field v for h such that

$$\boxed{\begin{cases} ① \;\; d\pi \circ v = \nabla b \\ ② \;\; v = \nabla h = \nabla b \oplus \varepsilon \nabla f_i \text{ on } \rho_i = 1 \end{cases}}$$

Hence, for $e \neq e' \in \operatorname{Crit} h$ define:

$$\boxed{V(e, e') = \{-v \text{ flowlines converging to } e, e'\}/\mathbb{R}}.$$

Because of ①, $V(e,e')$ projects via π to the moduli spaces $\mathcal{M}(b_i, b_j)$ for $b : B \to \mathbb{R}$, where $b_i = \pi(e)$, $b_j = \pi(e')$. Like for the $\mathcal{M}$ spaces,[6]

$$\boxed{\dim V(e,e') = |e| - |e'| - 1}$$

calculating the indices for h, since near the ends $-v = -\nabla h$ by ②.

Modifying g_E: Define the vertical and horizontal subspaces of TE by

$$\begin{aligned} V &= \ker d\pi, \\ H &= V^{\perp} \qquad \text{(perpendicular for } g_E) \end{aligned}$$

So in particular $V = TF$ and $H = TB_i$ over $B_i \times F \cong E|_{B_i}$. Define

$$\widetilde{g}_E = \begin{cases} g_E & \text{on } V \\ \pi^* g_B & \text{on } H \end{cases} \quad \text{and } V \perp H \text{ for } \widetilde{g}_E$$

$$v = \widetilde{\nabla} b + \varepsilon \textstyle\sum \rho_i \widetilde{\nabla} f_i \qquad (\widetilde{\nabla} = \text{ gradient for } \widetilde{g}_E)$$

Note that $\widetilde{g}_E = g_E$ on $\cup_i(\rho_i = 1)$.

Proof of ① *and* ②*:* ② is immediate.

$$\begin{aligned} db &= \widetilde{g}_E(\widetilde{\nabla} b, \bullet) \\ &= \widetilde{g}_E(\widetilde{\nabla} b, \text{project}_H \bullet) && \text{since } db = 0 \text{ on } V \\ &= (\pi^* g_B)(\text{project}_H \widetilde{\nabla} b, \text{project}_H \bullet) && \text{since } V \perp H \text{ for } \widetilde{g}_E \\ &= g_B(d\pi \widetilde{\nabla} b, d\pi \bullet) \\ \Rightarrow \quad & d\pi \cdot \widetilde{\nabla} b = \nabla b \\ \Rightarrow \quad & d\pi \cdot v = \nabla b && \text{since}^{7}\ d\pi \widetilde{\nabla} f_i = 0 \\ \Rightarrow \quad & ① \checkmark \end{aligned}$$

Proof v is gradient-like: $dh(v) = |\widetilde{\nabla} b|^2 - \text{order}(\varepsilon) > 0$ outside $\rho_i = 1$, and on $\rho_i = 1$ v is the gradient of h by ② ✓.

We now construct a Morse-like complex for $-v$. Because of $(*)$, we define

$$C_* = MC_*(h) = MC_*(b) \otimes MC_*(f),$$

---

[6] indeed, the same proof holds: our index calculation shows that only the asymptotics of the linearization of the flow matter, and at the ends the flow is a Morse flow: $-v = -\nabla h$ by ②.

[7] $\widetilde{g}_E(\widetilde{\nabla} f_i, H) = df_i(H) = df(TB_i) = 0$ over B_i, and outside B_i have $\rho_i = 0$.

with differential

$$\begin{aligned} de &= \sum_{\dim V(e,e')=0,\ e\neq e'} \#V(e,e')\cdot e' \\ &= (d_0+d_1+d_2+\cdots)\,e \\ d_p e &= \sum_{|\pi e|-|\pi e'|=p,\ \dim V(e,e')=0} \#V(e,e')\cdot e'. \end{aligned}$$

Define the filtration:

$$F_p = \bigoplus_{|\pi e|\leq p,\ e\in \mathrm{Crit}\,h} \mathbb{Z}/2\cdot e.$$

Observe: $F_{\text{negative}}=0$, $F_{\dim B}=C_*$, $F_{p-1}\subset F_p$.

Crucial claim. F_p is a subcomplex: $dF_p\subset F_p$.
Proof. if $\exists\ -v$ traj u, then $\pi\circ u$ is a $-\nabla b$ traj.
$\Rightarrow |\pi e|-|\pi e'| = \dim W(\pi e,\pi e')\geq 0$
$\Rightarrow |\pi e'|\leq|\pi e|\leq p$ ✓

$\Rightarrow \boxed{E^0_{p,*-p}=F_p/F_{p-1}=MC_p(b)\otimes MC_{*-p}(f)}$ with $d=d^0$ on E^0.

Claim. $d_0=\partial_{\text{fibre}}$ counts $-\nabla f$ trajectories in the fibres.
Pf. $|\pi e|=|\pi e'|\Rightarrow W(\pi e,\pi e')=\emptyset$ unless $\pi e=\pi e'$, in which case $\pi\circ u$ =constant.✓

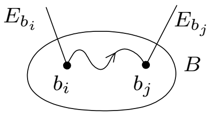
$\Rightarrow \boxed{E^1_{p,q}\cong MC_p(b)\otimes MH_q(f)}$

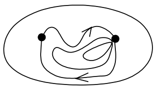
Warning: *this isomorphism is not canonical, because we made choices of trivializations. So let us be more precise:*

$$\begin{aligned} E^1_{p,q} &= \bigoplus_{b_i\in\mathrm{Crit}\,b} R_q(b_i) \\ R_q(b_i) &= MH_q(E_{b_i}, h|_{E_{b_i}} = b(b_i)+\varepsilon f_i|_{E_{b_i}}) \\ &\equiv MH_q(E_{b_i}, f_i|_{E_{b_i}}) \end{aligned}$$

and non-canonically $R_q(b_i)\cong MH_q(F,f)$ by using the choice of trivializations.

Using B simply connected. For simply connected B you can identify fibres by following a path in B, and if you change the path then you get a homotopic identification: hence the homology does not notice the change. Pictorial idea:[8]





Write $F_p=C^p\oplus F_{p-1}$, where C^p is generated by the e with $|\pi e|=p$. Then

$$F_p = C^p\oplus F_{p-1} \xrightarrow{d=\left[\begin{smallmatrix} d_0 & 0 \\ \partial' & \partial'' \end{smallmatrix}\right]} C^p\oplus F_{p-1}$$

---

[8] on the right, compare parallel transports P_1,P_2 along two paths: homotope to a constant the loop that concatenates the two paths; obtain a chain homotopy between $P_2^{-1}\circ P_1$ and id.

where ∂', ∂'' are $d_1 + d_2 + \cdots$ composed with projection to C^p, F_{p-1} respectively.

$$\begin{array}{rcl} F_{p-1} = C^{p-1} \oplus F_{p-2} & \xrightarrow{i} & F_p = C^p \oplus F_{p-1} \\ (a,b) & \mapsto & (0, a+b) \end{array}$$

$$\begin{tikzcd}
H_*(F_{p-1}) \arrow[rr, "i^1"] & & H_*(F_p) \arrow[dl, "j^1"] \\
& H_*(F_p/F_{p-1}) = E^1_{p,*-p} \arrow[ul, "k^1", "/" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
{[(a,b)]} \arrow[r, "i^1"] & {[(0,a+b)]} & {[(\alpha,\beta)]} \arrow[dll, "j^1"] \\
{[\alpha]} & &
\end{tikzcd}$$

Recall k^1 is the boundary of the LES, so study the SES's:

$$\begin{tikzcd}
0 \arrow[r] & (F_{p-1})_* \arrow[r] \arrow[d, "d"] & (F_p)_* \arrow[r] \arrow[d, "d"] & (F_p/F_{p-1})_* \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & (F_{p-1})_{*-1} \arrow[r] & (F_p)_{*-1} \arrow[r] & (F_p/F_{p-1})_{*-1} \arrow[r] & 0
\end{tikzcd}$$

and diagram chase what happens to α:

$$\begin{tikzcd}
& & (\alpha, 0) \arrow[r] \arrow[d, "d"] & \alpha \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & (\partial'\alpha, 0) \arrow[r] & d(\alpha,0) = (0, \partial'\alpha) \arrow[r] & d_0\alpha = 0 \arrow[r] & 0
\end{tikzcd}$$

so, by definition of the boundary k^1, in the triangle above we get

$$\begin{tikzcd}
{[(\partial'\alpha, 0)]} & \\
& {[\alpha]} \arrow[ul, "k^1"', "/" marking]
\end{tikzcd}$$

So $d^1[\alpha] = j^1 k^1 [\alpha] = [\partial'\alpha] = [d_1\alpha] \in H_{*-1}(F_{p-1}/F_{p-2})$ (here d_1 and ∂' agree since we quotient by F_{p-2}). Thus

$$\boxed{d^1[\alpha] = [d_1\alpha]}$$

We need to understand d_1:

$$d_1(b_i \otimes y) = \sum_{|b_j| = p-1, \text{ any } y' \in \operatorname{Crit} f} \# V_0(b_i \otimes y, b_j \otimes y') \cdot b_j \otimes y'$$

counts the 0-dimensional V spaces, and recall each $u \in V_0(b_i \otimes y, b_j \otimes y')$ lies over the $-\nabla b$ trajectory $\pi \circ u$ from b_i to b_j.

Note $d_1 : R(b_i) \to R(b_j)$ is a chain map (with respect to d_0) since:[9]

$$0 = d^2 = d_0^2 + (d_1 d_0 + d_0 d_1) + \cdots$$

and $d_0^2 = \partial_{\text{fibre}}^2 = 0$, hence $d_1 d_0 + d_0 d_1 = 0$. *(Exercise: you can also prove this last equality by a breaking analysis like in 7.4)*

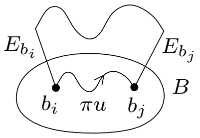
Idea. d_1 counts trajectories which have an index drop in the base, so we expect the trajectories to be "constant" fibrewise. This makes sense if E is trivial (that is how we proved the Künneth thm), but otherwise the notion of "locally constant" depends on choices of local trivializations. So we need to keep track of choices.

---

[9] we break d^2 up according to the filtration, so each summand must vanish.

Claim. *Along a trivialization over the path $\pi \circ u$ the solutions u are continuation solutions of a Morse flow, and the count d_1 of such rigid solutions defines the same identification between $R(b_i)$ and $R(b_j)$ as the one induced on ordinary homology by parallel translation along any path joining b_i, b_j.*

Proof.



Pick a trivialization $\mathbb{R} \times F$ agreeing[10] with the given ones at b_i, b_j, so

$$h = \begin{cases} \varepsilon f + \text{constant} & \text{at } -\infty \\ \varepsilon f + \text{constant} & \text{at } +\infty \end{cases}$$

The count of isolated $-v$ flowlines in this trivialization (which project to the ∂_s flow in $\mathbb{R}$) then defines a map similar to a continuation map. Indeed, by covering the path πu by small charts, and extending the trivialization to these charts, we can homotope the metric $\widetilde{g}_E$ to make it a direct sum metric $g_B \oplus g_F$ (which it already is at the ends of the path πu), so that v is the gradient of a homotopy $h_s = b + \varepsilon f_s$, where f_s at the ends equals f. But now this homotopy can be homotoped to $h_s = b + \varepsilon f + c(s)$, where $c(s)$ only depends on s and at the ends equals the constants in the above expression for h at $\pm\infty$.

Observe that $\nabla(b + \varepsilon f + c(s)) = \nabla(b + \varepsilon f)$, so just as in the case of a constant hpy (6.3 (2)), one proves that $b + \varepsilon f + c(s)$ induces the identity continuation map. Hence, our original count of flowlines is chain homotopic to the identity. Hence on Morse homology it equals the identity. Thus the map agrees with the parallel transport map which defined the various trivializations (see footnote 10). ✓

Conclusion: Recover d_1 by finding the isolated $-\nabla b$ trajectories on B, and doing parallel transport in the fibres to get the map between the $R(b_i)$'s. More precisely:

$$\boxed{d_1 \text{ on } E^1 = \bigoplus_{b_i} R(b_i) \text{ can be identified with } \partial_{\text{base}} \text{ on } MC_*(b) \otimes MH_*(f)}$$

$\Rightarrow E^2_{p,q} = MH_p(b) \otimes MH_q(f)$

$\Rightarrow \boxed{E^r_{p,q} \Rightarrow H_*(C_*, d)}$

Finally, the last step of the proof of the Leray-Serre theorem, is:

$$H_*(C_*, d) \cong MH_*(h)$$

This is proved by a parametrized moduli space argument like in 6.3 (4): you homotope $-v$ to $-\widetilde{\nabla} h$. Note that $-v = -\widetilde{\nabla} h$ except in the regions where the $\rho_i \neq 0, 1$ which are small subsets of $B_i \setminus (\rho_i = 1)$. □

---

[10] Need to choose trivializations carefully: pick a trivialization $B_i \times F$ for some b_i, then parallel transport this over chosen paths to define the trivializations for the other $B_j \times F$'s. Finally we use the fact $\pi_1 B = 0$ to obtain the required trivialization with prescribed ends.

LECTURE 24.

PART III, MORSE HOMOLOGY, 2011

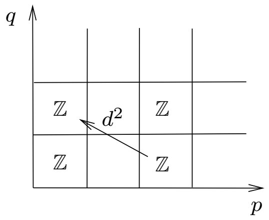
HTTP://MORSEHOMOLOGY.WIKISPACES.COM

7.8. **Example of applying the Leray-Serre theorem.**

Let $E = \{v \in TS^2 : |v| = 1\}$ (the sphere bundle of TS^2).[1]

$$\begin{tikzcd} S^1 \arrow[r] & E \arrow[d, "\pi"] \\ & S^2 \end{tikzcd} \qquad E^2_{p,q} = MH_p(S^2) \otimes MH_q(S^1).$$

Represent E^2 graphically as:



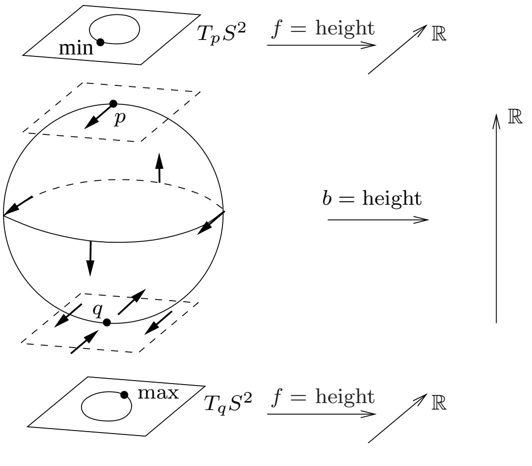
So we can already deduce:[a]

$$\begin{aligned} H^0(E) &= \mathbb{Z} \\ H^1(E) &= \mathbb{Z}/\operatorname{im} d^2 \\ H^2(E) &= \ker d^2 \\ H^3(E) &= \mathbb{Z} \end{aligned}$$

---

[a] here please take on trust that one can do Morse homology over $\mathbb{Z}$ by keeping track of orientation signs.

We take b = height function on S^2, and f = height function on the fibre S^1. To find d^2, we need to understand how parallel transport relates the critical points of index $1, 2$. Consider how a vector at the North pole p of S^2 gets parallel transported to the South pole q when moving along four great half-circles meeting at 90° at p.



We see that two[2] of the parallel transports of the vector point in the direction of the maximum in the S^1 fibre over q. Indeed, this shows that there are exactly two great half-circles from p to q such that the minimum in the fibre over p gets parallel

---

Date: May 3, 2011, © Alexander F. Ritter, Trinity College, Cambridge University.

[1] Secretly, one knows that $E \cong SO(3) \cong \mathbb{R}P^3$.

[2] secretly, this "two" is the Euler characteristic of the base S^2.

transported to the maximum in the fibre over q.
$\Rightarrow d^2 =$ multiplication by 2

$$\begin{array}{rcccccccc} \Rightarrow H_*(B) & = & \mathbb{Z} & \oplus & \mathbb{Z}/2 & \oplus & 0 & \oplus & \mathbb{Z} \\ * & = & 0 & & 1 & & 2 & & 3 \end{array}$$

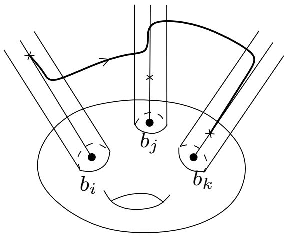
Rmk. *One can similarly do this for the higher dimensional case* $S^{n-1} \to S(TS^n) \to S^n$*. More generally, this method should in principle yield the Gysin sequence.*

8. Morse-Bott theory

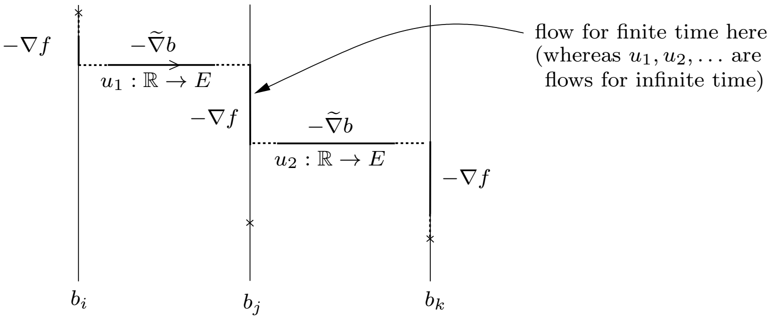
8.1. **Motivation.** *Question.* In the construction of the Leray-Serre spectral sequence, what happens if we let:

$$\begin{array}{l} \text{supports of } \rho_i \text{ shrink to } b_i \\ \text{and } \varepsilon \to 0. \end{array}$$

Answer. the trajectories become more vertical near the critical fibres, and more "horizontal" away from them:



So the trajectories converge to a combination of $-\nabla f$ flows along fibres and "quantum jumps" between the fibres given by $-\widetilde{\nabla} b$ flows:



8.2. **Morse-Bott functions.**

A smooth function $b : M \to \mathbb{R}$ is called *Morse-Bott* if

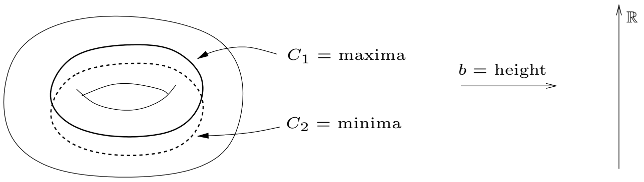
1. $C = \operatorname{Crit} b = \bigsqcup_i C_i$ is a finite disjoint union of connected submfds $C_i \subset M$
2. $\operatorname{Hess}_p b = D_p(db) : T_pM \to T_p^*M$ nondegenerate transversely to C_i, meaning
$$T_pC_i = \ker \operatorname{Hess}_p b \quad \forall p \in C_i.$$
Equivalently: $\operatorname{Hess}_p b$ induces an invertible self-adjoint map on normal bdles
$$\operatorname{Hess}_p b : \nu_{C_i} \to \nu_{C_i}.$$

Examples.

1. $b =$ *Morse,* $C = \{$*critical points*$\}$
2. $b = 0$, $C = M$.

(3) *$b(x,y,z) = -x^2 + y^2$ and $C = z$-axis inside $\mathbb{R}^3$, but not $b = -x^3 + y^2$.*

(4) *a torus lying flat with the height function:*



(5) *Fibre bundle*

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] & \\ & B \arrow[r, "b"] & \mathbb{R} \end{tikzcd}$$

Suppose $b : B \to \mathbb{R}$ is Morse. Then $b \circ \pi : E \to \mathbb{R}$ is Morse-Bott with $C_i = \pi^{-1}(b_i)$ the fibres over the critical points b_i of b.

8.3. **Morse-Bott chain complex.** Choose auxiliary Morse functions

$$\boxed{f = \sqcup f_i : C = \sqcup C_i \to \mathbb{R}}$$

and a generic metric $g_C = \sqcup g_{C_i}$ on C. Write ∇f for the gradient of f w.r.t. g_C.

Def. *Define the grading of $p \in \mathrm{Crit}(f_i) \subset C_i$ by:*

$$\boxed{|p| = \mathrm{ind}_b(p) + \mathrm{ind}_f(p) = \mathrm{ind}\, C_i + \mathrm{ind}_f(p)}$$

Note that $\mathrm{ind}_b(p)$ is independent of $p \in C_i$ and is the index of $\mathrm{Hess}_p b : \nu_{C_i} \to \nu_{C_i}$.

Key Idea: *you are pretending that you perturbed b to $b + \varepsilon \sum \rho_i f_i$.*

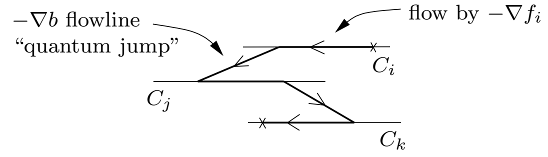
Example. *In example (3): $\mathrm{ind}_b(C) = 1$ because of the $-x^2$. For $f : C \to \mathbb{R}$, $(0,0,z) \mapsto -z^2$ get $|(0,0,0)| = -2$, which equals the index for $-x^2 + y^2 - \varepsilon z^2$.*

Def. *Define the* Morse-Bott complex *by*

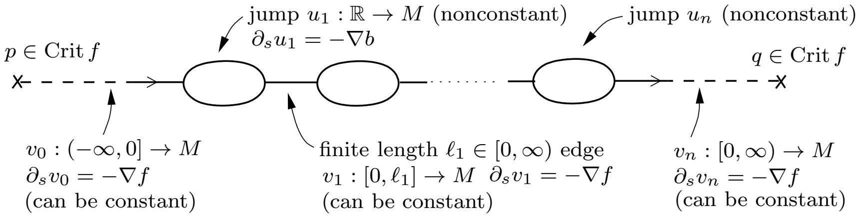
$$\boxed{\begin{aligned} BC_* &= \bigoplus_{p \in Crit\, f} \mathbb{Z}/2 \cdot p \\ &= \bigoplus_i MC_*(f_i)\,[\mathrm{ind}\, C_i] \end{aligned}}$$

The $[\mathrm{ind}\, C_i]$ is a shift in grading: $|p\,[\mathrm{ind}\, C_i]| = \mathrm{ind}_{f_i}(p) + \mathrm{ind}\, C_i$, so grading 0 becomes grading $\mathrm{ind}\, C_i$.

8.4. **Morse-Bott differential.** ∂ counts rigid Bott trajectories:



Useful Notation to summarize a Bott flowline:



Def. *The moduli space of Bott flowlines with* $n \geq 1$ *jumps is:*

$$W^n(p,q) = \{(u_1, \ldots, u_n; \ell_1, \ldots, \ell_{n-1}) : u_j \in W(p_j, q_j; b), p_j \neq q_j \in C \text{ such that } p_1 \in W^u(p, f), q_n \in W^s(q, f), \text{and } q_j, p_{i+1} \text{ are connected by a finite time } -\nabla f \text{ flowline } v_j : [0, \ell_j] \to C, \ell_j \in [0, \infty)\}$$

$W^0(p,q) = W(p,q;f)$ *the moduli space of* $-\nabla f$ *flowlines* $\mathbb{R} \to C$.

Def. *The moduli space of Bott trajectories is*

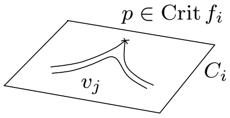
$$B(p,q) = \bigcup_{n \in \mathbb{N}} W^n(p,q)/\mathbb{R}^n,$$

where $\mathbb{R}^n$ *acts by shifting the* s *coordinates in* $u_1, \ldots, u_n$.

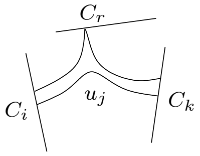
8.5. **Breaking of Bott trajectories.**

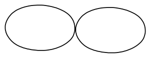
Under C^0_{loc}-convergence, a Bott flowline can break in two ways:

(1) some $\ell_j \to \infty$ and therefore v_j breaks on some critical level set C_i:



(2) or some u_j breaks:





where on the right we indicated the abbreviated notation for the breaking.

The broken Bott flowlines in (2) for $W^n(p,q)$ are precisely the boundary points of $W^{n+1}(p,q)$ arising when $\ell_j = 0$. So when we compactify $B(p,q)$ we do not need to artificially add these limit broken Bott trajectories since they are already present. However, we still need to enlarge the topology so that it is recognized as a limit in the sense of (2). So we just artificially add the breakings of type (1):

$$\overline{B}(p,q) = B(p,q) \sqcup \bigsqcup_{n \geq 2} B(p,p_2) \times B(p_2,p_3) \times \cdots B(p_n,q).$$

$\partial\overline{B}(p,q) = \sqcup B(p,p_2) \times B(p_2,p_3) \times \cdots B(p_n,q)$ are called the *broken Bott trajectories.*

8.6. **Energy estimates for Bott trajectories.**

Define the energy:

$$E(v_0, u_1, v_1, \ldots, u_n, v_n) = \sum \text{energies} = E(v_0) + E(u_1) + E(v_1) + \cdots + E(u_n) + E(v_n).$$

Lemma.

(1) *b decreases along a Bott trajectory*

(2) *$\exists \delta > 0$ such that to go from one C_i to another C_j a Bott trajectory must consume energy $\geq \delta > 0$.*
(3) *There are at most $(f(p) - f(q))/\delta$ jumps, so $W^n(p,q) = \emptyset$ for large n.*

Proof. b is constant along the v_i, and b decreases along u_i. (2) is proved like 3.3: $|\nabla b| > \delta > 0$ outside small nbhds of the C_i's, etc. and (3) follows from (2). □

For generic metrics g_M on M, g_C on C, one can prove the corresponding transversality, compactness and gluing results for $B(p,q)$ like we did for $\mathcal{M}(p,q)$, thus:

$$\boxed{\begin{array}{l} B(p,q) \text{ smooth mfd} \\ \dim B(p,q) = |p| - |q| - 1 \\ \overline{B}(p,q) \text{ is a compact mfd with corners} \end{array}}$$

8.7. **Morse-Bott homology.** Recall

$$BC_* = \bigoplus_i MC_*(f_i)[\operatorname{ind} C_i].$$

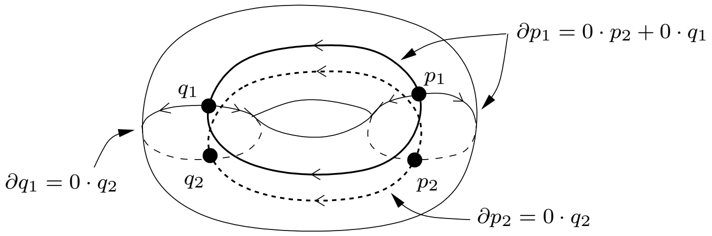
Define

$$\begin{aligned} &\partial : BC_* \to BC_{*-1} \\ &\partial p = \sum_{\dim B(p,q)=0,\ p\neq q} \#B(p,q) \cdot q. \end{aligned}$$

The proof of $\partial^2 = 0$ follows just like for Morse homology from the results in 8.6. Hence we obain the Morse-Bott homology:

$$BH_*(b,f) = \frac{\ker \partial}{\operatorname{im} f}$$

Example. *In example (4) above, using height functions on the circles C_1, C_2:*



This shows $\partial = 0$, so over $\mathbb{Z}/2$:

$$\begin{array}{rcccccccc} BH_* = BC_* & = & \mathbb{Z}/2 & \oplus & (\mathbb{Z}/2)^2 & \oplus & \mathbb{Z}/2 & \cong & H_*(\mathit{torus}) \\ * & = & 0 & & 1 & & 2 & & \end{array}$$

8.8. **Invariance of Morse-Bott homology.** At this stage, one has to redo some of the work done for Morse homology:

(1) Build continuation maps,[3] when homotopying b, f_i, g_M, g_C.
(2) Prove invariance using continuation map properties.
(3) Invariance (2) implies[4] $BH_*(b,f) \cong BH_*(\text{Morse function } F, 0) \cong MH_*(F) \cong H_*(M)$, thus:

$$\boxed{BH_*(b,f) \cong H_*(M)}$$

---

[3] these are a little tricky, since homotopying b can change C drastically.

[4] Also, $BH_*(b,f) \cong BH_*(0, \text{Morse function } F \text{ on } M)$ works.

8.9. **Filtration by action.** For $a \in \mathbb{R}$,

$$F_a = F_a(BC_*) = \bigoplus_{y \in \mathrm{Crit}(f),\ f(y) \leq a} \mathbb{Z}/2 \cdot y$$

Key observations: $dF_a \subset F_a$, $F_a = 0$ for $a \ll 0$, $F_a = BC_*$ for $a \gg 0$.

Now make it a discrete filtration, using the $\delta > 0$ of the energy estimates:

$$\cdots \subset F_0 \subset F_\delta \subset F_{2\delta} \subset \cdots \subset F_{p \cdot \delta} \subset \cdots$$

$\Rightarrow E^0_{p,*-p} = F_{(p+1)\delta}/F_{p\delta}$, and $d = d_0$ on E^0 since if you make a quantum jump, then you fall inside $F_{p\delta}$, so d only counts $-\nabla f$ trajectories in C.
$\Rightarrow E^1_{p,*-p} = \bigoplus_i MH_*(f_i)[\mathrm{ind}\, C_i]$
So we deduce:

Thm. *There exists a spectral sequence* $\boxed{E^1 = \bigoplus_i MH_*(f_i)[\mathrm{ind}\, C_i] \Rightarrow BH_*(b, f)}$

So there is also a spectral sequence $\boxed{\bigoplus_i H_*(C_i)[\mathrm{ind}\, C_i] \Rightarrow H_*(M)}$

Cor. *The Euler-characteristic* $\chi(M) = \sum (-1)^{\mathrm{ind}\, C_i} \chi(C_i)$.

Proof. This follows from the Theorem and from Lemma 7.3:

$$\chi(M) = \chi(H_*(M)) = \chi(E^\infty) = \chi(E^1) = \chi\left(\oplus_i H_*(C_i)[\mathrm{ind}(C_i)]\right) = \sum (-1)^{\mathrm{ind}(C_i)} \chi(C_i).$$

□

8.10. **Filtration by the index of b.** Make the following

Assumption. $B(p, q) = 0$ if $\mathrm{ind}_b(p) < \mathrm{ind}_b(q)$.

For example, this holds in example (5) above. Define

$$F_p = \bigoplus_{\mathrm{ind}_b C_i \leq p} \ \bigoplus_{y \in \mathrm{Crit}(f_i)} \mathbb{Z}/2 \cdot y.$$

Then $d = d_0 + d_1 + d_2 + \cdots$, where

d_0 = counts $-\nabla f$ flowlines in C
d_1 = allow one quantum jump
d_2 = allow two quantum jumps
$\cdots$

Thm. *There exists a spectral sequence of the same form as above.*

Example. *In example (5), we obtain the Leray-Serre spectral sequence. Use* $b : B \to \mathbb{R}$ *Morse on the base, and* $f_i : E_{b_i} \to \mathbb{R}$ *Morse on the fibres over* $b_i \in \mathrm{Crit}(b)$.

$$F_p = \bigoplus_{|b_i| \leq p,\ y \in Crit\, f} \mathbb{Z}/2 \cdot y$$
$$E^1 = \bigoplus_{b_i \in \mathrm{Crit}(b)} MH_*(f_i)\,[\,|b_i|\,]$$

For $\pi_1(B) = 0$, *get* $d^1 = \partial_{\mathrm{base}}$. *So*

$$E^2_{p,q} = MH_p(B) \otimes MH_q(F) \Rightarrow H_*(E).$$

Note this has the enormous advantage that we do not have to construct a special metric and a pseudo-gradient vector field.

9. Where to go from here

I recommend three interesting survey papers:

- Michael Hutchings, *Lecture Notes on Morse homology.*

  This is available online. It is a very elegant treatment of many interesting topics. It covers parts of this course, but sometimes using a different approach (some proofs in Morse homology can be simplified if one uses smooth dependence of ODE's on initial conditions, but unfortunately these proofs do not generalize to Floer theory so we avoided this approach).

- Dietmar Salamon, *Lectures on Floer homology.*

  This is available online. It is the best place to learn the basics of Floer homology. Always short and to the point, which is wonderful.

- Kenji Fukaya, *Morse homotopy,* A^∞*-category and Floer homologies.*

  Available online (the fonts are a little strange). This is excellent to get a feel for the ideas involved in Floer theory.

For research directions on more advanced topics, I recommend three books:

- Dusa McDuff and Dietmar Salamon, *J-Holomorphic Curves and Quantum Cohomology*, 1994 (not the similarly called 2004 version).

  This is a great book and is very readable.

- Paul Seidel, *Fukaya categories and Picard-Lefschetz theory.*

  This is a very advanced book. It is the key reference for A^∞-algebras, Lagrangian Floer homology, Lefschetz fibrations, Fukaya categories. This is useful if you become a mathematician in the area of symplectic topology.

- Peter Kronheimer and Tomasz Mrowka, *Monopoles and Three-Manifolds.*

  This is a very detailed treatment of Seiberg-Witten Floer homology. It always motivates ideas using Morse homology, which is a great approach.