LECTURE 3.

PART III, MORSE HOMOLOGY, 2011

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1.3. Sard's theorem.

Fact.[1] *For smooth $f : M \to N$,*

$$\boxed{\text{Almost every point of } N \text{ is a regular value of } f}$$

This means: $\{\text{critical values}\} = f(\{\text{critical points}\})$ is a set of *measure zero*[2] in N. Equivalently: $\{\text{regular values}\} \subset N$ has *full measure*, so these points are "generic".

Cor. *$\{$regular values$\} \subset N$ is dense.*

Pf. Non-empty open sets in $\mathbb{R}^n$ have measure > 0. □

Rmk. *M, N need not be compact. The result only uses that M is second countable.*[3]

Fact. *For C^k-maps*[4] *$f : M^m \to N^n$, the above fact holds provided $k > m - n$. (Here M, N need not be smooth, just need C^k-mfds: the transition maps are C^k.)*

Examples.

(1) $f : \mathbb{R}^m \to \mathbb{R}$, $x \mapsto \sum x_i^2 - 1$
0 regular value, so $f^{-1}(0) = S^{m-1}$ mfd of dim $= m - 1$.

(2) f : *Matrices$_{n\times n}$ $\to$ Symmetric Matrices$_{n\times n}$*, $A \mapsto A^T A$
I regular value, so $f^{-1}(0) = O(n)$ mfd of dim $= n^2 - \frac{n(n+1)}{2}$.

(3) *Hwk.*[5] *Sard $\Rightarrow$ homotopy groups $\pi_i(S^n) = 0$ for $i < n$.*

1.4. Transversality.

Motivation:

$$\begin{array}{rcl} q \in N \text{ regular value} & \Rightarrow & f^{-1}(q) \subset M \text{ submfd} \\ ① \text{ submfd } Q \subset N \text{ satisfying} \ldots ? & \Rightarrow & f^{-1}(Q) \subset M \text{ submfd} \\ ② \text{ submfds } Q_1, Q_2 \subset N \text{ satisfying} \ldots ? & \Rightarrow & Q_1 \cap Q_2 \subset N \text{ submfd} \end{array}$$

① Pretend N/Q made sense ☺

$\Rightarrow F : M \xrightarrow{f} N \to N/Q \ni \overline{q} = Q/Q$

[1] If you are curious about its non-examinable proof, see Milnor's *Topology from the Differentiable Viewpoint*, or Guillemin & Pollack, *Differential Topology*.

[2] A subset S of $\mathbb{R}^n$ has *measure zero* if $\forall \varepsilon > 0$, $\exists$countable covering of S by cubes C_i, with $\sum \mathrm{vol}(C_i) < \varepsilon$. A subset S of a mfd N has *measure zero* if for any chart $\varphi : U \to \mathbb{R}^n$, $\varphi(S \cap U)$ has measure 0 (it's enough to require this for a covering $\varphi_i : U_i \to \mathbb{R}^n$). Example: $\mathbb{Q} \subset \mathbb{R}$. Useful facts: countable unions of measure 0 sets have measure 0; C^1-maps between subsets of $\mathbb{R}^n$ always map measure 0 sets to measure 0 sets.

[3] *Second countable*= there is a countable covering by charts. This is always part of the definition of manifold. Consequence: any covering has a countable subcover.

[4] k-times continuously differentiable maps, with $k \geq 1$ so "regular/critical points" are defined.

[5] *Non-examinable:* the proof essentially shows Sard implies the cellular approximation theorem.

$\Rightarrow f^{-1}(Q) = F^{-1}(\overline{q})$
$\Rightarrow f^{-1}(Q)$ is mfd if $\overline{q}$ regular value of F

if d_pF surjective $\forall p \in F^{-1}(\overline{q})$

if $d_pF(T_pM) = T_{\overline{q}}(N/Q)$

if $\boxed{d_pf(T_pM) + T_qQ = T_qN \qquad \forall p \in f^{-1}(q), \forall q \in Q}$

Def. *$f : M \to N$ is transverse to Q if the above box holds. Write $f \pitchfork Q$.*

Thm.

$$\boxed{f \pitchfork Q \Rightarrow \begin{cases} f^{-1}(Q) \subset M \textit{ submfd of codim} = \operatorname{codim} Q \\ T_pf^{-1}(Q) = \ker(T_pM \xrightarrow{d_pf} TN \to TN/TQ) = \ker(D_pf : T_pM \to \nu_{Q,q}) \end{cases}}$$

Pf. Locally $Q \subset N$ is[6] $\mathbb{R}^a \subset \mathbb{R}^n$, so "$N/Q$" is well-defined locally: $\mathbb{R}^n/\mathbb{R}^a$. □

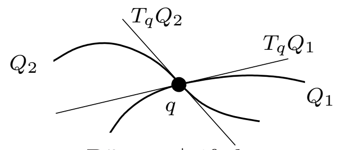
Explanation: $\nu_Q = TN/TQ =$*normal bundle* to $Q \subset N$, fibre $\nu_{Q,q} = T_qN/T_qQ$. D_pf is abuse of notation:[7] $\boxed{Df_p \cdot X = \text{vertical projection of } d_pf \cdot X \text{ at } q = f(p) \in Q}$

② For $f : Q_1 \underset{\text{inclusion}}{\hookrightarrow} N$ and $Q = Q_2 \subset N$,

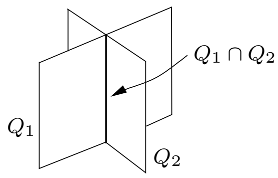
$$f^{-1}(Q) = Q_1 \cap Q_2 \subset N.$$

Def. *Q_1, Q_2 are* transverse *submfds of N, written $Q_1 \pitchfork Q_2$, if*

$$\boxed{T_qQ_1 + T_qQ_2 = T_qN \qquad \forall q \in Q_1 \cap Q_2}$$



Examples. $N \pitchfork$ any submfd! Two vector subspaces $\subset \mathbb{R}^n$ are $\pitchfork$ if they span $\mathbb{R}^n$.



Cor.

$$\boxed{Q_1 \pitchfork Q_2 \Rightarrow \begin{cases} Q_1 \cap Q_2 \subset N \textit{ submfd} \\ \textit{of codim} = \operatorname{codim} Q_1 + \operatorname{codim} Q_2 \\ T_q(Q_1 \cap Q_2) = T_qQ_1 \cap T_qQ_2 \end{cases}}$$

Rmk.

1. $\dim Q_1 + \dim Q_2 < \dim N$ *then* $Q_1 \pitchfork Q_2 \Leftrightarrow Q_1 \cap Q_2 = \emptyset$

2. $\dim Q_1 + \dim Q_2 = \dim N$ *then* $Q_1 \pitchfork Q_2 \Leftrightarrow \begin{cases} Q_1 \cap Q_2 \textit{ finite set}^{8} \\ TQ_1 \oplus TQ_2 \cong TN \textit{ at } q \in Q_1 \cap Q_2 \ (*) \end{cases}$

In case 2. you can define an intersection number

$$Q_1 \cdot Q_2 = \#(Q_1 \cap Q_2) \textit{ mod } 2 \in \mathbb{Z}/2\mathbb{Z}$$

If Q_1, Q_2, N oriented:[9]

$$Q_1 \cdot Q_2 = \#(Q_1 \cap Q_2) \in \mathbb{Z},$$

[6] Hwk 3: $Q \to N$ immersion $\Rightarrow$ locally has form $(x_1, \ldots, x_a) \mapsto (x_1, \ldots, x_a, 0, \ldots, 0) \in \mathbb{R}^n$.

[7] f is not a section of ν_Q, but the construction of that vertical projection is analogous.

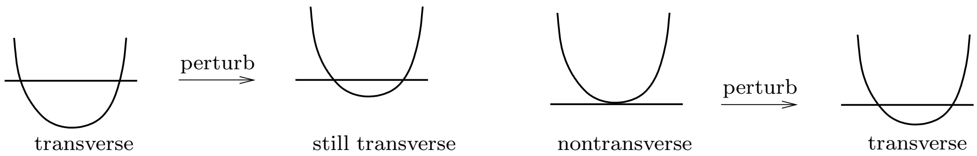
[8] assuming Q_1, Q_2 are compact submanifolds. Otherwise, replace with "discrete set".

[9] *Non-examinable:* it suffices that Q_1 is oriented, and Q_2 is co-oriented (= normal bundle $\nu_{Q_2} = TN/TQ_2$ is oriented). Assign $+1$ to $p \in Q_1 \cap Q_2$ if an oriented basis of T_pQ_1 gives rise to an oriented basis of ν_{Q_2}, and -1 else. When Q_1, Q_2, N are oriented, this sign agrees with the one above, if we orient so that $TN|_{Q_2} \cong \nu_{Q_2} \oplus TQ_2$ preserves orientation ("normals first").

where # counts with sign +1 if the iso (∗) is orientation-preserving, −1 otherwise.

Next time, we will deduce that one can always achieve $Q_1 \pitchfork Q_2$ *after perturbing* Q_1 *(or* Q_2*), and in case **2.** the value* $Q_1 \cdot Q_2$ *is independent of the perturbation.*

Motivation for stability and genericity. Transversality is stable and generic: Stable: perturbing preserves the property, generic: it can be achieved by perturbing.



1.5. **Stability.**

Recall a (smooth) *homotopy* f_t of $f : M \to N$ means a smooth map

$$H : M \times [0,1] \to N \qquad \text{with} \begin{cases} f_t(x) = H(x,t) \\ f_0 = f \end{cases}$$

Call f_0, f_1 (smoothly) homotopic.

Def. *A "property" P is* stable *for a class C of maps* $f : M \to N$*, if*

$$\left.\begin{array}{l} f \in C \text{ satisfies } P \\ f_t \text{ homotopy} \end{array}\right\} \Rightarrow f_t \text{ satisfies } P \text{ for each } t < \varepsilon \quad (\varepsilon > 0 \text{ depending on } f, f_t)$$

Rmk.

(1) Locally stable *means* $\forall p \in M$, $\exists$ *nbhd* $U \ni p$ *such that P is stable for the restrictions* $\{f|_U : f \in C\}$

(2) *For compact M, one can often deduce stability from local stability, by covering M by such U, taking a finite subcover, taking min of* ε*'s.*

(3) *Can use more general parameters* $t \in S =$ *metric space.*

Stability Theorem. *M compact ⇒ the following classes are stable:*

$\{$*local diffeos*$\}$
$\{$*regular maps*$\}$
$\{$*maps* $\pitchfork$ *to a given topologically-closed submfd* $Q \subset N\}$

Pf. The definition of these classes locally involve the non-vanishing of some (sub) determinant of some differential. Use Rmk (2) to globalize. □

Cor. *Transversality is stable and it is an open condition.*

Pf. Stability by Thm. Open: if not, find non-transverse $f_n \to f$ as[10] $n \to \infty$. Produce a homotopy H of f with $H(1/n,t) = f_n(t)$. H contradicts stability. □

Rmk. *Here is a more direct proof that transversality is an open condition:*
Claim 1. *regular points of any smooth map f of mfds forms an open set.*
Pf. *Locally at regular p,* $d_p f = (I \ \ 0)$*. So for q close to p,* $d_q f = (T \ \ *)$ *for some invertible T since invertibility is an open condition.*[11] *So q is regular.* □
Transversality can be expressed as a regularity condition, so it is also open.

[10] the convergence is in C^∞. Also C^1 is enough: we just need the derivatives to converge.

[11] If s is an operator with small norm ($\|s\| < 1$ is enough), then $(I+s)^{-1} = I - s + s^2 - s^3 + \cdots$ is a well-defined operator. If L is invertible and $\|s\| < \|L\|$ then $(L+s)^{-1} = (I + L^{-1}s)^{-1}L^{-1}$.

1.6. **Local to global examples.**

Thm. *Any compact mfd N can be embedded in some $\mathbb{R}^k$.*

Pf. Cover N by all possible charts[12] $\varphi : B(2) \to N$.
Pick finitely many φ_i for which $\varphi_i(B(1))$ cover N.
Let $\beta =$ bump function[13] $B(2) \to [0,1]$, $\beta = 1$ on $B(1)$, $\beta = 0$ near $\partial B(2)$.
$\Rightarrow N \hookrightarrow \mathbb{R}^{(n+1)\cdot\#\text{charts}}$

$$p \mapsto (\,\beta(\varphi_i^{-1}(p)) \cdot \varphi_i^{-1}(p),\ \beta(\varphi_i^{-1}(p))\,)_{i=1,2,\ldots} \quad \text{(zero entry for } i \text{ if } p \notin \mathrm{im}\varphi_i).$$

Note we are keeping track of the β values to ensure global injectivity. $\square$

Cultural Rmk. *Whitney proved $N^n \hookrightarrow \mathbb{R}^{2n}$. Transversality techniques from this course can easily prove $N^n \hookrightarrow \mathbb{R}^{2n+1}$ (if you're curious, see Guillemin & Pollack).*

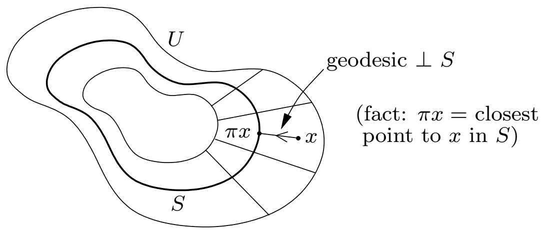
Def. *A* tubular neighbourhood *is a nbhd U of S with a regular retraction*

$$\pi : U \to S.$$

(Retraction *just means $\pi|_S = id_S$*).

Thm. *Any submanifold $S \subset M$ has a tubular nbhd $U \subset M$.*

Pf. Pick a Riemannian metric for M, use exp map. $\square$

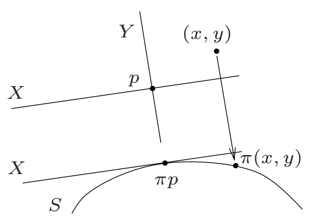


Rmk.

(1) $U \overset{exp^{-1}}{\cong}$ *nbhd of zero section of normal bundle ν_S*
$\pi \cong$ *projection*

(2) *Converse:*[14] $\boxed{\text{A closed subset } S \subset \mathbb{R}^k \text{ is a submfd} \Leftrightarrow S \text{ is a smooth retract}^{15}}$

Pf. implicit function theorem for regular $\pi : U \to S$. $\square$
Non-examinable details of Pf:



For $p \in U$ near S, let $X = d_p\pi(T_pU) \subset \mathbb{R}^k$ a v.subspace (secretly $T_{\pi(p)}S$). Then $\mathbb{R}^k = X \oplus Y$ some v.subspace Y. After lin change of coords,

$$d_p\pi = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] : X \oplus Y \to X \oplus Y,$$

with $p = (0,0) \in X \oplus Y = \mathbb{R}^k$. Define

$$F : X \oplus Y \to X \oplus Y, F(x,y) = \pi(x,y) + y.$$

$d_pF = I \Rightarrow$ InvFnThm $\Rightarrow F^{-1}(s) = (g(s,0),0)$ for $s \in S$ defines chart $s \mapsto g(s,0)$ at $\pi(p) \in S$.

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[12] $B(r) =$ open ball of radius r, centre 0, in $\mathbb{R}^n$.

[13] You gain nothing from writing out explicitly a bump function you already know exists: *Non-examinable:* for $b > a > 0$, let $\alpha(x) = e^{-1/x}$ for $x > 0$, 0 for $x \leq 0$; let $\gamma(x) = \alpha(x-a)\cdot\alpha(b-x)$; let $\delta(x) = \int_x^b \gamma / \int_a^b \gamma$. Then $\beta(x) = \delta(|x|)$ is 1 on $|x| \leq a$, 0 on $|x| \geq b$, $\beta(x) \in (0,1)$ for $a < |x| < b$.

[14] the same proof shows this holds for C^r-mfds, π a C^r-map, $r \geq 1$ (not just $r = \infty$).

[15] Smooth retract$=$ $\exists$open nbhd U of S, $\exists$smooth $\pi : U \to \mathbb{R}^k$ with $\pi(U) \subset S$, $\pi|_S = \mathrm{id}_S$.