LECTURE 18.

PART III, MORSE HOMOLOGY, 2011

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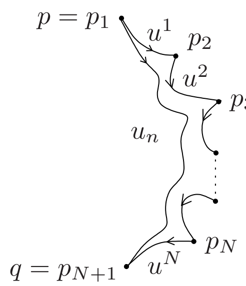
5.3. **Convergence to broken trajectories (continued).** Recall that by the reparametrization trick, for any sequence $u_n \in W(p,q)$ without a convergent subsequence, $\exists s_n \in \mathbb{R}$ with $w_n = u_n(\cdot + s_n) \to w$ in C^0_{loc} with $f(w(\mathbb{R})) \cap f(u(\mathbb{R})) = \emptyset$.

Thm. *$u_n \in W(p,q) \Rightarrow \exists$ subseq u_n such that:*

- $\exists\, s_n^i \in \mathbb{R} \qquad i = 1, \ldots, N$
- $\exists\, u^i \in W(p_i, p_{i+1}) \qquad p = p_1, q = p_{N+1}$
- $f(p_1) > f(p_2) > \cdots > f(p_{N+1})$

with

$$\boxed{u_n^i = u_n(\cdot + s_n^i) \to u^i \text{ in } W(p_i, p_{i+1})}$$



Proof. Cover $[f(p), f(q)]$ by closures of disjoint intervals obtained by the reparametrization trick. This is a finite cover by Trick 3.3.[1] □

Def. *Call $(u^1, u^2, \ldots, u^N) \in W(p_1,p_2) \times \cdots \times W(p_N, p_{N+1})$ a* broken flowline.

5.4. **Compactness theorem.**

Rmk. *In the Theorem, $u_n^i \in W(p,q)$ are different lifts of the same $[u_n] \in \mathcal{M}(p,q)$.*

Def. *In the Theorem, denote $v_n = [u_n] = [u_n^i] \in \mathcal{M}(p,q)$, $v^i = [u^i] \in \mathcal{M}(p_i, p_{i+1})$. Then we summarize the conclusion of the Theorem by the* broken limit *symbol*

$$v_n \rightrightarrows v^1 \# \cdots \# v^N$$

and we call $v^1 \# \cdots \# v^N \in \mathcal{M}(p, p_2) \times \cdots \mathcal{M}(p_N, q)$ an (N-times) broken trajectory.

Cor. *$v_n \in \mathcal{M}(p,q) \Rightarrow \exists$ subseq $v_n \rightrightarrows v^1 \# \cdots \# v^N$ with $v^i \in \mathcal{M}(p_i, p_{i+1})$ $(f(p) = f(p_1) > \cdots > f(p_{N+1}) = f(q),\ p = p_1,\ q = p_{N+1})$.*

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[1] you consume energy $\geq$ length of interval $\geq \delta > 0$.

Rmk. *From now on, assume transversality holds (it does for a generic metric). So*

$$|p| = |p_1| > |p_2| > \cdots > |p_{N+1}| = |q|$$

since $\mathcal{M}(p_i, p_{i+1}) = \emptyset$ *if*[2] $|p_i| \leq |p_{i+1}|$ (*note* $\dim \mathcal{M}(p_i, p_{i+1}) = |p_i| - |p_{i+1}| - 1 < 0$).

Repeat the Key idea 5.0 for the compactification of $\mathcal{M}(p, q)$:

(1) sequences $u_n \in \mathcal{M}(p, q)$ which do not have a convergent subsequence:
 those with a subsequence $\rightrightarrows$ broken trajectory

(2) artificially add limit points to $\mathcal{M}(p, q)$:

$$\overline{\mathcal{M}}(p, q) = \mathcal{M}(p, q) \cup \partial \mathcal{M}(p, q)$$
$$\partial \mathcal{M}(p, q) = \bigcup_{N \geq 2, |p| > |p_2| > \cdots > |q|} \mathcal{M}(p, p_2) \times \cdots \times \mathcal{M}(p_N, q)$$

(3) enlarge the topology to make them limit points:
 topology of $\rightrightarrows$ convergence to broken trajectories

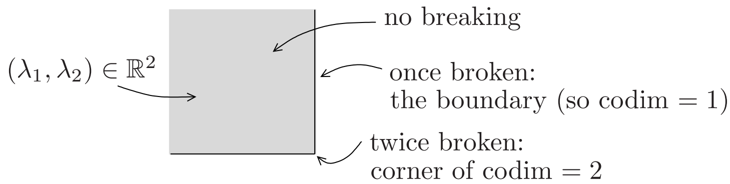
Upshot: Theorem. $\overline{M}(p, q)$ *is compact.*

Two problems:

- 5.3 $\not\Rightarrow$ every broken flowline arises as a $\rightrightarrows$ limit
- $\overline{\mathcal{M}}(p, q)$ smooth mfd (with corners)?

Answer: Yes, by the gluing theorem! (next section)

We will only study once-broken trajectories, so there are no corners. But, for example, you should think of a 2-dimensional moduli space as follows:



5.5. **Gluing theorem.** For once broken flowlines (for simplicity):

$$\dim W(p, q) = |p| - |q| = 2$$
$$\dim \mathcal{M}(p, q) = 1$$

Thm. *(Assuming transversality) For all* $a \in Crit(f)$ *with* $|p| - |a| = 1 = |a| - |q|$, *there is a* gluing map

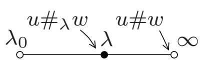
$$\boxed{\begin{array}{rrcl} \# : & W(p,a) \times W(a,q) \times (\lambda_0, \infty) & \rightarrow & W(p,q) \\ & (u, w, \lambda) & \mapsto & u \#_\lambda w \end{array}}$$

(1) $\#$ *induces a smooth embedding on* $\mathcal{M}(\cdot, \cdot)$ *spaces*

(2) $u \#_\lambda w \rightrightarrows u \# w$ *as* $\lambda \to \infty$

(3) *if* $v_n \rightrightarrows u \# w$ *then for* $n \gg 0$, $v_n = [u \#_{\lambda_n} w] \in \mathcal{M}(p, q)$, *for some* $\lambda_n \to \infty$

Cor. $\dim \mathcal{M}(p, q) = 1 \Rightarrow \overline{\mathcal{M}(p, q)}$ *smooth compact* 1*-mfd with bdry* $\partial \mathcal{M}(p, q)$.

Proof. Thm $\Rightarrow \exists$ (collar nbhd of $u \# w$) $\cong$ 

$\subset \mathbb{R}$. ∎

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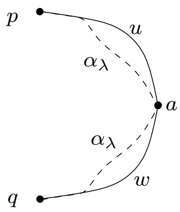
[2] $p_i \neq p_{i+1}$ since $f(p_i) > f(p_{i+1})$.

Sketch of Proof of Theorem[3]



Step 1. construct a smooth *approximate solution* of $F(u) = 0$:

$$\alpha_\lambda(s) = \begin{cases} u(s+2\lambda) & \text{for } s \leq -\lambda \\ a & \text{for } s \in [-\lambda+1, \lambda-1] \\ w(s-2\lambda) & \text{for } s \geq \lambda \end{cases}$$

and we use $\exp_a(\cdot)$ to interpolate this data.[4]

Then:

- $F(\alpha_\lambda(s)) \neq 0$ since you would need[5] ∞ time s to reach the crit pt a
- $(*)$ $F(\alpha_\lambda(s)) \to 0$ as $\lambda \to \infty$ since

$$\begin{aligned} &F(u(\cdot+2\lambda)) = 0 = F(w(\cdot-2\lambda)) \\ &F(s \mapsto a) = -\nabla f_a = 0 \\ &F(\text{interpolation}) \approx -\nabla f_a = 0 \end{aligned}$$

Step 2. $(*) \Rightarrow \exists$ "unique" actual solution $u\#_\lambda w$ close to α_λ,

$$F(u\#_\lambda w) = 0.$$

This "$\Rightarrow$" is proved using the contraction mapping theorem and the implicit function theorem. "Unique" is imprecise: one can construct a cts bijection $\alpha_\lambda \to u\#_\lambda w$.

Step 3. $\alpha_\lambda(s) \rightrightarrows u\#w$, indeed make s-shifts by -2λ and $+2\lambda$ when you lift α_λ.

Ideas used in Step 2. $L_u = D_uF$, $L_w = D_wF$, $L_\lambda = D_{\alpha_\lambda}F$

Rmk. *D_uF, D_wF, L_λ are Fredholm (Thm 4.14*[6]*)*

Technical Fact:

$$\begin{array}{l} L_u, L_w \text{ surjective} \\ \text{(by transversality)} \end{array} \Rightarrow \begin{cases} ① \ L_\lambda \text{ surjective for } \lambda \gg 0 \\ ② \ \exists c > 0 \text{ s.t. for } \lambda \gg 0: \\ \quad \boxed{\|L_\lambda^\star V\|_{1,2} \leq c \cdot \|L_\lambda L_\lambda^\star V\|_2} \quad \forall V \in W^{1,2}(\mathbb{R}, \alpha_\lambda^* TM) \end{cases}$$

① One can patch[7] together elements in $\ker L_u, \ker L_w$ to obtain approximate solutions to $L_\lambda V = 0$, and one proves that for $\lambda \gg 0$ this defines an isomorphism:

$$\boxed{\begin{array}{rcl} \ker L_u \oplus \ker L_w & \xrightarrow{\sim} & \ker L_\lambda \\ V_u \oplus V_w & \mapsto & (\text{orthogonal projection}) \cdot (V_u \#_\lambda V_w) \end{array}}$$

where $\#$ is the patching. This we call ***linear gluing***. It is quite simple to prove because it just involves linear subspaces. This linear gluing map arises as the differential of the gluing map, and this isomorphism is used to prove the embedding property in (2).

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[3]This would take too many Lectures to prove in detail, and the details are not enlightening.

[4]*Non-examinable:* $\exp_a(\beta(-s-\lambda+1)\cdot u(s+2\lambda))$ for $s \in [-\lambda, -\lambda+1]$; $\exp_a(\beta(s-\lambda+1)\cdot w(s-2\lambda))$ for $s \in [\lambda-1, \lambda]$, where $\beta : \mathbb{R} \to [0,1]$ is increasing with $\beta = 0$ on $s \leq 0$, $\beta = 1$ on $s \geq 1$.

[5]Hwk 22. ex. 2

[6]recall the theorem only used that the path was C^k, not that $F(\text{path}) = 0$.

[7]*Non-examinable:* For operators L, K which are asymptotically constant at $+\infty, -\infty$ respectively, then for $\lambda \gg 0$ we can glue $L(\cdot+2\lambda)\#K(\cdot-2\lambda) = L\#_\lambda K$, then $\ker L \oplus \ker K \xrightarrow{\sim} \ker(L\#_\lambda K)$ is the orthog projection of the patching $V\#_\lambda W = V(\cdot+2\lambda) + W(\cdot-2\lambda)$ (for fixed s this is small for $\lambda \gg 0$ since the solutions V, W decay to zero fast at the ends). This map is an iso for $\lambda \gg 0$.

By invariance of the Fredholm index under homotopying paths (indeed we know it is the difference of the Morse indices of the ends):[8]

$$\boxed{\operatorname{index}(L_u) + \operatorname{index}(L_w) = \operatorname{index}(L_\lambda) \quad (\lambda \gg 0)}$$

so $\dim\operatorname{coker} L_\lambda = \dim\operatorname{coker} L_u + \dim\operatorname{coker} L_w = 0$, so L_λ is surjective. ✓

② Why that inequality? For A, B Hilbert,

$$L : A \to B \text{ Fredholm and surjective} \Rightarrow A = K \oplus^{\perp} A_0 \longrightarrow B, \qquad L^* : B \to A_0, \qquad R = (L|_{A_0})^{-1} : B \to A_0$$

where[9] $\boxed{A_0 = \operatorname{im} L^*}$ and "R" stands for right-inverse since $LR = I$.

Cor. $L : A \to B$ *Fred and surj* $\Leftrightarrow \exists$ *bdd right inverse and* $\dim\ker L < \infty$

Lemma. $\|L^*b\| \leq c \cdot \|LL^*b\|\ \forall b \Leftrightarrow \|Rb\| \leq c \cdot \|b\|\ \forall b$

Proof. Both are equivalent to: $\|a\| \leq c \cdot \|La\|\ \forall a \in A_0$. □

Upshot: Combining inequality ② with the Lemma:[10]

$\Rightarrow$ L_λ have uniformly bounded right inverses.

$\overset{\text{Hwk 19}}{\Rightarrow}$ $\exists$ unique actual solution $\exp_{\alpha_\lambda}(L_\lambda^\star V)$ (some unique $V \in W^{1,2}$) and all nearby actual solutions are of form $\exp_{\alpha_\lambda}(k \oplus g(k))$ where $k \in K$ is small and $g : K \to A_0$ is a smooth implicit function, $g(0) = \exp_{\alpha_\lambda}(L_\lambda^\star V)$.

So we define $\boxed{u \#_\lambda w = \exp_{\alpha_\lambda}(L_\lambda^\star V)}$

Rmk. *The key is that L^* provides a way to obtain uniqueness. $L_\lambda^\star V$ is constrained to be inside A_0, whereas if you allow vectors in the whole of A, such as $k \oplus g(k)$, then you no longer get uniqueness.*[11] *This is crucial also in Hwk 19: the contraction mapping principle (Picard's method) is applied to A_0, not the whole of A.*

Hwk 19: Picard's method.

For $F : A \to B$ a C^1-map of Hilbert spaces, by Taylor:

$$F(x) = c + L \cdot x + N(x)$$

where $c = F(0)$, $L = d_0F$ linear, N non-linear. Assume L Fred & surj, so as above:

$$L : K \oplus A_0 \to B \qquad R : B \to A_0 \qquad LR = I.$$

Assume the following two estimates hold:

(1) $\|Rc\| \leq \frac{\varepsilon}{2}$
(2) $\|RN(x) - RN(y)\| \leq C \cdot (\|x\| + \|y\|) \cdot \|x - y\|$ for all $x, y \in \operatorname{ball}_\varepsilon(0)$, $\varepsilon \leq \frac{1}{3C}$.

then

- by the contraction mapping theorem for $P : A_0 \to B$, $P(x) = -Rc - RN(x)$, there is a unique $a_0 \in A_0 \cap \operatorname{ball}_\varepsilon(0)$ with $F(a_0) = 0$.

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[8] or use formal adjoints to get isos of cokernels like for linear gluing of kernels.

[9] $(\operatorname{im} L^*)^\perp = \ker L = K$, and A_0 is closed since it is the complement of a finite dim'l subspace.

[10] which works in our setup for the formal adjoint $L_\lambda^\star$ instead of L_λ^*.

[11] Unsurprisingly, since when the Morse index difference is large, there is a large dimensional family of actual solutions, so the actual solution $u\#_\lambda v$ is not isolated. Indeed, the family is parametrized by K via $\exp_{\alpha_\lambda}(k \oplus g(k))$.

- by the implicit function theorem at a_0, there is a C^1-map $g : K \to A_0$ such that $F(k \oplus g(k)) = 0$ for small $k \in K$ (with $0 \oplus g(0) = a_0$).

Application: *We apply Picard's method to F = local expression of the vertical part of our section $\mathcal{F} = \partial_s + \nabla f : U \to E$ in a chart around $\alpha_\lambda \in U$ (so α_λ is 0 in the chart). So*

$$F : W^{1,2}(\mathbb{R}, \alpha_\lambda^* TM) \to L^2(\mathbb{R}, \alpha_\lambda^* TM),$$
$$c = F(0) = \mathcal{F}(\alpha_\lambda),$$
$$L = d_0 F = D_{\alpha_\lambda} \mathcal{F} = L_\lambda.$$

Thus g defines a parametrization of all the actual solutions $\mathcal{F}(u \#_{\lambda(k)} w) = 0$ close to the approximate solution $\mathcal{F}(\alpha_\lambda) \approx 0$, where $\lambda(0) = \lambda$.