LECTURE 19.

PART III, MORSE HOMOLOGY, 2011

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6. Morse Homology

6.1. **Definition.** M closed mfd, $f : M \to \mathbb{R}$ Morse, g generic metric ($\Rightarrow$transversality).

The *Morse(-Smale-Witten) complex* is the $\mathbb{Z}/2$-vector space generated by the critical points of f:

$$\boxed{MC_k = \bigoplus_{p \in \mathrm{Crit}(f),\ |p|=k} \mathbb{Z}/2 \cdot p}$$

where $k \in \mathbb{Z}$ is the $\mathbb{Z}$-grading by the Morse index.

The *Morse differential* $\partial : MC_k \to MC_{k-1}$ is defined on generators p by

$$\boxed{\partial p = \sum_{\dim \mathcal{M}(p,q)=0,\ p \neq q} \#\mathcal{M}(p,q) \cdot q}$$

and extend ∂ linearly to MC_*. Note[1] $\dim \mathcal{M}(p,q) = 0$ is equivalent to $|q| = |p| - 1$.

Rmk. *The sum is well-defined because $\mathcal{M}(p,q)$ is a 0-dimensional compact manifold, hence finite, so can count*[2] *the number of elements $\#\mathcal{M}(p,q)$. Proof: It is a smooth manifold by transversality, and it is compact by the following argument:*[3]

$$\begin{aligned} \dim \mathcal{M}(p,q) = 0 \quad &\Rightarrow \quad \dim \partial\mathcal{M}(p,q) < 0 \\ &\Rightarrow \quad \partial\mathcal{M}(p,q) = \emptyset \\ &\Rightarrow \quad \mathcal{M}(p,q) = \overline{\mathcal{M}}(p,q) \text{ compact 0-dim mfd } \checkmark \end{aligned}$$

Thm. $\partial^2 = 0$.

Proof. $|p| = k$. Compute:

$$\begin{aligned} \partial^2 p &= \partial \sum_{|a|=k-1} \#\mathcal{M}(p,a) \cdot a \\ &= \sum_{|a|=k-1} \sum_{|q|=k-2} \#\mathcal{M}(p,a) \cdot \#\mathcal{M}(a,q) \cdot q \\ &= \sum_{|a|=k-1,\ |q|=k-2} \#(\mathcal{M}(p,a) \times \mathcal{M}(a,q)) \cdot q \\ &= \sum_{|a|=k-1,\ |q|=k-2} \#\partial\mathcal{M}(p,q) \cdot q \end{aligned}$$

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[1] *Cultural Remark:* In more general situations, $\mathcal{M}(p,q)$ may have components of different dimensions, and you only count the $u \in \mathcal{M}(p,q)$ in the 0-dimensional part $\mathcal{M}_0(p,q)$.

[2] *Non-examinable:* To work over $\mathbb{Z}$ instead of $\mathbb{Z}/2$ you must count the elements of $\mathcal{M}(p,q)$ with orientation signs ± 1. Orientations of moduli spaces are an unpleasant technical detail which we decided to omit from this course (compare with sign headaches in singular homology arguments).

[3] Exercise. Can you think of a simple argument which only involves using tranversality, the compactness thm and dimension arguments, but which does not use the gluing theorem?

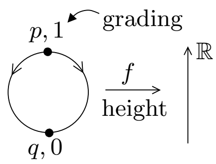
Finally observe:

$\dim \mathcal{M}(p,q) = |p| - |q| - 1 = k - (k-2) - 1 = 1$
$\Rightarrow \overline{\mathcal{M}}(p,q)$ compact 1-mfd with boundary
$\Rightarrow \overline{\mathcal{M}}(p,q)$ is a disjoint union of finitely many circles and closed intervals
$\Rightarrow \#\partial\mathcal{M}(p,q)$ even, so 0 modulo 2
$\Rightarrow \partial^2 p = 0$
$\Rightarrow \partial^2 = 0$ by linearity. □

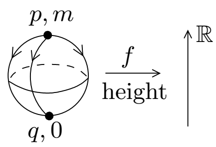
Def. $\boxed{MH_*(M,f,g) = \dfrac{\ker\partial}{\operatorname{im}\partial}}$ *is the Morse homology of* (M,f,g).

Rmk. *If you are given a metric g, then a priori you need to perturb g unless you know/check that transversality holds (see Hwk 1). Key Trick: perturbing g does not affect Crit(f) and indices, this often helps.*[4]

Examples (all homologies are over $\mathbb{Z}/2$):

(1) 

$$\partial p = q + q = 0$$
$$\begin{array}{rclccl} MH_* & = & \mathbb{Z}/2 & \oplus & \mathbb{Z}/2 & \cong H_*(S^1) \\ * & = & 0 & & 1 & \leftarrow \text{ grading} \end{array}$$

(2) 

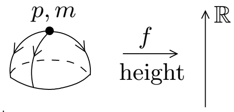
$\partial = 0$ as the indices are ≥ 2 apart (see Rmk)
$$\begin{array}{rclccl} MH_* & = & \mathbb{Z}/2 & \oplus & \mathbb{Z}/2 & \cong H_*(S^m) \\ * & = & 0 & & m & \leftarrow \text{ grading} \end{array}$$

(3) **Thm.** $MH_*(M,f,g) \cong H_*(M)$.
Proof. Next time we will prove *invariance*:
$$MH_*(M,f_1,g_1) \cong MH_*(M,f_2,g_2).$$
So
$$\begin{array}{rcl} MH_*(f) & \cong & MH_*(\text{self-indexing Morse function}) \\ & \cong & H_*^{\text{cellular}}(M) \quad (3.10 \text{ \& Hwk } 19) \\ & \cong & H_*(M). \end{array}$$

(4) M compact mfd with boundary:

Ensure $f|_{\partial M} = \text{constant min} < f|_{\text{interior}}$
$(\Rightarrow \nabla f \pitchfork \partial M \Rightarrow \text{no crit pts on } \partial M)$
$\Rightarrow \mathcal{M}(p,q)$ stay away from ∂M (f decreases along flowlines)
$\Rightarrow MH_* = \dfrac{\ker\partial}{\operatorname{im}\partial} \cong H_*(M,\partial M)$ (proved like in (3))

Example: $M = D^m$ disc.

D^m 

$f|_{\partial M} = \min$

$$\begin{array}{rcl} MH_* & = & \mathbb{Z}/2 \cdot p \quad (\text{in degree } m) \\ & \cong & H_*(D^m, \partial D^m) \\ & & (\text{have } H_0 = 0) \end{array}$$

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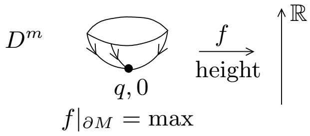
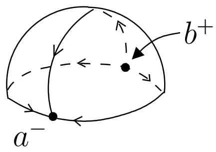
[4] Key example: if all indices are even then, after perturbing, the 0-dimensional moduli spaces are empty for index reasons. So there is no differential. So $MH_* = MC_*$, which you know.



$$\begin{aligned} MH_* &= \mathbb{Z}/2 \cdot q \quad \text{(in degree 0)} \\ &\cong H_*(D^m) \\ &\quad \text{(compare Handle attaching)} \end{aligned}$$

But both are useless for LES of pair $(M, \partial M)$: cannot recover $MH_*(\partial M)$.
Instead of making $\nabla f \pitchfork \partial M$ we will now try ∇f tangent to ∂M.

6.2. **Morse homology for mfds with bdry.** [*Non-examinable*]
Assume $\boxed{\nabla f \text{ is tangent to } \partial M}$ (that is: $\nabla f \in T(\partial M)$). This ensures that the flow of a point in ∂M stays in ∂M and $f|_{\partial M}$ is Morse, so hope to recover $MH_*(\partial M)$.



Write $\boxed{a = a^- \text{ if } df(\text{outward normal}) < 0 \text{ at } a}$
$\Rightarrow W^u(a) \subset \partial M$ (exercise)
Write $\boxed{b = b^+ \text{ if } df(\text{outward normal}) > 0 \text{ at } b}$
$\Rightarrow W^u(b)$ intersects interior, $\partial W^u(b) = W^u(b) \cap \partial M$

$\exists$ Similar statements for W^s reversing the roles of $+, -$ (*Proof*: switch sign of f).

$$\Rightarrow \boxed{MC_* = MC_*^0 \oplus MC_*^- \oplus MC_*^+}$$

respectively generated by $p \in \operatorname{int} M$, a^-'s, b^+'s.

Bad case:[5]

$$\begin{aligned} a^-, b^+ \quad &\Rightarrow \quad W^u(a) \subset \partial M,\ W^s(b) \subset \partial M \\ &\Rightarrow \quad \text{cannot hope } W^u(a) \pitchfork W^s(b) \text{ in } M \\ &\Rightarrow \quad \text{require } \pitchfork \text{ just in } \partial M. \end{aligned}$$

Therefore:

$$\boxed{\begin{array}{ll} \dim \mathcal{M}(a^-, b^+) = |a| - |b| & \text{bad case} \\ \dim \mathcal{M}(p, q) = |p| - |q| - 1 & \text{otherwise (as usual)} \end{array}}$$

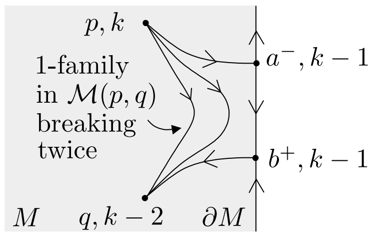
Getting $MH_*(\partial M)$:
$p, q \in \partial M \Rightarrow \boxed{B(p,q) = \{[u] \in \mathcal{M}(p,q) : u \subset \partial M\} = \mathcal{M}(p, q, f|_{\partial M})}$
$p \in \partial M \Rightarrow \operatorname{index}_{f|_{\partial M}}(p) = \begin{cases} |p| & \text{if } p^- \\ |p| - 1 & \text{if } p^+ \end{cases}$
Therefore $MC_k(\partial M, f|_{\partial M}) = MC_k^- \oplus MC_{k+1}^+$, with differential

$$Bp = \sum_{\dim B(p,q) = 0,\ p \neq q} \#B(p,q) \cdot q$$

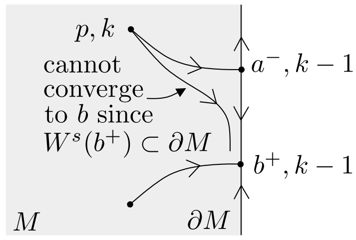
whose homology recovers $MH_*(\partial M)$.

Getting $MH_*(M, \partial M)$:
There are 2 new bad phenomena:

[5]We are tweaking the definition of Morse-Smale to suit the situation.



① ∃ flowlines between same index bdry pts! (Bad case)



② Gluing fails for this twice broken trajectory!

Upshot: $\boxed{\partial^2 \neq 0}$

The argument in 6.1 for $\partial^2 = 0$ will yield:

$$\sum_{r \in \operatorname{int} M} \#\mathcal{M}(p,r)\cdot\#\mathcal{M}(r,q) + \sum_{|a^-|=|b^+|=k-1} \#\mathcal{M}(p,a)\cdot\#B(a,b)\cdot\#\mathcal{M}(b,q) = 0 \pmod 2 \quad (*)$$

where $p, q \in \operatorname{int} M$, $|p| = k$, $|q| = k-2$.

Miracle: ① essentially fixes ②. The two problems suggest that one should not keep both MC^+ and MC^-, one should use only one of the two. Try keeping

$$\boxed{MC^0_* \oplus MC^+_*}$$

Notation:[6] $B : MC^-_k \oplus MC^+_{k+1} \to MC^-_{k-1} \oplus MC^+_k$,

$$B = \begin{bmatrix} B^-_- & B^+_- \\ B^-_+ & B^+_+ \end{bmatrix}$$

Similar notation for ∂. Then $(*)$ can be rewritten as:

$$\partial^0_0 \partial^0_0 + \partial^+_0 B^-_+ \partial^0_- = 0$$

Define a differential d by combining ∂ with B's, so that a once-broken trajectory breaking at an a^- point is considered as if it were just one flowline.[7]

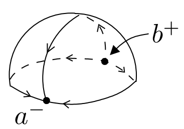
$$\boxed{d = \begin{bmatrix} \partial^0_0 & \partial^+_0 \\ B^-_+\partial^0_- & B^+_+ + B^-_+\partial^+_- \end{bmatrix} : MC^0_k \oplus MC^+_k \to MC^0_{k-1} \oplus MC^+_{k-1}}$$

$$\Rightarrow d^2 = \begin{bmatrix} \partial^0_0\partial^0_0 + \partial^+_0 B^-_+ \partial^0_- & \bullet \\ \bullet & \bullet \end{bmatrix}$$

By $(*)$, the first entry is 0. Similar arguments show the other entries are zero. So

$$\Rightarrow \boxed{d^2 = 0}$$

Example



$MC^0_* = 0$
$MC^+_* = \mathbb{Z}/2 \cdot b$ in degree 2
$\Rightarrow H_*(d) = \mathbb{Z}/2 \cdot b \cong H_*(D^2, \partial D^2)$

Thm. $\boxed{H_*(d) \cong H_*(M, \partial M)}$

[6] The top index is "from", the bottom index is "to", so B^+_- goes from MC^+ to MC^-.

[7] This is to fix the two problems by pretending that ① is a once-broken trajectory, and that the first breaking in ② is not a breaking.

Proof Sketch. First you show that MH_* changes by an iso if you change f. Then you construct your favourite f by the methods of Hwk 7: one for which

$$df(\text{outward normal}) \leq 0$$

near ∂M (so all critical $a \in \partial M$ are of type a^-, and no trajectory from the interior will get arbitrarily close to ∂M unless it ends there). The claim then follows by examples (3) & (4). □

If you instead try just keeping

$$\boxed{MC_*^0 \oplus MC_*^-}$$

then the appropriate differential is

$$\boxed{\delta = \begin{bmatrix} \partial_0^0 & \partial_0^+ B_+^- \\ \partial_-^0 & B_-^- + \partial_-^+ B_+^- \end{bmatrix}}$$

In the above Example, $H_*(\delta)$ is generated by a in degree $|a| = 0$.

Thm. $\boxed{H_*(\delta) \cong H_*(M)}$

Proof idea. Make all critical $b \in \partial M$ to be of type b^+. □

Def. *The homologies of B, d, δ (also denoted $\overline{\partial}, \hat{\partial}, \check{\partial}$) are called:*

$$\begin{array}{lcll} \overline{MH} & \cong & H_*(\partial M) & \text{“}MH\text{ bar”} \\ \widehat{MH} & \cong & H_*(M, \partial M) & \text{“}MH\text{ from”} \\ \widecheck{MH} & \cong & H_*(M) & \text{“}MH\text{ to”} \end{array}$$

The hat tells you the movement of flowlines: from/to the boundary ∂M.

Thm. LES of pair $(M, \partial M)$

$$\cdots \to \overline{MH}_* \to \widecheck{MH}_* \to \widehat{MH}_* \to \overline{MH}_{*-1} \to \cdots$$

at the chain level, the maps are:

$$\cdots \to MC^- \oplus MC^+ \xrightarrow{\begin{bmatrix} 0 & \partial_0^+ \\ I & \partial_-^+ \end{bmatrix}} MC^0 \oplus MC^- \xrightarrow{\begin{bmatrix} I & 0 \\ 0 & B_+^- \end{bmatrix}} MC^0 \oplus MC^+ \xrightarrow{\begin{bmatrix} \partial_-^0 & \partial_-^+ \\ 0 & I \end{bmatrix}} MC^- \oplus MC^+ \to \cdots$$

An excellent reference for further details is the CUP book *Monopoles and Three-Manifolds* by Kronheimer & Mrowka.