C2.6 Introduction to Schemes

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Feedback and corrections are welcome!

EXERCISE SHEET 1

1) i) For ring R, and $\mathfrak{a},\mathfrak{b} \subseteq R$ radical ideals, prove: $\mathfrak{a} \subseteq \mathfrak{b} \Leftrightarrow V(\mathfrak{a}) \supseteq V(\mathfrak{b})$

ii) Show that the presheaf of constant real functions is not a sheaf on X when $X = 2$ points with the discrete topology.

iii) Show that the sheafification of the pre-sheaf of constant functions is the sheaf of locally constant functions. (Optional: What happens in the case $X = \mathbb{Q}$? (with subspace topology $\mathbb{Q} \subseteq \mathbb{R}$))

iv) A scheme X is irreducible $\Leftrightarrow$ every non-empty open subset is dense
(irreducible means: $X = C_1 \cup C_2$ for closed $C_i \Rightarrow C_i = X$ some i)

v) R Noetherian $\Rightarrow$ every subset of $\operatorname{Spec} R$ is quasi-compact.

2) Let $(X, \mathcal{O}_X)$ be a scheme. For $s \in \mathcal{O}_X(U)$ show: $s_x = 0 \in \mathcal{O}_{X,x}\ \forall x \Rightarrow s = 0$, and prove:
X reduced $\Leftrightarrow$ all stalks $\mathcal{O}_{X,x}$ are reduced (reduced means $\mathcal{O}_X(U)$ reduced all open $U \subseteq X$, i.e. nilpotent-free)

3) Let $X = \operatorname{Spec} R$, prove

i) X irreducible $\Leftrightarrow$ R has a unique minimal prime $\mathfrak{p}$ $\leftarrow$ Hint: nilradical.
$\Leftrightarrow$ X has a unique generic point $\mathfrak{p}$ (meaning $\overline{\{\mathfrak{p}\}} = X$)

ii) X reduced and irreducible $\Leftrightarrow$ R integral domain
(you may assume as known that localisation preserves the "reduced" property)

4) $(X, \mathcal{O})$ scheme.

i) If R local ring, $\operatorname{Mor}(\operatorname{Spec} R, X) \overset{1:1}{\longleftrightarrow} \bigsqcup_{x \in X} \operatorname{Hom}_{\text{local rings}}(\mathcal{O}_{X,x}, R)$

Hint: if $\mathfrak{m} \overset{\varphi}{\mapsto} x$ show that $x \in \varphi(\mathfrak{p})$ any $\mathfrak{p}$

ii) If K field, $\operatorname{Mor}(\operatorname{Spec} K, X) \overset{1:1}{\longleftrightarrow} \bigsqcup_{x \in X} \{\text{field extensions } k(x) \hookrightarrow K\}$

$k(x)$ "$= \mathcal{O}_x / \mathfrak{m}_x$" (where $\mathfrak{m}_x \subseteq \mathcal{O}_x$ is unique max ideal)

iii) The Zariski tangent space at x is defined as:

$$T_x = (\mathfrak{m}_x / \mathfrak{m}_x^2)^*$$

$\leftarrow$ vectorspace dual over field $k(x) = \mathcal{O}_x / \mathfrak{m}_x$

Let X be a scheme over a field k, meaning we are given a morph $X \longrightarrow \operatorname{Spec} k$. Convince yourself that this means that locally X is Spec of a k-algebra, not just Spec of a ring. Show:

$$\operatorname{Mor}\left(\operatorname{Spec}\left(k[\varepsilon]/\varepsilon^2\right), X\right) \overset{1:1}{\longleftrightarrow} \bigsqcup_{\substack{x \in X : k(x) \cong k \\ \text{as } k\text{-algebras}}} T_x$$

Here we mean morphisms of schemes over k so commute with maps to $\operatorname{Spec} k$, so maps of sheaves are k-alg. homs

Rmk: if locally X is Spec of f.g. k-algebras, and k alg closed then $k(x) \cong k$ at closed points $x \in X$

Comment on what happens for $X = \operatorname{Spec} k[x]/x^2$ $\leftarrow$ (compare Sec. 0.2 of Notes)

5) A non-affine scheme

In differential geometry, a classic example of a non-Hausdorff space that locally looks Euclidean is the line with two origins:

$$(\mathbb{R}\times 1 \sqcup \mathbb{R}\times 2) \,/\, ((x,1)\sim(x,2) \text{ except if } x=0)$$

Notice: $\mathcal{O}_1=(0,1)\neq\mathcal{O}_2=(0,2)$ are two origins, but the space near $\mathcal{O}_i$ is still homeomorphic to $\mathbb{R}$ via $\mathbb{R}\times i$

It is not Hausdorff since any two neighbourhoods of $\mathcal{O}_1,\mathcal{O}_2$ intersect.

In algebraic geometry, $\operatorname{Spec} k[x]$ is the line k (field) with the Zariski topology and $\operatorname{Spec}(k[x]_{(x)})$ is the "germ of the line at $0\in k$".

i) Let $R=k[x]_{(x)}$. Show that $\operatorname{Spec} R=\{(0),(x)\}$ with $\mathcal{O}_{\operatorname{Spec} R}$: $\emptyset\to 0$, $\operatorname{Spec} R\to R$, $\underset{D_x}{\{(0)\}}\to\underset{\operatorname{Frac} R}{K(x)}$

ii) Let $X=\{o_1,o_2,\ell\}$ three points with the basis of open sets $D_1=\{o_1,\ell\}$, $D_2=\{o_2,\ell\}$, $D_{12}=\{\ell\}$. Define the presheaf $\mathcal{O}$ by $\mathcal{O}(X)=\mathcal{O}(D_1)=\mathcal{O}(D_2)=k[x]_{(x)}$, $\mathcal{O}(D_{12})=k(x)$ $(=\operatorname{Frac} k[x]_{(x)})$, $\mathcal{O}(\emptyset)=0$, restriction homs $\mathcal{O}(X)\xrightarrow{id}\mathcal{O}(D_i)$ and $\mathcal{O}(X_i)\xrightarrow{incl}\mathcal{O}(D_{12})$

Show that $(X,\mathcal{O})$ is a scheme and that it is <u>not</u> affine

6) $\mathcal{A}$ abelian category (Although category theory, this particular exercise is important in C2.6)

i) Show $h^X:=\operatorname{Hom}_{\mathcal{A}}(X,\cdot):\mathcal{A}\to Ab$ is a left exact functor

Fact (Not difficult but you don't need to write it up) Yoneda's lemma: $\boxed{\operatorname{Nat}(h^X,F)\cong F(X)}$ (Nat = natural transformations)

(natural in X,F, for any functor F)

namely via image of $id\in\operatorname{Hom}_{\mathcal{A}}(X,X)=h^X(X)\longrightarrow F(X)$

Rmk Similarly $h_X:=\operatorname{Hom}_{\mathcal{A}}(\cdot,X)$ is left exact contravariant functor, called functor of points of X.

(follows by (i) since $h_X=\operatorname{Hom}_{\mathcal{A}^{op}}(X,\cdot)$ ← recall "op" means you reverse directions of arrows) and $\operatorname{Nat}(h_X,F)\cong F(X)$.

ii) Show: $h^X(A)\to h^X(B)\to h^X(C)$ exact $\forall X\in\mathcal{A}$ $\Longrightarrow$ $A\to B\to C$ is exact

Rmk Similarly $h_X(C)\to h_X(B)\to h_X(A)$ exact $\forall X\in\mathcal{A}$ $\Rightarrow$ $A\to B\to C$ exact.

iii) Show that $h^\bullet:\mathcal{A}\to Ab^{\mathcal{A}}$, $X\mapsto h^X$ ← (category whose objects are functors $\mathcal{A}\to Ab$ & morphs are natural transformations)

is a fully faithful contravariant functor, called "contravariant Yoneda embedding"

Rmk Similarly $h_\bullet:\mathcal{A}\longrightarrow Ab^{\mathcal{A}^{opp}}$ (covariant) called Yoneda embedding

iv) Let $F:\mathcal{A}\to\mathcal{B}$ be a <u>left</u> adjoint functor to $G:\mathcal{B}\to\mathcal{A}$ ← ($\mathcal{A},\mathcal{B}$ abelian cats, F,G additive functors)

meaning $\operatorname{Hom}_{\mathcal{B}}(FX,Y)\cong\operatorname{Hom}_{\mathcal{A}}(X,GY)$ are iso abelian groups.

(natural in X,Y)

Prove that F is <u>right</u> exact and G is left exact.

Rmk (iii) & (iv) also hold if replace Ab by just Sets. (except last statement about exactness becomes: F preserves colimits, G preserves limits)