An Introduction to Smoothed Particle Hydrodynamics
By Martin Robinson, Multiscale Mechanics, University of Twente

Outline
1. Introduction to SPH
   - Kernel Interpolation
   - First and second derivatives
   - Derivation of main SPH equations

Smoothed Particle Hydrodynamics
- Smoothed Particle Hydrodynamics is a Lagrangian scheme
- The fluid is discretised into particles that move with the fluid velocity

Introduction
- Grew out of the astrophysics community in the late seventies

Some advantages:
- Lack of mesh simplifies problems involving complex, moving and deforming geometries (eg. Solid fracture), free-surfaces and multi-phase interfaces.
- Advection is obtained from the movement of the particles and the time history of fluid particles is easy to obtain (eg. Mixing visualization/measurement).
- Strong conservation properties for energy, linear and angular momentum.
- No computational effort where there is no mass

Common Application Areas:
- Astrophysics (eg. Binary star systems, shocks, impact of planetesimals)
- Marine or coastal hydrodynamics (eg. Breaking waves, sloshing in gas tankers, gravity currents)
- Fluid Structure Interaction.
- Industrial flows (eg. die casting, resin transfer moulding, sag mills)
- Deformation or fracture of brittle or elastic solids
**SPH Basics – Kernel Interpolation**

- Smoothed Particle Hydrodynamics is a Lagrangian scheme, whereby the fluid is discretised into particles that move with the fluid velocity.
- Navier-Stokes equations are discretised using kernel interpolation between particles.

\[
A_i(x) = \int A(x') W(x' - x, h) \, dx'
\]

\[
A_i(x) = \sum_j A_j W(x_j - x, h) \frac{m_j}{\rho_j}
\]

\[
A_i = \sum_j A_j W(x_i - x_j, h) \frac{m_j}{\rho_j}
\]

**Integral Interpolant**
**Summation Interpolant**
**Calculated on particle a**

**SPH Basics – Kernel Interpolation**

- SPH Kernels are Gaussian-like with a compact support.
- Cubic Spline kernel commonly used:

\[
W(q) = \begin{cases} 
\frac{4}{3} - 6q^2 + 3q^3 & \text{for } 0 \leq q < 1 \\
(2 - q)^3 & \text{for } 1 \leq q < 2 \\
0 & \text{for } q > 2 
\end{cases}
\]

\[
\beta = \frac{1}{6}, \frac{5}{14\pi}, \frac{1}{4\pi} \text{ for one, two and three dimensions}
\]

**Example – interpolate density**

\[
A_v = \sum_j m_j \frac{A_j}{\rho_j} W_{ab}
\]

\[
\rho_v = \sum_j m_j \frac{\rho_j}{\rho_j} W_{ab}
\]

Let \( A = \rho \)

**Example – interpolate velocity**

\[
A_v = \sum_j m_j \frac{A_j}{\rho_j} W_{ab}
\]

\[
v_v = \sum_j m_j \frac{v_j}{\rho_j} W_{ab}
\]

Let \( A = v \)

**Example – interpolate density**

\[
\rho_v = \sum_j m_j W_{ab}
\]

- Physical interpretation: SPH particles represent a constant mass of fluid.
- Density field is particle mass smoothed according to the kernel.
- Particle smoothing length can vary according to density:

\[
h_i = \left( \frac{m_i}{\rho_i} \right)^{1/3}
\]

Ideally, iterate the calculations of \( h \) and \( \rho \) each timestep until they converge.
Example – interpolate velocity (optimized version)

\[ v_i = \frac{1}{\rho_i} (v \rho)_i \]

\[ A_i = \sum_m \frac{m_i}{\rho_i} v_i \rho_i W_{ab} \]

\[ = \frac{1}{\rho_i} \sum_m m_i v_i \rho_i W_{ab} \]

\[ = \frac{1}{\rho_i} \sum_m m_i v_i \rho_i W_{ab} \]

Shepard Correction

- Use Shepard correction to make interpolation more accurate near boundaries/free surface:

\[ A_i(x) = \frac{1}{\sum_m \rho_i W_{ab}} \sum m_i A_i \rho_i W_{ab} \]

- Correction denominator is \(<1\) for kernel summations with full support, but \(<1\) near a free surface or some boundaries

Interpolation of Derivatives

- First approach:

\[ \nabla A_i = \sum_m \frac{m_i}{\rho_i} \nabla A_i \nabla W_{ab} \]

- But, does not vanish when \( A(x) \) is constant! Instead use:

\[ \nabla A_i = \frac{1}{\Phi} (\nabla (\Phi A) - A \nabla \Phi) \]

\[ = \frac{1}{\Phi} \sum_m \frac{m_i}{\rho_i} (A_i - A) \nabla W_{ab} \]

\[ \nabla A_i \cdot (a_i - a_j) \cdot \nabla W_{ab} \]

Example – Continuity Equation

\[ \frac{DP}{dt} = -\rho \nabla \cdot v \]

- Use SPH derivatives to evaluate divergence of velocity

\[ \nabla \cdot (a_i - a_j) \cdot \nabla W_{ab} \]

\[ \frac{d\rho}{dt} = \rho \sum_m \frac{m_i}{\rho_i} (v_i - v_j) \cdot \nabla W_{ab} \]

\[ = \sum_m m_i v_i \cdot \nabla W_{ab} \]

Example – Internal Energy

- From first law of thermodynamics:

\[ dTds = dU - PdV \]

\[ 0 \geq \frac{du}{dt} - \frac{P}{\rho^2} \frac{dp}{dt} \]

Let \( a = v \)

Let \( \Phi = p \)

\[ \frac{d\rho}{dt} = \frac{\rho}{\rho_i} \sum_m m_i v_i \cdot \nabla W_{ab} \]
Example – The Pressure Gradient

- Momentum equation from Euler equations:
  \[
  \frac{DV}{Dt} = -\frac{1}{\rho} \nabla P
  \]
  \[
  \nabla A = \frac{1}{\Phi} (V(\Phi A) - A \nabla \Phi)
  \]
  \[
  \frac{1}{\rho} \nabla P = \frac{1}{\rho} \rho \left( \nabla (P/\rho) - \nabla(1/\rho) \right)
  \]
  \[
  \nabla (P/\rho) - \frac{\rho}{\rho^2} \nabla \rho
  \]

Let \( A = P \)
Let \( \Phi = \frac{1}{\rho} \)

Artificial Viscosity

- Artificial viscosity originally developed to stabilise shocks and prevent the build-up of acoustic energy due to integration errors.
  \[
  \frac{dV}{dt} = -\sum \frac{m_i}{\rho_i} \left( \frac{P_i}{\rho_i^2} + \frac{P}{\rho^2} + \Pi_{ab} \right) \nabla W_{ab}
  \]
- Modern SPH viscosity terms approximate the viscous term in the Navier-Stokes equations

Second Derivatives in SPH

- Better approach: construct a second derivative in SPH to approximate \( \mu \nabla^2 \nu \)
  \[
  \left( \frac{d\nu}{dx} \right)^2 \approx \sum \frac{m_i}{\rho_i} \frac{dW_i}{dx_i}
  \]
- But:
  \- Sensitive to particle disorder
  \- Sign of this expression can change with particle separation due to second derivative of the kernel.

Artificial Viscosity – some examples

- \( \Pi_{ab} = -\frac{\varepsilon}{\rho_{ab}} \frac{v_{ab} \cdot r_{ab}}{r_{ab}^2 + \delta r_{ab}^2} \)
  Monaghan and Gingold (1983)
- \( \Pi_{ab} = -\frac{16\mu_1}{\rho (\mu_1 + \mu_2 \mu_1)} \frac{v_{ab} \cdot r_{ab}}{r_{ab}^2 + \delta r_{ab}^2} \)
  Cleary (1999)

- Conserves linear and angular momentum.
- But derivation is rather ad-hoc.

Second Derivatives in SPH

- More stable and accurate -> start with an integral approximation (Cleary and Monaghan 1999, heat conduction)
  \[
  \nabla \cdot \kappa \nabla T + O(h^2) = \int \left( \kappa(T) + \kappa(T') \right) \nabla (T - T') \cdot \frac{1}{|r - r'|} \nabla W(|r - r'|) \, dr' = \sum \frac{m_i}{\rho_i} \left( k_i + k_i' \right) \frac{dW_i}{\delta r_i}
  \]
  \[
  \nabla \cdot \kappa \nabla T = \sum \frac{m_i}{\rho_i} \left( k_i + k_i' \right) \left( \nabla v_i - \frac{1}{\rho_i} \frac{dW_i}{\delta r_i} \right)
  \]
- (Morris 1997) applied this to the viscosity term
  \[
  \nabla \cdot \mu \nabla v = \sum \frac{m_i}{\rho_i} \left( k_i + k_i' \right) \left( \nabla v_i - \frac{1}{\rho_i} \frac{dW_i}{\delta r_i} \right)
  \]

Note: Does not conserve angular momentum
Summary

- Navier Stokes Equations for a Newtonian fluid:
  \[
  \frac{D\rho}{Dt} = -\nabla P + \rho \frac{\partial v}{\partial t} + \mu \nabla^2 v
  \]
- Calculate SPH derivatives using kernel interpolation:
  \[
  \frac{dv_i}{dt} = \sum m_i \left( \frac{p}{\rho_i} \mathbf{a}_i + \Pi_i \right) \mathbf{V}_W - \sum m_i \mathbf{V}_W \frac{dp_i}{dt}
  \]
- Use integral approximation for the second derivative to find the viscous term \( \Pi \):
  \[
  \rho \frac{\partial v}{\partial t} = \frac{1}{\rho} \sum \frac{m_i}{\rho_i} (\mu_i + \mu) (\mathbf{v}_i - \mathbf{v}) \frac{1}{\rho_i} \frac{\partial W_{ab}}{\partial t}
  \]

Example SPH algorithm

- Let's use velocity verlet integration.....
- For each timestep:
  \[
  v^{i+1} = v + \Delta t \frac{v}{2},
  \]
  \[
  h^{i+1} = F_0(p^{i+1}) = \sigma(p^{i+1}) \frac{\rho^{i+1}}{\rho^{i}} v^{i+1},
  \]
  \[
  \rho^{i+1} = F_1(v^{i+1}, h^{i+1}) = \sum m_i W
  \]
  \[
  v^{i+1} = v^{i+1} + \Delta t \frac{\partial h^{i+1}}{\partial t},
  \]
  \[
  r^{i+1} = r^{i+1} + \frac{\Delta t}{2}
  \]

Example SPH algorithm

- Need to update density and h field each timestep. Here we use the summation form of the density equation.
- For each timestep:
  \[
  r^{i+1} = r + \frac{\Delta t}{2} \frac{\mathbf{v}}{\rho},
  \]
  \[
  h^{i+1} = F_0(p^{i+1}) = \sigma(p^{i+1}) \frac{\rho^{i+1}}{\rho^{i}} \mathbf{v},
  \]
  \[
  \rho^{i+1} = F_1(v^{i+1}, h^{i+1}) = \sum m_i W
  \]
  \[
  v^{i+1} = v^{i+1} + \Delta t \frac{\partial h^{i+1}}{\partial t},
  \]
  \[
  r^{i+1} = r^{i+1} + \frac{\Delta t}{2}
  \]

Example SPH algorithm

- Also need an Equation Of State....
- For each timestep:
  \[
  s^{i+1} = s + \frac{\Delta t}{2} \frac{\mathbf{v}}{\rho},
  \]
  \[
  h^{i+1} = F_0(p^{i+1}) = \sigma(p^{i+1}) \frac{\rho^{i+1}}{\rho^{i}} \mathbf{v},
  \]
  \[
  \rho^{i+1} = F_1(v^{i+1}, h^{i+1}) = \sum m_i W
  \]
  \[
  v^{i+1} = v^{i+1} + \Delta t \frac{\partial h^{i+1}}{\partial t},
  \]
  \[
  r^{i+1} = r^{i+1} + \frac{\Delta t}{2}
  \]

Example SPH algorithm

- Finally add viscosity.....
- For each timestep:
  \[
  s^{i+1} = s + \frac{\Delta t}{2} \frac{\mathbf{v}}{\rho},
  \]
  \[
  h^{i+1} = F_0(p^{i+1}) = \sigma(p^{i+1}) \frac{\rho^{i+1}}{\rho^{i}} \mathbf{v},
  \]
  \[
  \rho^{i+1} = F_1(v^{i+1}, h^{i+1}) = \sum m_i W
  \]
  \[
  v^{i+1} = v^{i+1} + \Delta t \frac{\partial h^{i+1}}{\partial t},
  \]
  \[
  r^{i+1} = r^{i+1} + \frac{\Delta t}{2}
  \]

Time step conditions

- Time step calculated on a Courant–Friedrichs–Lewy (CFL) condition:
  \[
  \Delta t < \frac{h}{v_{s}}
  \]
  \[
  v_{s} = \frac{1}{2} (c_s + c_e - 2w) \frac{\mathbf{v}}{\rho}
  \]
- \( v_{s} \) is a signal velocity equal to the speed of information exchange between particles
Time step conditions

- Viscous timescale
  \[ \Delta t \ll \frac{h^2}{v} \]
- Acceleration timescale
  \[ \Delta t \ll \frac{h}{\sqrt{div \mathbf{f}}} \]

Finding Neighbours

1. Domain divided into “bins” of side length \(2h_{\text{max}}\)
2. Particles placed in bins
3. Each particle’s neighbours can be found in its own bin and neighbouring bins
4. Search these bins, keeping particles within 2h.

The End

- Exercise 9: 1D Sod Shock using SPH....