Lectures on classical algebraic geometry (C3.4 Algebraic Geometry). All lectures.

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Contents

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1 Introduction

Classical algebraic geometry is the study of the sets of of simultaneous solutions of collections of polynomial equations in several variables with coefficients in an algebraically closed field. Such sets are called algebraic varieties. So eg the set of simultaneous solutions of the equations $x^2 + y^2 - 1 = 0, xy = 0$ in $\mathbb{C}^2$ is an algebraic variety.

Because they are so easy to define, algebraic varieties appear in almost every area of mathematics. They play a crucial role in number theory, in topology, in differential geometry and complex geometry (ie the theory of complex manifolds). When the base field is $\mathbb{C}$, an algebraic variety defines a complex manifold provided it has "no kinks" (we shall give a precise definition later).

A basic reference for classical algebraic geometry is chap. I of D. Mumford's book *The Red Book of Varieties and Schemes* (Springer Lecture Notes in Mathematics 1358). Another reference is chap. I of R. Hartshorne's book *Algebraic Geometry* (Springer). One might also consult the book by M. Reid *Undergraduate algebraic geometry* (London Mathematical Society Student Texts **12**, Cambridge University Press 1988). An updated free online version of M. Reid's lectures can be found under

`https://homepages.warwick.ac.uk/staff/Miles.Reid/MA4A5/UAG.pdf`

The natural generalisation of classical algebraic geometry is the theory of schemes, which will be taught in Hilary Term. In Grothendieck's theory of schemes, the base field can be replaced by any commutative ring but the absence of Hilbert's Nullstellensatz, which is at the root of the material presented here, means that different techniques have to be used.

There are three important tools, which will not be presented in this course:

- The theory of sheaves
- Cohomological techniques
- The technique of base change

These tools are very powerful but there will not be enough time to present them in these lectures . Also, the best framework for them is the theory of schemes (although they could also be used in the restricted setting of this text).

There is also a tool from Commutative Algebra, which will not be used here but which is very useful in Algebraic Geometry: the tensor product of modules over a ring. Tensor products are ubiquitous in the theory of schemes.

The prerequisites for this course are the part A course *Rings and Modules* and the part B course *Commutative Algebra*. It is assumed that the reader is familiar with the terminology used in the notes of the commutative algebra course. We shall often quote results proven in that course, referring to it as "CA". I have put the CA notes on the web page of the present course for easy reference.

Throughout the course, we shall work over a fixed algebraically closed field k. As in the CA course, a ring will be a commutative ring with unit, unless stated otherwise. The reader may assume that for any $n \geqslant 1$, the ring of polynomials $k[x_1, \ldots, x_n]$ is a UFD (Unique Factorisation Domain). It can also be assumed that the localisation $k[x_1, \ldots, x_n]_S$ is a UFD for any multiplicative set $S \subseteq k[x_1, \ldots, x_n]$.

2 Hilbert's Nullstellensatz and algebraic sets

Let $n \geqslant 0$ and let $R_n := k[x_1, \dots, x_n]$. Let $\Sigma \subseteq R_n$. The *algebraic set* associated with Σ is

$$\mathrm{Z}(\Sigma) = \text{zero set of } \Sigma := \{(t_1, \dots, t_n) \in k^n \,|\, \forall P \in I : P(t_1, \dots, t_n) = 0\}$$

Note the following simple fact. Let ΣR_n be the ideal generated by Σ in R_n. Then we have $\mathrm{Z}(\Sigma) = \mathrm{Z}(\Sigma R_n)$.

We now recall two basic results in commutative algebra.

Theorem 2.1 (Hilbert's basis theorem; see Th. 7.6 in CA)**.** *The ring $k[x_1, \dots, x_n]$ is noetherian.*

Recall that a noetherian ring is a ring all of whose ideals are finitely generated. In particular, by the remark above any algebraic set in k^n is the zero set of a finite number of polynomials.

Theorem 2.2 (Hilbert's strong Nullstellensatz; see Cor. 9.5 in CA)**.** *For any ideal $I \subseteq R_n$ we have*

$$\mathfrak{r}(I) = \{P \in R_n \,|\, \forall (t_1, \dots, t_n) \in \mathrm{Z}(I) : P(t_1, \dots, t_n) = 0\}$$

Recall that $\mathfrak{r}(I)$ is the radical (or nilradical) of I, ie the intersection of all the prime ideals of R_n containing I. One can show that $\mathfrak{r}(I)$ consists of all the elements Q of R_n, such that $Q^l \in I$ for some $l = l(Q)$ (see Prop. 3.2 in CA). Recall also that a radical ideal is an ideal which coincides with its own radical.

If $A \subseteq k^n$ is subset, we shall write

$$\mathcal{I}(A) := \{P \in R_n \,|\, \forall (t_1, \dots, t_n) \in A : P(t_1, \dots, t_n) = 0\}.$$

The set $\mathcal{I}(A)$ is clearly and ideal in R_n. Note that in terms of the operator $\mathcal{I}(\cdot)$, the strong HNS implies that $\mathcal{I}(\mathrm{Z}(I)) = \mathfrak{r}(I)$ for any ideal of R_n.

We may now prove the basic

Proposition 2.3. *Let $V \subseteq k^n$ be an algebraic set and let $I \subseteq R_n$ be an ideal. Then the identities $\mathrm{Z}(I) = \mathrm{Z}(\mathfrak{r}(I))$, $\mathcal{I}(\mathrm{Z}(I)) = \mathfrak{r}(I)$ and $\mathrm{Z}(\mathcal{I}(V)) = V$ hold.*

In particular, the two maps

$$\{\text{algebraic sets in } k^n\} \underset{Z}{\overset{\mathcal{I}}{\rightleftarrows}} \{\text{radical ideals in } R_n\}$$

are inverse to each other. Note that in this correspondence, we have $V_1 \subseteq V_2$ iff $\mathrm{Z}(V_1) \supseteq \mathrm{Z}(V_2)$ for any two algebraic sets V_1 and V_2 (why?).

Proof. (of Proposition 2.3) The identity $\mathrm{Z}(I) = \mathrm{Z}(\mathfrak{r}(I))$ follows from the definitions and the identity $\mathcal{I}(\mathrm{Z}(I)) = \mathfrak{r}(I)$ was noted just before the statement of the proposition. We thus only have to prove that $\mathrm{Z}(\mathcal{I}(V)) = V$. To see this, note that by definition we have $V \subseteq \mathrm{Z}(\mathcal{I}(V))$. On the other hand, by definition $V = \mathrm{Z}(J)$ for some ideal J in $k[x_1, \dots, x_n]$. By construction, we have $J \subseteq \mathcal{I}(V)$, so $\mathrm{Z}(J) = V \supseteq \mathrm{Z}(\mathcal{I}(V))$. Hence $V = \mathrm{Z}(\mathcal{I}(V))$. $\square$

We also note the following identities, whose proof is left as an exercise for the reader:

(1) $\mathcal{I}(V_1 \cup V_2) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$

(2) $\mathcal{I}(\cap_i V_i) = \mathfrak{r}(\sum_i \mathcal{I}(V_i))$

(3) $\mathrm{Z}(I_1 \cap I_2) = \mathrm{Z}(I_1) \cup \mathrm{Z}(I_2)$

(4) $\mathrm{Z}(\sum_i I_i) = \cap_i \mathrm{Z}(I_i)$

(where the V_i are algebraic sets, the I_i are ideals and the symbol $\sum$ refers to the sum of ideals).

In view of the properties (4) and (3) above, the algebraic sets in k^n can be viewed as the closed sets of a topology on k^n, called the *Zariski topology*. If $V \subseteq k^n$ is an algebraic set, we endow V with the topology induced by the Zariski topology of k^n. This topology is called the Zariski topology of V.

We can refine the correspondence above as follows.

Say that an algebraic set $V \subseteq k^n$ is *reducible* if $V = V_1 \cup V_2$, where $V_1, V_2 \subseteq k^n$ are non empty algebraic sets, $V_1 \not\subseteq V_2$ and $V_2 \not\subseteq V_1$. An algebraic set $V \subseteq k^n$ is said to be *irreducible* if it is not reducible. One verifies from the definition that an algebraic set is irreducible iff all its non empty open subsets are dense.

For the following two lemmata, we shall need the following result from CA:

Theorem 2.4. *Let R be a noetherian commutative ring and let $I \subseteq R$ be a radical ideal. Then there is unique finite set of prime ideals $\{\mathfrak{p}_l\}$ such that $I = \cap_l \mathfrak{p}_l$ and such that for all indices l we have $\mathfrak{p}_l \not\supseteq \cap_{j \neq l} \mathfrak{p}_j$. Furthermore, the $\mathfrak{p}_l$ are the prime ideals of R, which are minimal for the inclusion relation among the prime ideals containing I.*

Proof. This follows from the Lasker-Noether theorem (see Prop. 7.8 in CA) and the remark after Th. 6.7 in CA. □

Lemma 2.5. *Let $V \subseteq k^n$ be an algebraic subset. Then V is irreducible iff $\mathcal{I}(V)$ is a prime ideal.*

Proof. "$\Leftarrow$": Suppose that V is reducible. Then $V = V_1 \cup V_2$, where V_1 and V_2 are two algebraic subsets not contained in each other (and in particular not empty). By property (1) above, we have $\mathcal{I}(V) = \mathcal{I}(V_1) \cap \mathcal{I}(V_2)$, where $\mathcal{I}(V_1)$ and $\mathcal{I}(V_2)$ are two ideals not contained in each other. In particular, there is $a_1 \in \mathcal{I}(V_1)$ such that $a_1 \notin \mathcal{I}(V_2)$ and $a_2 \in \mathcal{I}(V_2)$ such that $a_2 \notin \mathcal{I}(V_1)$. In particular $a_1, a_2 \notin \mathcal{I}(V)$. On the other hand $a_1 a_2 \in \mathcal{I}(V)$ so that $\mathcal{I}(V)$ is not prime.

"$\Rightarrow$": Suppose that $\mathcal{I}(V)$ is not prime. Let $\{\mathfrak{p}_l\}_{l \in \Lambda}$ be the set of prime ideals in R, which are minimal among the prime ideals containing $\mathcal{I}(V)$. By Theorem 2.4 we know that Λ is finite and that $\mathcal{I}(V) = \cap_l \mathfrak{p}_l$. Hence $\#\Lambda > 1$ since $\mathcal{I}(V)$ is not prime. Let l_1 be any element of Λ. By Theorem 2.4 again (or Prop. 6.1 (ii) in CA and the minimality of the $\mathfrak{p}_l$) we have $\mathfrak{p}_{l_1} \not\supseteq \cap_{l \neq l_1} \mathfrak{p}_l$. On the other hand, we also have $\mathfrak{p}_{l_1} \not\subseteq \cap_{l \neq l_1} \mathfrak{p}_l$ by minimality. Hence $\mathrm{Z}(\mathfrak{p}_{l_1}) \not\subseteq \mathrm{Z}(\cap_{l \neq l_1} \mathfrak{p}_l)$ and $\mathrm{Z}(\mathfrak{p}_{l_1}) \not\supseteq \mathrm{Z}(\cap_{l \neq l_1} \mathfrak{p}_l)$. Finally, we have $\mathrm{Z}(\mathcal{I}(V)) = V = \mathrm{Z}(\mathfrak{p}_{l_1}) \cup \mathrm{Z}(\cap_{l \neq l_1} \mathfrak{p}_l)$ by (3) above and Proposition 2.3 so that V is reducible. □

Lemma 2.6. *Let $V \subseteq k^n$ be an algebraic set. Then there is a unique finite collection $\{V_l\}_{l \in \Lambda}$ of irreducible algebraic subsets of k^n such that*

(1) $V = \cup_l V_l$;

(2) $\forall l : V_l \not\subseteq \cup_{j \neq l} V_j$.

Furthermore, the V_l are the irreducible algebraic sets in k^n, which are maximal among the irreducible algebraic sets contained in V.

Proof. In view of the remark after Prop. 2.3, the properties (1)...(4) above and Lemma 2.5, this is equivalent to Theorem 2.4 for $R = R_n$. $\square$

Example. The algebraic set defined by the equation $x_1x_2 = 0$ in k^2 has the irreducible components $x_1 = 0$ and $x_2 = 0$. Indeed, the two components are clearly not contained in each other and $x_i = 0$ defines an irreducible algebraic set because $k[x_1, x_2]/(x_i) \simeq k[x]$ and thus the ideal (x_i) is prime in $k[x_1, x_2]$ (use Lemma 2.5).

For the next proposition, we shall need the weak Nullstellensatz (see Cor. 9.2 in CA). This states the following. Let $L \hookrightarrow K$ be a map of fields and suppose that K is finitely generated as an L-algebra. Then the map $L \to K$ is finite (meaning that K is a finite field extension of L via $L \hookrightarrow K$).

Proposition 2.7. *Let $V \subseteq k^n$ be an algebraic set defined by a radical ideal I. Let $\bar{v} = \langle v_1, \dots, v_n \rangle \in V$ and let $\mathfrak{m}$ be a maximal ideal of R_n. Suppose that $\mathfrak{m} \supseteq I$. Then*

(1) $\mathcal{I}(\{\bar{v}\}) \supseteq I$ and $\mathcal{I}(\{\bar{v}\})$ is a maximal ideal of R_n;

(2) $\mathrm{Z}(\mathfrak{m})$ consists of one point $\bar{u} = \langle u_1, \dots, u_n \rangle \in V$ and $\bar{u} \in V$;

(3) $\mathfrak{m} = (x_1 - u_1, \dots, x_n - u_n)$ where $\bar{u}$ is as in (2).

Proof. (1) The fact that $\mathcal{I}(\{\bar{v}\}) \supseteq I$ follows from the definitions. Now note that a point of V is an algebraic subset of V, which is minimal among all the algebraic subsets of V, which are not empty. From the correspondence between algebraic sets and radical ideals, we thus see that $\mathcal{I}(\{\bar{v}\})$ is a radical ideal, which is maximal among all the non-trivial ideals radical ideals containing I. Such an ideal is a maximal ideal of R_n (why?).

(2) and (3) From the weak Nullstellensatz, we see that the natural map $k \hookrightarrow R_n/\mathfrak{m}$ is a finite field extension of k. Hence the natural map $k \hookrightarrow R_n/\mathfrak{m}$ is an isomorphism, since k is algebraically closed. Let $u_1, \dots, u_n \in k$ be the elements of k corresponding to $x_1 \pmod{\mathfrak{m}}, \dots, x_n \pmod{\mathfrak{m}}$ via this isomorphism. Then $\mathfrak{m} = (x_1 - u_1, \dots, x_n - u_n)$ (since we have $\mathfrak{m} \supseteq (x_1 - u_1, \dots, x_n - u_n)$ and the ideal $(x_1 - u_1, \dots, x_n - u_n)$ is maximal) and in particular $\mathrm{Z}(\mathfrak{m}) = \{\bar{u}\}$. $\square$

The last proposition in particular provides a correspondence between the points of V and the maximal ideals of R_n containing $\mathcal{I}(V)$, or equivalently with the maximal ideals of $R_n/\mathcal{I}(V)$. In other words, if we write for any ring R

$$\mathrm{Spm}(R) := \{\text{maximal ideals of } R\}$$

then there is a natural bijection $\mathrm{Spm}(R_n/\mathcal{I}(V)) \simeq V$.

Lemma 2.8. *Let $V \subseteq k^n$ be an algebraic set. Under the bijection $\mathrm{Spm}(R_n/\mathcal{I}(V)) \simeq V$, the closed subsets of V correspond the subsets of $\mathrm{Spm}(R_n/\mathcal{I}(V))$ of the form*

$$\mathrm{Z}(S) := \{\mathfrak{m} \in \mathrm{Spm}(R_n/\mathcal{I}(V)) \mid \mathfrak{m} \supseteq S\}$$

where $S \subseteq R_n/\mathcal{I}(V)$. The closed subsets of V are in one to one correspondence with the radical ideals of $R_n/\mathcal{I}(V)$ via $\mathrm{Z}(\cdot)$.

Proof. Left to the reader. Unroll the definitions. $\square$

Note that the set $\{\mathfrak{m} \in \mathrm{Spm}(R_n/\mathcal{I}(V)) \mid \mathfrak{m} \supseteq S\}$ corresponds in V to the set $\mathrm{Z}(S') \cap V$ for any lifting of S to R_n. So the notation $\mathrm{Z}(S)$ will not lead to any confusion. Also, if $C \subseteq V$ is a closed subset, then we have $C = \mathrm{Z}(\mathcal{I}(C) \pmod{\mathcal{I}(V)}) = \mathrm{Z}(\mathcal{I}(C)) \cap V$. So we will sometimes use the shorthand $\mathcal{I}(C)$ for $\mathcal{I}(C) \pmod{\mathcal{I}(V)} \subseteq R_n/\mathcal{I}(V)$ if C is a closed subset of V. With this notation, the properties (1),. . . , (4) listed above are also valid for the correspondence described in Lemma 2.8.

3 Regular maps between algebraic sets

Let $n, t \geqslant 0$. A map $\phi : k^n \to k^t$ is said to be *polynomial* if there are elements $P_1(x_1, \dots, x_n), \dots, P_t(x_1, \dots, x_n) \in R_n = k[x_1, \dots, x_n]$, such that

$$\phi(a_1, \dots, a_n) = \langle P_1(a_1, \dots, a_n), \dots, P_t(a_1, \dots, a_n)\rangle$$

for all $\langle a_1, \dots, a_n\rangle \in k^n$.

Note that the polynomials P_i define a map of k-algebras $\phi^* : R_t \to R_n$ by the formula

$$\phi^*(Q(y_1, \dots, y_t)) := Q(P_1(x_1, \dots, x_n), \dots, P_t(x_1, \dots, x_n))$$

and on the other hand, if we are given a map of k-algebras $\Phi : k[y_1, \dots, y_t] = R_t \to R_n = k[x_1, \dots, x_n]$, then we can define polynomials $T_1(x_1, \dots, x_n), \dots, T_t(x_1, \dots, x_n) \in R_n$ by the formula

$$T_i(x_1, \dots, x_n) := \Phi(y_i)$$

and these two processes are obviously inverse to each other. So to give polynomials P_i as above is equivalent to giving a map of k-algebras $R_t \to R_n$. Note that from definitions we see that the composition of two polynomials maps is a polynomial map.

If $\Phi : R_t \to R_n$ is a map of k-algebras, we shall write $\mathrm{Spm}(\Phi) : k^n \to k^t$ for the corresponding polynomial map (defined as above from the polynomials arising from Φ).

Lemma 3.1. *The map*

$$\mathrm{Spm} : \{\textit{maps of } k\textit{-algebras } R_t \to R_n\} \to \{\textit{polynomial maps } k^n \to k^t\}$$

is bijective.

Proof. The surjectivity of Spm is a tautology so we only have to prove injectivity. Let $\Phi_1, \Phi_2 : R_t \to R_n$ be two maps of k-algebras. Suppose that $\mathrm{Spm}(\Phi_1) = \mathrm{Spm}(\Phi_2)$. We have to prove that $\Phi_1 = \Phi_2$. Suppose that Φ_1 (resp. Φ_2) is defined by polynomials $P_{11}(x_1, \dots, x_n), \dots, P_{1t}(x_1, \dots, x_n)$ (resp. $P_{21}(x_1, \dots, x_n), \dots, P_{2t}(x_1, \dots, x_n)$). Let $i \in \{1, \dots, t\}$. If $\mathrm{Spm}(\Phi_1) = \mathrm{Spm}(\Phi_2)$ then the polynomial $P_{1i} - P_{2i}$ vanishes for all the values of its variables. This implies that $P_{1i} = P_{2i}$ (why?). Since i was arbitrary, we conclude that $\Phi_1 = \Phi_2$. □

In view of the lemma, for any polynomial map $\phi : k^n \to k^t$, there is a unique map of k-algebras $\phi^* : R_t \to R_n$ such that $\mathrm{Spm}(\phi^*) = \phi$. Note that the operation $(\cdot)^*$ (resp. $\mathrm{Spm}(\cdot)$) is compatible with composition of polynomial maps (resp. composition of maps of k-algebras). This follows from the definitions.

Let now $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic sets in k^n and k^t, respectively. A map $\psi : V \to W$ is said to be *regular* if there is a polynomial map $\phi : k^n \to k^t$ such that $\phi(V) \subseteq W$ and such that $\psi(v) = \phi(v)$ for all $v \in V$. Note that if ψ is given, there might be several different ϕ inducing ψ (what is an obvious example of this phenomenon?). Note also that a regular map is continuous for the Zariski topology (why?). Also, a composition of regular maps is regular (unroll the definitions).

We shall attempt to generalise Lemma 3.1 to algebraic sets.

For this, we make the following definition.

Definition 3.2. *Let $V \subseteq k^n$ be an algebraic set. The coordinate ring $\mathcal{C}(V)$ of V is the ring*

$$\mathcal{C}(V) := R_n/\mathcal{I}(V).$$

Note that since $\mathcal{I}(V)$ is a radical ideal (see above - this also follows directly from the definitions), the ring $\mathcal{C}(V)$ is a reduced ring, ie the only nilpotent element of $\mathcal{C}(V)$ is the zero element. We also recall that any finitely generated algebra over a field is a Jacobson ring (see Cor. 9.4 in CA). In particular, $\mathcal{C}(V)$ is a Jacobson ring. Recall that a Jacobson ring R is a ring such that for any ideal $I \subseteq R$, we have

$$\cap_{\mathfrak{m}\in \mathrm{Spm}(R), \mathfrak{m}\supseteq I} = \cap_{\mathfrak{p}\in \mathrm{Spec}(R), \mathfrak{p}\supseteq I} =: \mathfrak{r}(I)$$

where $\mathrm{Spec}(R)$ is the set of prime ideals of R (see section 4 in CA).

Let again $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic sets in k^n and k^t, respectively. Let $\psi : V \to W$ be a regular map and let $\phi : k^n \to k^t$ be a polynomial map inducing ψ, as above. Suppose that $\phi = \mathrm{Spm}(\Phi)$ for the map of k-algebras $\Phi : R_t \to R_n$.

Lemma 3.3. *We have $\Phi(\mathcal{I}(W)) \subseteq \mathcal{I}(V)$.*

Proof. Since R_n is a Jacobson ring, the ideal $\mathcal{I}(V)$ is the intersection of all the maximal ideals containing it. We thus only have to prove that for any maximal ideal $\mathfrak{m}$ containing $\mathcal{I}(V)$, we have $\Phi^{-1}(\mathfrak{m}) \supseteq \mathcal{I}(W)$. Now by Prop. 2.7, we know that the maximal ideals containing $\mathcal{I}(V)$ are precisely the ideals of the form $(x_1 - v_1, \dots, x_n - v_n)$, where $\bar{v} := \langle v_1, \dots, v_n\rangle \in V$. So we have to prove that $\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n)) \supseteq \mathcal{I}(W)$, when $\langle v_1, \dots, v_n\rangle \in V$. So fix $\langle v_1, \dots, v_n\rangle \in V$. Note first that $\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n))$ is maximal in $\mathcal{C}(W)$ because there is by construction an injection of k-algebras

$$\mathcal{C}(W)/\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n)) \hookrightarrow \mathcal{C}(V)/(x_1 - v_1, \dots, x_n - v_n) \simeq k$$

so that $\mathcal{C}(W)/\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n)) \simeq k$ (isomorphism of k-algebras). On the other hand, any maximal ideal in $R_t = k[y_1, \dots, y_t]$ is likewise of the form $(y_1 - u_1, \dots, y_t - u_t)$ by Proposition 2.7. So in order to determine the ideal $\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n))$ we only need to find $u_1, \dots, u_t \in k$ such that $\Phi(y_i - u_i) \in (x_1 - v_1, \dots, x_n - v_n)$. By the correspondence between algebraic sets and radical ideals, the condition $\Phi(y_i - u_i) \in (x_1 - v_1, \dots, x_n - v_n)$ is equivalent to the condition that the polynomial $\Phi(y_i - u_i)$ vanishes on $\langle v_1, \dots, v_n\rangle$. We compute

$$\Phi(y_i - u_i)(\langle v_1, \dots, v_n\rangle) = \Phi(y_i)(\langle v_1, \dots, v_n\rangle) - u_i = \phi_i(\langle v_1, \dots, v_n\rangle) - u_i$$

where ϕ_i is the projection of the map $\phi : k^n \to k^t$ to the i-th coordinate. We thus see that $\Phi(y_i - u_i)$ vanishes on $\langle v_1, \dots, v_n\rangle$ for all $i \in \{1, \dots, t\}$ iff $\phi(\langle v_1, \dots, v_n\rangle) = \langle u_1, \dots, u_t\rangle$. Hence

$$\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n)) = (y_1 - \phi_1(\bar{v}), \dots, y_t - \phi_t(\bar{v})).$$

Since $\phi(\bar{v}) \in W$ by assumption, we conclude that $\Phi^{-1}((x_1 - v_1, \dots, x_n - v_n)) \supseteq \mathcal{I}(W)$, which is what we wanted to prove. □

From the lemma, we see that Φ induces a map of k-algebras $\Phi_{V,W} : \mathcal{C}(W) \to \mathcal{C}(V)$.

Note the following important fact, we was established in the course of the proof of Lemma 3.3:

Lemma 3.4. *If $\bar{v} := \langle v_1, \ldots, v_n \rangle \in V$ then the maximal ideal corresponding to $\psi(\bar{v})$ is the ideal $\Phi^*_{V,W}((x_1 - v_1, \ldots, x_n - v_n) \pmod{\mathcal{I}(V)})$. In particular, $\Phi^{-1}_{V,W}$ send maximal ideals to maximal ideals and $\Phi_{V,W}$ entirely determines $\psi : V \to W$.*

We now have the

Proposition 3.5. *The map $\Phi_{V,W} : \mathcal{C}(W) \to \mathcal{C}(V)$ depends only on ψ.*

Proof. Suppose that ψ is also induced by another polynomial map $\phi' : k^n \to k^t$, associated with a map of k-algebras $\Phi' : R_t \to R_n$. Let $\Phi'_{V,W} : \mathcal{C}(W) \to \mathcal{C}(V)$ be the map of k-algebras induced by ϕ' via Lemma 3.3. Let $r \in \mathcal{C}(W)$ and let $\mathfrak{m} \in \mathrm{Spm}(V))$. By the above remark and the assumptions, we have $(\Phi')^{-1}_{V,W}(\mathfrak{m}) = \Phi^{-1}_{V,W}(\mathfrak{m}) \in \mathrm{Spm}(\mathcal{C}(W))$. Let $\mathfrak{n} := (\Phi')^{-1}_{V,W}(\mathfrak{m}) = \Phi^{-1}_{V,W}(\mathfrak{m})$. We have commutative diagrams

$$\begin{array}{ccc} \mathcal{C}(W) & \xrightarrow{\Phi_{V,W}} & \mathcal{C}(V) \\ \downarrow & & \downarrow \\ \mathcal{C}(W)/\mathfrak{n} & \longrightarrow & \mathcal{C}(V)/\mathfrak{m} \\ \simeq \uparrow & & \simeq \uparrow \\ k & \xrightarrow{=} & k \end{array}$$

and also

$$\begin{array}{ccc} \mathcal{C}(W) & \xrightarrow{\Phi'_{V,W}} & \mathcal{C}(V) \\ \downarrow & & \downarrow \\ \mathcal{C}(W)/\mathfrak{n} & \longrightarrow & \mathcal{C}(V)/\mathfrak{m} \\ \simeq \uparrow & & \simeq \uparrow \\ k & \xrightarrow{=} & k \end{array}$$

In particular, we see that $\Phi_{V,W}(r) \pmod{\mathfrak{m}} = \Phi'_{V,W}(r) \pmod{\mathfrak{m}}$. Since $\mathfrak{m}$ was an arbitrary maximal ideal of $\mathcal{C}(V)$, we conclude that $\Phi_{V,W}(r) - \Phi'_{V,W}(r)$ lies in the Jacobson radical of $\mathcal{C}(V)$. Since $\mathcal{C}(V)$ is a Jacobson ring and is reduced, we thus see that $\Phi_{V,W}(r) = \Phi'_{V,W}(r)$. Since $r \in \mathcal{C}(W)$ was arbitrary, we conclude that $\Phi_{V,W} = \Phi'_{V,W}$. □

From the last lemma, we see that we may write $\Phi_{V,W} =: \psi^*$.

Lemma 3.6. *Let $\Lambda : \mathcal{C}(W) \to \mathcal{C}(V)$ be a map of k-algebras. Then there is a regular map $\lambda : V \to W$ such that $\lambda^* = \Lambda$.*

Proof. Let $\Lambda_0 : R_t \to R_n$ be a map of k-algebras such that the diagram

$$\begin{array}{ccc} R_t = k[y_1, \ldots, y_t] & \xrightarrow{\Lambda_0} & R_n = k[x_1, \ldots, x_n] \\ \downarrow & & \downarrow \\ \mathcal{C}(W) & \xrightarrow{\Lambda} & \mathcal{C}(V) \end{array}$$

commutes. We may obtain such a map by choosing representatives in R_n of $\Lambda(y_i \pmod{\mathcal{I}(W)})$ for each $i \in \{1, \ldots, x_t\}$. By construction, we then have $\Lambda_0(\mathcal{I}(W)) \subseteq \mathcal{I}(V)$. Applying Lemma 3.4, Lemma 3.1

and Proposition 2.7, we conclude that Λ_0 arises from a polynomial map $\mathrm{Spm}(\Lambda_0) : k^n \to k^t$ such that $\mathrm{Spm}(\Lambda_0)(V) \subseteq W$. By construction, we have $(\mathrm{Spm}(\Lambda_0)|_V)^* = \Lambda$ so we may choose $\lambda = \mathrm{Spm}(\Lambda_0)|_V$. □

From the last lemma, Lemma 3.4 and Proposition 3.5, we see that given a map of k-algebras $\Lambda : \mathcal{C}(W) \to \mathcal{C}(V)$, there is a unique regular map $\mathrm{Spm}(\Lambda) : V \to W$ such that $\mathrm{Spm}(\Lambda)^* = \Lambda$ (note that this generalises the operator $\mathrm{Spm}(\cdot)$ defined before Lemma 3.1). On the other hand, by Proposition 3.5, Lemma 3.4 and the previous lemma, given a regular map $\lambda : V \to W$, the map of k-algebras $\lambda^* : \mathcal{C}(W) \to \mathcal{C}(V)$ is the unique one such that $\mathrm{Spm}(\lambda^*) = \lambda$.

We conclude that there is a bijection from the set of regular maps $V \to W$ to the set of maps of k-algebras $\mathcal{C}(W) \to \mathcal{C}(V)$, which sends $\lambda : V \to W$ to λ^* and who inverse is given by $\mathrm{Spm}(\cdot)$.

Finally note that any finitely generated reduced k-algebra is isomorphic as a k-algebra to the coordinate ring of some algebraic set (why?).

All this leads to an intrinsic characterisation of algebraic sets and regular maps between them. We may view algebraic sets as a category whose objects are pairs (V, n) $(n \geqslant 0)$, where V is the zero set in k^n of a set of k-polynomials in n variables, and where the arrows from (V, n) to (W, t) are the maps from V to W which are restrictions of polynomial maps from k^n to k^t.

The following theorem summarises the previous discussion.

Theorem 3.7. *The category of algebraic sets is equivalent to the category of finitely generated reduced k-algebras.*

Note that in this equivalence, a finitely reduced k-algebra R is not naturally associated with an algebraic set. However if $V_1 \subseteq k^{n_1}$ and $V_2 \subseteq k^{n_2}$ are two algebraic sets such that $\mathcal{C}(V_1) \simeq \mathcal{C}(V_2)$, then the algebraic sets $V_1 \subseteq k^{n_1}$ and $V_2 \subseteq k^{n_2}$ are isomorphic, and any such isomorphism corresponds to precisely one isomorphism of k-algebras $\mathcal{C}(V_1) \simeq \mathcal{C}(V_2)$. Also, if $V \subseteq k^n$ is an algebraic set, there is a canonical identification between the set V and the set $\mathrm{Spm}(\mathcal{C}(V))$. Finally, we see from Lemma 2.8 that the topology induced on $\mathrm{Spm}(\mathcal{C}(V))$ by this identification is determined by $\mathcal{C}(V)$ only.

4 Varieties

Let $V \subseteq k^n$ be an algebraic set. Note that from Theorem 3.7, there is a natural identification between the regular maps from V to k (where k is viewed as an algebraic set) and the elements of $\mathcal{C}(V)$. Indeed the elements of $\mathcal{C}(V)$ are in one-to-one correspondence with the morphisms of k-algebras $k[x] \to \mathcal{C}(V)$ and in turn these morphisms correspond to regular maps $V \to k$. More concretely, let $f \in \mathcal{C}(V) = R_n/\mathcal{I}(V)$ and let $\widetilde{f}$ be an arbitrary lifting of f to $R_n = k[x_1, \ldots, x_n]$. The regular function $V \to k$ corresponding to f is then the restriction of the map $k^n \to k$ given by the polynomial $\widetilde{f}$.

We would also like to make sense of regular maps from open subsets of V to k.

We first note the

Lemma 4.1. *Any open set in V is a union of open subsets of the form $V \backslash \mathrm{Z}(f)$, for $f \in \mathcal{C}(V)$.*

Proof. Left to the reader. Unroll the definitions. □

Definition 4.2. *Let $U \subseteq V$ be an open subset. A function $u : U \to k$ is said to be regular if for any regular map of algebraic sets $\tau : T \to V$ such that $\tau(T) \subseteq U$, the function $\tau \circ u$ is regular on T (ie corresponds to an element of $\mathcal{C}(T)$).*

To show that this definition is useable, we shall need the following

Lemma 4.3. *Suppose that the regular map $h : V' \to V$ makes $\mathcal{C}(V')$ isomorphic to $\mathcal{C}(V)[f^{-1}]$ as a $\mathcal{C}(V)$-algebra for some $f \in \mathcal{C}(V)$. Then*

(1) h is injective and h is a homeomorphism onto $V \backslash \mathrm{Z}(f)$;

(2) if $g : V'' \to V$ is a regular map such that $g(V'') \subseteq V \backslash \mathrm{Z}(f)$, then there is a unique regular map $g' : V'' \to V'$ such that $g = h \cdot g'$.

Proof. (1) The injectivity follows from the fact that for any maximal ideal $\mathfrak{m}$ of $\mathcal{C}(V)[f^{-1}]$, $\mathfrak{m}$ is generated by the image of $\mathfrak{m} \cap \mathcal{C}(V) = (h^*)^{-1}(\mathfrak{m})$ in $\mathcal{C}(V)[f^{-1}]$ (recall that $(h^*)^{-1}(\mathfrak{m})$ is maximal by Lemma 3.4 - one could also appeal to Cor. 10.4 in CA). See Lemma 5.6 in CA for this.

Next, we show that $h(V') \subseteq V \backslash \mathrm{Z}(f)$. In terms of maximal ideals, this translates to the statement that $f \notin (h^*)^{-1}(\mathfrak{m}) = \mathfrak{m} \cap \mathcal{C}(V)$ for all the maximal ideals of $\mathcal{C}(V)[f^{-1}]$. By the general properties of localisations (see Lemma 5.6 in CA), $\mathfrak{m} \cap \mathcal{C}(V)$ does not meet the multiplicative set generated by f, so in particular $f \notin \mathfrak{m} \cap \mathcal{C}(V)$. This shows that $h(V') \subseteq V \backslash \mathrm{Z}(f)$.

We now show that $h|_{V'} : V' \to V \backslash \mathrm{Z}(f)$ is surjective. For this, note that if $\mathfrak{n}$ is a maximal ideal of $\mathcal{C}(V)$ such that $f \notin \mathfrak{n}$ then there is a unique prime ideal $\mathfrak{n}_0$ of $\mathcal{C}(V)[f^{-1}]$ such that $\mathfrak{n}_0 \cap \mathcal{C}(V) = (h^*)^{-1}(\mathfrak{n}_0) = \mathfrak{n}$ (again use Lemma 5.6 in CA). The ideal $\mathfrak{n}_0$ is also maximal. To see this, let $\mathfrak{n}_0' \supseteq \mathfrak{n}_0$ be a maximal ideal. Since we have $(h^*)^{-1}(\mathfrak{n}_0') \supseteq \mathfrak{n}$, we thus have $(h^*)^{-1}(\mathfrak{n}_0') = \mathfrak{n}$. Since as before $\mathfrak{n}_0$ (resp. $\mathfrak{n}_0'$) is generated by the image of $\mathfrak{n}$ (resp. $(h^*)^{-1}(\mathfrak{n}_0')$) in $\mathcal{C}(V)[f^{-1}]$, we see that $\mathfrak{n}_0 = \mathfrak{n}_0'$, ie $\mathfrak{n}_0$ is maximal.

To show that h is a homeomorphism onto $V \backslash \mathrm{Z}(f)$, it is sufficient to show that image of any closed subset of V' maps to a closed subset of $V \backslash \mathrm{Z}(f)$. In terms of ideals, this translates to the statement that for any ideal J of $\mathcal{C}(V)[f^{-1}]$, there exists an ideal $I = I(J)$ of $\mathcal{C}(V)$ such that $\mathfrak{m} \supseteq J$ iff $\mathfrak{m} \cap \mathcal{C}(V) \supseteq I$ for all $\mathfrak{m} \in \mathrm{Spm}(\mathcal{C}(V)[f^{-1}])$. Letting J be an ideal of $\mathcal{C}(V)[f^{-1}]$, define $I := J \cap \mathcal{C}(V)$. If $\mathfrak{m} \supseteq J$ we clearly have $\mathfrak{m} \cap \mathcal{C}(V) \supseteq I$. On the other hand, if $\mathfrak{m} \cap \mathcal{C}(V) \supseteq I$, we have $\mathfrak{m} = (\mathfrak{m} \cap \mathcal{C}(V)) \cdot \mathcal{C}(V) \supseteq I \cdot \mathcal{C}(V) = J$ (again use Lemma 5.6 in CA, in particular (ii) in the proof of Lemma 5.6). So for any ideal J of $\mathcal{C}(V)[f^{-1}]$, we may choose $I = I(J) = I \cap \mathcal{C}(V)$.

(2) We first translate this into a statement of commutative algebra. We are given a map of k-algebras $g^* : \mathcal{C}(V) \to \mathcal{C}(V'')$ such that for any maximal ideal $\mathfrak{m}$ of $\mathcal{C}(V'')$, we have $f \notin (g^*)^{-1}(\mathfrak{m})$. We would like to show that there is a map of $\mathcal{C}(V)$-algebras from $\mathcal{C}(V)[f^{-1}]$ to $\mathcal{C}(V'')$. In view of the universal property of localisation (see Lemma-Definition 5.1 in CA), it is sufficient for this to show that $g^*(f)$ is a unit in $\mathcal{C}(V'')$. Now suppose for contradiction that $g^*(f)$ is not a unit in $\mathcal{C}(V'')$. Then $g^*(f)$ is contained in a maximal ideal $\mathfrak{m}$ of $\mathcal{C}(V'')$. Hence $f \in (g^*)^{-1}(\mathfrak{m})$, which is a contradiction. □

We give a description of regular functions in terms of the ambient space in the next corollary.

Corollary 4.4. *Let $f \in \mathcal{C}(V)$. The regular functions on $V \backslash \mathrm{Z}(f)$ are the restrictions of the functions $k^n \to k$ which are of the form $\frac{P(x_1,\dots,x_n)}{(F(x_1,\dots,x_n))^l}$ ($l \geqslant 0$), where $P(x_1, \dots, x_n) \in R_n$ and $F(x_1, \dots, x_n) \in R_n$ is any lifting of f to R_n.*

Proof. Note first that $\mathcal{C}(V)[f^{-1}] \simeq \mathcal{C}(V)[t]/(tf - 1)$ as a $\mathcal{C}(V)$-algebra (see Lemma 5.3 in CA). Hence

$\mathcal{C}(V)[f^{-1}]$ corresponds to the algebraic set Z in k^{n+1} given by the ideal generated by the sets $\mathcal{I}(V)$ and $tF(x_1,\dots,x_n)-1$ in $k[x_1,\dots,x_n,t]$. The polynomial map $\phi: k^{n+1}\to k^n$ inducing the map of k-algebras $\mathcal{C}(V)\to\mathcal{C}(V)[t]/(tf-1)$ is simply given by the formula $\phi(\langle v_1,\dots,v_n,z\rangle)=\langle v_1,\dots,v_n\rangle$. The inverse of the map $Z\overset{\phi|_Z}{\to}V\backslash \mathrm{Z}(f)$ is given by the formula $\langle v_1,\dots,v_n\rangle\mapsto\langle v_1,\dots,v_n,F(v_1,\dots,v_n)^{-1}\rangle$ (it must be this map since the map $Z\overset{\phi|_Z}{\to}V\backslash \mathrm{Z}(f)$ is bijective and $(v_1,\dots,v_n,F(v_1,\dots,v_n)^{-1})\in Z$ by construction). Hence a regular map on $V\backslash \mathrm{Z}(f)$ is given by the evaluation of a polynomial in the variables $x_1,\dots,x_n,t$ on the vector $\langle v_1,\dots,v_n,F(v_1,\dots,v_n)^{-1}\rangle$ (for $\langle v_1,\dots,v_n\rangle\in V\backslash \mathrm{Z}(f)$). This is the conclusion of the corollary. □

Note that the last lemma implies that the regular functions on $V\backslash \mathrm{Z}(f)$ are all quotients of restrictions of regular functions on V by powers of f. Also, Lemma 4.3 implies that if $h\in\mathcal{C}(V)$ is a regular function on V and $h|_{V\backslash \mathrm{Z}(f)}=0$, then $f\cdot h=0$ (by the definition of localisation and the fact that $\mathcal{C}(V)$ is reduced).

Proposition 4.5. *Let U be an open subset of the algebraic set $V\subseteq k^n$. A function $a:U\to k$ is regular iff for any point $\bar u\in U$, there is a polynomial $F\in R_n$, such that $F(\bar u)\neq 0$ and a polynomial $P\in R_n$ such that a coincides with P/F in a neighbourhood of $\bar u$.*

This implies in particular that if a function $a:U\to k$ is regular and nowhere vanishing, then $1/a:U\to k$ is also a regular function. In other words, the units in the ring of regular functions $U\to k$ are the nowhere vanishing regular functions.

Proof. (of Proposition 4.5). We first show the following.

Let $W\subseteq k^t$ be an algebraic set. Let $f_1,\dots,f_l\in\mathcal{C}(W)$ and suppose that $(f_1,\dots,f_l)=\mathcal{C}(W)$. Let $h:W\to k$ be a function (not assumed regular) and suppose that for each $i\in\{1,\dots,l\}$ there is an integer $n_i\geqslant 0$ and an element $c_i\in\mathcal{C}(W)$ such that $h|_{W\backslash \mathrm{Z}(f_i)}=c_i/f_i^{n_i}$. We claim that the function h is then regular on W (ie arises from an element of $\mathcal{C}(W)$, or in other words by restriction to W of a polynomial map $k^t\to k$).

To prove this, note first that we may assume that all the n_i are equal to some $m\geqslant 1$. Indeed, if we let $m:=1+\sup_i n_i$ then we may write $h|_{W\backslash \mathrm{Z}(f_i)}=c_if_i^{m-n_i}/f_i^m$ for all i. Now notice that for all $i,j\in\{1,\dots,l\}$ we have

$$h|_{W\backslash \mathrm{Z}(f_if_j)}=c_i/f_i^m=c_j/f_j^m$$

so that $(f_if_j)^m(c_i/f_i^m-c_j/f_j^m)=f_j^mc_i-c_jf_i^m=0$ on $W\backslash \mathrm{Z}(f_if_j)$. We deduce from the remark preceding the proposition that

$$(f_if_j)f_j^mc_i=(f_if_j)c_jf_i^m$$

on V. Now let $b_i\in\mathcal{C}(W)$ be functions such that

$$\sum_i b_if_i^{2m}=1$$

(note that we also have $(f_1^{2m},\dots,f_l^{2m})=\mathcal{C}(W)$ - prove this or see Lemma 12.2 in CA). Let

$$\widetilde{h}:=\sum_i b_if_i^mc_i.$$

We compute

$$\widetilde{h}f_j^{2m}=\sum_i b_if_i^mf_j^{2m}c_i=\sum_i b_i(f_if_j)^mf_j^mc_i=\sum_i b_i(f_if_j)^mf_i^mc_j=(\sum_i b_if_i^{2m})f_j^mc_j=f_j^mc_j$$

so that $\widetilde{h}|_{W\backslash \mathrm{Z}(f_j)}=c_j/f_j^m$. Hence $\widetilde{h}=h$. This completes the proof of the claim.

Coming back to the proposition, note that the "$\Rightarrow$" direction of the equivalence stated in the proposition is clear from Lemma 4.1 and Corollary 4.4. Thus we only have to prove the "$\Leftarrow$" direction of the equivalence. Since the topology of U is quasi-compact (this will be proven in exercise sheet 2, Q4 (4) - you can also prove this directly), we may reword this implication as follows. Let $g_1, \dots, g_l \in \mathcal{C}(V)$ and suppose that $U = \cup_i(V \backslash \mathrm{Z}(g_i))$. Let $V' \subseteq k^{n'}$ be an algebraic set and let $H : V' \to V$ be a regular map such that $H(V') \subseteq U$. Suppose that for all $i \in \{1, \dots, l\}$ we have $a|_{V\backslash \mathrm{Z}(g_i)} = d_i/g_i^{n_i}$ for some $n_i \geqslant 0$ and some $d_i \in \mathcal{C}(V)$. The "$\Leftarrow$" direction of the equivalence of the proposition is then the statement that $a \circ H = H^*(a)$ is a regular function on V'. So we only have to prove this last statement under the just stated assumptions.

Note first that by construction, for all $i \in \{1, \dots, l\}$ we have

$$H^*(a)|_{V'\backslash \mathrm{Z}(H^*(g_i))} = H^*(d_i)/H^*(g_i)^{n_i}.$$

Also, since $H(V') \subseteq U$, we have $(H^*(g_1), \dots, H^*(g_l)) = \mathcal{C}(V')$. Hence we may apply the preceding claim to $W = V'$, $f_i = H^*(g_i)$ and $h = H^*(a)$ to conclude that $H^*(a)$ is regular on V'. $\square$

Note that in view of the previous proposition, the following property holds trivially: if $U' \subseteq U$ is an inclusion of open subsets of V, then the restriction to U' of a regular function on U is also regular. We encapsulate this property in the following

Definition 4.6. *Let T be a topological space. A sheaf of functions $\mathcal{O}_T$ on T with values in k is an assignement, which associates with each open subset O of T a sub k-algebra $\mathcal{O}_T(O)$ of* $\mathrm{Maps}(O, k)$*, with the following property: for any open covering $\{O_i\}$ of an open subset O, a function $f : O \to k$ lies in $\mathcal{O}_T(O)$ iff $f|_{O_i} \in \mathcal{O}_T(O_i)$ for all i.*

Here $\mathrm{Maps}(O, k)$ is the set of functions from O to k, with its natural k-algebra structure (given by pointwise multiplication and addition).

Note that if O is an open subset of topological space endowed with a sheaf of k-valued functions, O inherits a sheaf of k-valued functions from T.

Proposition 4.5 implies that for any algebraic set $V \subseteq k^n$, the regular functions on Zariski open subsets of V define a sheaf of functions $\mathcal{O}_V$ with values in k on V.

There is a natural notion of mapping between topological spaces endowed with sheaves of k-valued functions:

Definition 4.7. *Let $(T, \mathcal{O}_T)$ and $(T', \mathcal{O}_{T'})$ be two topological spaces endowed with sheaves of functions with values in k. A morphism (sometimes loosely called a map) from $(T, \mathcal{O}_T)$ to $(T', \mathcal{O}_{T'})$ is a continuous map $a : T \to T'$ such that for any open subset $U' \subseteq T'$ and any element $f \in \mathcal{O}_{T'}(U')$, the function $f \circ a|_{a^{-1}(U')}$ on $a^{-1}(U')$ lies in $\mathcal{O}_T(a^{-1}(U'))$.*

We will also need the following definition.

Let T be a topological space endowed with a sheaf of functions $\mathcal{O}_T$ with values in k. Let $t \in T$. Let $\widehat{\mathcal{O}}_{T,t} := \cup_{O \text{ open}, t \in O} \mathcal{O}_T(O)$ (where all the $\mathcal{O}_T(O)$ are considered to be disjoint from each other). Define an equivalence relation on $\widehat{\mathcal{O}}_{T,t}$ by declaring two functions in $\widehat{\mathcal{O}}_{T,t}$ equivalent if they coincide in some open neighbourhood of t. The set of equivalence classes in $\widehat{\mathcal{O}}_{T,t}$ has a natural k-algebra structure and we denote it by $\mathcal{O}_{T,t}$. The k-algebra $\mathcal{O}_{T,t}$ is called *the local ring at* t. Note that by definition, for any open neighbourhood O of t, there is a natural map of k-algebras $\mathcal{O}_T(O) \to \mathcal{O}_{T,t}$. Also, there is a natural map of k-algebras $\mathcal{O}_{T,t} \to k$, which is given by evaluation at t.

If we are given a morphism from $(T, \mathcal{O}_T)$ to $(T', \mathcal{O}_{T'})$ as in the last definition, the pull-back of functions gives a map of k-algebras $\mathcal{O}_{T,a(t)} \to \mathcal{O}_{T,t}$ for any $t \in T$.

From the very definition of regularity, we see that any regular map from an algebraic set to another induces a morphism between the associated topological spaces with sheaves of k-valued functions.

We are now ready to define a general variety.

Definition 4.8. *Let T be a topological space endowed with a sheaf of functions with values in k. We say that T is a variety if there is a finite open covering $\{U_i\}$ of T, such that U_i with its induced sheaf of k-valued functions is isomorphic to an algebraic set endowed with its sheaf of regular functions. A morphism of varieties is a morphism of the corresponding topological spaces with sheaves of k-valued functions.*

Lemma 4.9. *Let $V \subseteq k^n$ be an algebraic set and let $(V, \mathcal{O}_V)$ be the associated topological space with sheaf of k-valued functions. Let $\bar{v} \in V$. Then the natural map of k-algebras $\mathcal{C}(V) = \mathcal{O}_V(V) \to \mathcal{O}_{V,\bar{v}}$ extends (necessarily uniquely) to an isomorphism of k-algebras $\mathcal{C}(V)_{\bar{v}} \simeq \mathcal{O}_{V,\bar{v}}$.*

Here we identified $\bar{v}$ with the corresponding maximal ideal $\mathcal{I}(\{\bar{v}\})$ when writing $\mathcal{C}(V)_{\bar{v}}$ (so that $\mathcal{C}(V)_{\bar{v}}$ is the localisation of $\mathcal{C}(V)$ at the multiplicative set $\mathcal{C}(V)\backslash\mathcal{I}(\{\bar{v}\})$).

Proof. We first show that the map $\mathcal{C}(V) \to \mathcal{O}_{V,\bar{v}}$ extends to a map of k-algebras $\mathcal{C}(V)_{\bar{v}} \to \mathcal{O}_{V,\bar{v}}$.

To show this, we have to show that a regular function $f \in \mathcal{C}(V)$, which does not vanish at $\bar{v}$, maps to a unit in $\mathcal{O}_{V,\bar{v}}$. Now by definition a unit in $\mathcal{O}_{V,\bar{v}}$ is represented by a regular function in a neighbourhood of $\bar{v}$, which vanishes nowhere in that neighbourhood (see the remark before Definition 4.6). Now since f does not vanish at $\bar{v}$, it is nowhere vanishing in the set $V\backslash \mathrm{Z}(f)$, which is a neighbourhood of $\bar{v}$. So the image of f in $\mathcal{O}_{V,\bar{v}}$ is a unit.

So we have a unique extension of the map $\mathcal{C}(V) \to \mathcal{O}_{V,\bar{v}}$ to a map of k-algebras $\mathcal{C}(V)_{\bar{v}} \to \mathcal{O}_{V,\bar{v}}$. We still have to show that this last map is injective and surjective.

We first show injectivity. Let $f/s \in \mathcal{C}(V)_{\bar{v}}$ (where $s \in \mathcal{C}(V)\backslash\mathcal{I}(\{\bar{v}\})$). Suppose that the image of f/s in $\mathcal{O}_{V,\bar{v}}$ vanishes. By definition, this means that the function f vanishes in a neighbourhood of $\bar{v}$. In particular, there exists an $h \in \mathcal{C}(V)$ such that f vanishes in $V\backslash \mathrm{Z}(h)$, where h does not vanish at $\bar{v}$ (use Lemma 4.1). In other words, the image of f in $\mathcal{C}(V)[h^{-1}]$ vanishes (use Lemma 4.4 and the commentary thereafter). Since $h \notin \mathcal{I}(\{\bar{v}\})$, the natural map $\mathcal{C}(V) \to \mathcal{C}(V)_{\bar{v}}$ factors through $\mathcal{C}(V)[h^{-1}]$ and hence the image of f in $\mathcal{C}(V)_{\bar{v}}$ also vanishes. This settles injectivity.

Now for surjectivity. By Lemma 4.1, an element $\widetilde{e} \in \mathcal{O}_{V,\bar{v}}$ is represented by a regular function on $V\backslash \mathrm{Z}(h)$, for some h which does not vanish at $\bar{v}$. Such a function corresponds to an element of $\mathcal{C}(V)[h^{-1}]$ and again since the natural map $\mathcal{C}(V) \to \mathcal{C}(V)_{\bar{v}}$ factors through $\mathcal{C}(V)[h^{-1}]$, we see that $\widetilde{e}$ lies in the image of $\mathcal{C}(V)_{\bar{v}}$. Since $\widetilde{e} \in \mathcal{O}_{V,\bar{v}}$ was arbitrary, the natural map $\mathcal{C}(V)_{\bar{v}} \to \mathcal{O}_{V,\bar{v}}$ is surjective. □

Note the following consequences of the last lemma. With the terminology of the lemma, the ring $\mathcal{O}_{V,\bar{v}}$ is local. Also, note that the natural evaluation map $\mathcal{O}_{V,\bar{v}} \to k$ is surjective, because all constant functions are regular on V. Hence the kernel of the map $\mathcal{O}_{V,\bar{v}} \to k$ is maximal. Hence this kernel coincides with the unique maximal ideal of $\mathcal{O}_{V,\bar{v}}$.

For Definition 4.8 to be coherent, we need to check that we can recover an algebraic set from its associated topological space with sheaf of k-valued functions:

Lemma 4.10. *Let $V \subseteq k^n$ and $W \subseteq k^t$ be two algebraic sets. Let $(V, \mathcal{O}_V)$ and $(W, \mathcal{O}_W)$ be the associated* *topological spaces with sheaves of k-valued functions. Let g be a morphism from $(V, \mathcal{O}_V)$ to $(W, \mathcal{O}_W)$. Then g is induced by a regular map $\psi : V \to W$.*

Proof. By definition, the morphism g provides a map of k-algebras $\mathcal{C}(W) \to \mathcal{C}(V)$. Furthermore, for any $\bar{v} \in V$, we have a commutative diagram of k-algebras

$$\begin{array}{ccc} \mathcal{C}(W) & \xrightarrow{g^*} & \mathcal{C}(V) \\ \downarrow & & \downarrow \\ \mathcal{O}_{W,g(\bar{v})} & \xrightarrow{g^*} & \mathcal{O}_{V,\bar{v}} \end{array}$$

From the remark after Lemma 4.9, the ring $\mathcal{O}_{V,\bar{v}}$ is a local ring and its maximal ideal consists of the elements represented by the regular functions h defined in a neighbourhood of $\bar{v}$ such that $h(\bar{v}) = 0$. A similar statement is true for $\mathcal{O}_{W,g(\bar{v})}$ and $g(\bar{v})$ in place of $\bar{v}$. In particular, the map $g^* : \mathcal{O}_{W,g(\bar{v})} \to \mathcal{O}_{V,\bar{v}}$ sends the maximal ideal of $\mathcal{O}_{W,g(\bar{v})}$ into the maximal ideal of $\mathcal{O}_{V,\bar{v}}$. Since the involved rings are local, this implies that the inverse image by g^* of the maximal ideal of $\mathcal{O}_{V,\bar{v}}$ is the maximal ideal of $\mathcal{O}_{W,g(\bar{v})}$. Using standard properties of localisations and Lemma 4.9, we conclude that the inverse image of $\mathcal{I}(\{\bar{v}\}) \subseteq \mathcal{C}(V)$ by $g^* : \mathcal{C}(V) \to \mathcal{C}(W)$ is $\mathcal{I}(\{\bar{g}(\bar{v})\})$. In particular, $g(\bar{v}) = \mathrm{Spm}(g^*)(\bar{v})$ (use Lemma 3.4). Hence g is induced by the map of k-algebras $g^* : \mathcal{C}(W) \to \mathcal{C}(V)$ and hence by a regular map $V \to W$ (by Theorem 3.7). □

In categorical terms, this implies that the category of algebraic sets embeds in the category of topological spaces with sheaves of k-valued functions by a fully faithful functor. We shall from now on call *affine variety* a variety isomorphic to a variety associated with an algebraic set.

We shall often abbreviate "topological space with sheaf of k-valued functions" as "Topskf" from now on.

5 Open and closed subvarieties

Proposition 5.1. *Let $(V, \mathcal{O}_V)$ be a variety. Let $U \subseteq V$ be an open subset and let $\mathcal{O}_U$ be the sheaf of k-valued functions induced by $\mathcal{O}_V$. Then $(U, \mathcal{O}_U)$ is a variety and the inclusion map is a morphism of varieties.*

Proof. Let $\{V_i\}$ be an open covering of V such that each V_i is isomorphic as a Topskf to an affine variety (where V_i is endowed with the sheaf of k-valued functions induced by V). Then $\{V_i \cap U\}$ is an open covering of U. Since $V_i \cap U$ is open in V_i, there is for each i a subset $E_i \subseteq \mathcal{C}(V_i)$ such that $\cup_{e \in E_i}(V_i \backslash \mathrm{Z}(e)) = V_i \cap U$ (use Lemma 4.1). Hence we only have to show that the open subset $V_i \backslash \mathrm{Z}(e)$ of V_i is isomorphic as a Topskf to an affine variety. But this follows from Lemma 4.3. □

An open subset of a variety is called an *open subvariety* if it is endowed with the structure of Topskf described in the last Proposition.

Let $(V, \mathcal{O}_V)$ be a variety. Let $Z \subseteq V$ be a closed subset. Endow Z with the topology induced by V. For any open subset O of Z, define a function $f : O \to k$ to be regular if there is collection of open subsets $\{U_i\}$ of V and regular functions $g_i : U_i \to k$ such that

- $(\cup_i U_i) \cap Z = O$;

- $g_i|_{O \cap U_i} = f|_{O \cap U_i}$.

In words, $f : O \to k$ is regular iff in the neighbourhood of every point of O the function f is the restriction of a regular function on some open subset of V. This endows Z with a structure of topological space with k-valued functions. We shall write $\mathcal{O}_Z$ for the corresponding sheaf of k-valued functions. The sheaf of k-valued functions $\mathcal{O}_Z$ on Z is said to be *induced* by $\mathcal{O}_V$.

Proposition 5.2. *The topological space Z with sheaf of k-valued functions $\mathcal{O}_Z$ is a variety. The inclusion map $Z \to V$ is a morphism of varieties.*

Proof. The inclusion map $Z \to V$ provides us with a morphism $(Z, \mathcal{O}_Z) \to (V, \mathcal{O}_V)$ of Topskf by construction. Hence we only have to show that $(Z, \mathcal{O}_Z)$ is a variety (see Definition 4.8). Let $\{V_i\}$ be a covering of V by open subsets such that $(V_i, \mathcal{O}_{V_i})$ is isomorphic as a Topskf to an affine variety. By definition, it is sufficient to show that for each i, the Topskf $Z \cap V_i$ is isomorphic to an affine variety. Hence we may assume that V is affine to begin with. Hence we are reduced to the situation where $V \subseteq k^n$ is an algebraic set and $Z \subseteq k^n$ is another algebraic set such that $Z \subseteq V$. Endow Z with the sheaf of functions $\mathcal{O}_Z$ induced by $\mathcal{O}_V$. We would like to show that $(Z, \mathcal{O}_Z)$ is isomorphic to an affine variety as a Topskf. Now note that by Proposition 4.5 the sheaf $\mathcal{O}_Z$ is precisely the sheaf of regular functions on Z viewed as an algebraic subset of k^n. So $(Z, \mathcal{O}_Z)$ is isomorphic to an affine variety as a Topskf. □

An closed subset of a variety V is called a *closed subvariety* if it is endowed with the structure of Topskf induced by V.

Lemma 5.3. *Let $(W, \mathcal{O}_W)$ and $(V, \mathcal{O}_V)$ be two varieties. Let Z (resp. O) be a closed subset (resp. open subset) of V. Endow Z (resp. O) with its structure of closed (resp. open) subvariety. Let $\lambda : W \to V$ be a morphism of Topskf such that $\lambda(W) \subseteq Z$ (resp. $\lambda(W) \subseteq O$). Then the induced map $W \to Z$ (resp. $W \to O$) is a morphism of Topskf.*

Proof. Left to the reader. Unroll the definitions. □

We also record a consequence of the proof of Proposition 5.2:

Lemma 5.4. *Let $V \subseteq W \subseteq k^n$, where V and W are algebraic sets in k^n. Let $(V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$ be the corresponding morphism of topological spaces with sheaves of k-valued functions. Then $\mathcal{O}_V$ is induced by $\mathcal{O}_W$.*

6 Projective space

Projective varieties arise when one tries to find an algebraic counterpart of the topological notion of compactness. We will revisit this later when we consider complete varieties.

Let $n \geqslant 0$. A *line through the origin* of k^{n+1} is by definition the vector subspace $[\bar{v}]$ of k^{n+1} generated by a vector $\bar{v} \in k^{n+1} \backslash \{0\}$. We define *projective space of dimension n* to be the set $\mathbb{P}^n(k)$ of lines through the origin of k^{n+1}. If $\bar{v} = \langle v_0, \dots, v_n \rangle \in k^{n+1} \backslash \{0\}$, we shall write $[v_0, \dots, v_n]$ for $[\langle v_0, \dots, v_n \rangle]$.

We shall endow $\mathbb{P}^n(k)$ with a variety structure. For $i \in \{0, \dots n\}$, define

$$U_i = \{[v_0, \dots, v_n] \in \mathbb{P}^n(k) \,|\, v_i \neq 0\}.$$

In the following, we shall write the symbol $\check{}$ over a term that is to be omitted. The map $u_i : k^n \to U_i$ such that

$$u_i(\langle v_0, \dots, \check{v}_i, \dots, v_n \rangle) := [v_0, \dots, v_{i-1}, 1, v_{i+1}, \dots v_n]$$

is clearly a bijection and we have

$$u_i^{-1}([v_0,\dots,v_n]) = \langle \frac{v_0}{v_i},\dots,\overbrace{\frac{v_i}{v_i}},\dots,\frac{v_n}{v_i}\rangle.$$

if $[v_0,\dots,v_n]\in U_i$.

If $j<i$ and $v_j\neq 0$, we compute

$$(u_j^{-1}\circ u_i)(\langle v_0,\dots,\check{v}_i,\dots,v_n\rangle) = u_j^{-1}([v_0,\dots,v_{i-1},1,v_{i+1},\dots v_n]) = \langle \frac{v_0}{v_j},\dots,\overbrace{\frac{v_j}{v_j}},\dots,\frac{1}{v_j},\frac{v_{i+1}}{v_j},\dots,\frac{v_n}{v_j}\rangle$$

and if $j>i$ and $v_j\neq 0$, we have similarly

$$(u_j^{-1}\circ u_i)(\langle v_0,\dots,\check{v}_i,\dots,v_n\rangle) = \langle \frac{v_0}{v_j},\dots,\frac{v_{i-1}}{v_j},\frac{1}{v_j},\dots,\overbrace{\frac{v_j}{v_j}},\dots,\frac{v_n}{v_j}\rangle$$

Hence, if $i\neq j$, the map $u_j^{-1}\circ u_i$ gives a map from the open subset of k^n

$$\mathcal{U}_{ij} := \{\langle v_0,\dots,\check{v}_i,\dots,v_n\rangle\in k^n \mid v_j\neq 0\}$$

into the open subset of k^n

$$\mathcal{U}_{ji} := \{\langle v_0,\dots,\check{v}_j,\dots,v_n\rangle\in k^n \mid v_i\neq 0\}$$

and $u_i(\mathcal{U}_{ij}) = U_i\cap U_j = u_j(\mathcal{U}_{ji})$. Let $u_{ij} := u_j^{-1}\circ u_i : \mathcal{U}_{ij}\to\mathcal{U}_{ji}$.

Note that if one sees $\mathcal{U}_{ij}$ as an open subvariety of k^n, then $\mathcal{U}_{ij}$ is an affine variety associated with the coordinate ring

$$k[x_0,\dots,\check{x}_i,\dots,x_n][x_j^{-1}] \simeq k[x_0,\dots,\check{x}_i,\dots,x_n][t]/(tx_j-1)$$

and similarly, $\mathcal{U}_{ji}$ is an affine variety associated with the coordinate ring

$$k[y_0,\dots,\check{y}_j,\dots,y_n][y_i^{-1}] \simeq k[y_0,\dots,\check{y}_j,\dots,y_n][t]/(zy_i-1)$$

One checks from the definitions that u_{ij} arises from the polynomial map which sends z to x_j and y_l to $x_l\cdot t$ if $l\neq i$ and to t if $l=i$. Hence u_{ij} defines a morphism of varieties from $\mathcal{U}_{ij}$ to $\mathcal{U}_{ji}$. One checks from the just given formula that u_{ij} and u_{ji} are inverse to each other, so u_{ij} is an isomorphism of varieties.

Now we define a topology on $\mathbb{P}^n(k)$ by declaring a subset $O\subseteq\mathbb{P}^n(k)$ to be open iff $u_i^{-1}(O)$ is open in k^n for all $i\in\{0,\dots,n\}$ (why does this define a topology?). Furthermore, if $O\subseteq\mathbb{P}^n(k)$ is open, we define a k-valued function $f:O\to k$ to be *regular* iff $f\circ u_i|_{u_i^{-1}(O)}$ is a regular function on $u_i^{-1}(O)$ for all i. Since $(k^n,\mathcal{O}_{k^n})$ is a Topskf, we see that with this definition, $\mathbb{P}^n(k)$ becomes a Topskf (why? - unroll the definitions). We shall write $\mathcal{O}_{\mathbb{P}^n(k)}$ for the just defined sheaf of k-valued functions on $\mathbb{P}^n(k)$.

Proposition 6.1. *The sets U_i are open in $\mathbb{P}^n(k)$ for all $i\in\{0,\dots,n\}$ and the maps $u_i : k^n\to\mathbb{P}^n(k)$ restrict to isomorphisms of Topskf between k^n and $(U_i,\mathcal{O}_{U_i})$, where $\mathcal{O}_{U_i}$ is the sheaf of k-valued functions induced on U_i by $\mathcal{O}_{\mathbb{P}^n(k)}$. In particular, the Topskf $(\mathbb{P}^n(k),\mathcal{O}_{\mathbb{P}^n(k)})$ is a variety.*

The U_i are called the *standard coordinate charts* of $\mathbb{P}^n(k)$. We shall sometimes write U_i^n for U_i to emphasise the dependence on n.

Proof. To show that U_i is open, we have to show that $u_j^{-1}(U_i)$ is open in k^n for all j. We have shown above that $u_j^{-1}(U_i) = \mathcal{U}_{ji}$ is open, so U_i is open.

Next, we have to show that the map u_i is a homeomorphism onto its image. The map u_i is continuous and injective by definition so we only have to show that u_i is an open map. So let $O \subseteq k^n$ be an open set. We have to show that $u_i(O)$ is open, or in other words that $u_j^{-1}(u_i(O))$ is open for all j. Now we have

$$u_j^{-1}(u_i(O)) = u_j^{-1}(u_i(O) \cap (U_i \cap U_j)) = u_j^{-1}(u_i(O \cap \mathcal{U}_{ij})) = u_{ij}(O \cap \mathcal{U}_{ij})$$

and $u_{ij}(O \cap \mathcal{U}_{ij})$ is open in $\mathcal{U}_{ji}$ since $u_{ij} : \mathcal{U}_{ij} \to \mathcal{U}_{ji}$ is a homeomorphism by the above. On the other hand $\mathcal{U}_{ji}$ is open in U_j, so $u_{ij}(O \cap \mathcal{U}_{ij})$ is also open in U_j. So u_i is a homeomorphism onto its image.

Finally, we have to show that if $O \subseteq U_i$ is an open set then $f : O \to k$ is a regular function iff $f \circ u_i|_{u_i^{-1}(O)}$ is a regular function on the open subset $u_i^{-1}(O)$ of k^n. By definition, if $f : O \to k$ is a regular function, then $f \circ u_i|_{u_i^{-1}(O)}$ is regular. So suppose that $f \circ u_i|_{u_i^{-1}(O)}$ is regular. We have to show that for all j the function $f \circ u_j|_{u_j^{-1}(O)}$ is regular on $u_j^{-1}(O)$ viewed as an open subset of k^n. Now we have by definition

$$u_j|_{\mathcal{U}_{ji}} = u_i|_{\mathcal{U}_{ij}} \circ u_{ji}$$

and since $u_j^{-1}(O) \subseteq \mathcal{U}_{ji}$ we thus have

$$u_j|_{u_j^{-1}(O)} = u_i|_{\mathcal{U}_{ij} \cap u_i^{-1}(O)} \circ u_{ji}|_{u_j^{-1}(O)}$$

where $u_{ji}|_{u_j^{-1}(O)}$ is viewed as a map from $u_j^{-1}(O)$ to $\mathcal{U}_{ij} \cap u_i^{-1}(O)$. The map $u_{ji}|_{u_j^{-1}(O)}$ is a morphism of Topskf since u_{ji} is a morphism of Topskf by the above. Also, the function $f \circ u_i|_{\mathcal{U}_{ij} \cap u_i^{-1}(O)}$ is a regular function by the definition of a sheaf of k-valued functions. Hence

$$f \circ u_j|_{u_j^{-1}(O)} = (f \circ u_i|_{\mathcal{U}_{ij} \cap u_i^{-1}(O)}) \circ u_{ji}|_{u_j^{-1}(O)}$$

is a regular function. This completes the proof. □

Example. The space $\mathbb{P}^1(k)$ only has two coordinate charts, the charts U_0 and U_1. By inspection, we see that $\mathbb{P}^1(k) \backslash U_i$ consists of only one point. So one can see $\mathbb{P}^1(k)$ as the "compactification" of k obtained by adding a "point at ∞" to k. If $k = \mathbb{C}$, the space $\mathbb{P}^1(k)$ can be naturally identified (as a set) with the Riemann sphere of complex analysis.

7 Projective varieties

What are the closed subsets of projective space? To answer this question, we shall need the following definitions.

A polynomial $P(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is said to be *homogenous* if it is a sum of monomials of the same degree. Any polynomial $P(x_0, \dots, x_n)$ has a canonical decomposition

$$P = \sum_{i=0}^{\deg(P)} P_{[i]}$$

where $P_{[i]}$ is the sum of the monomials of degree i appearing in P (so that in particular $P_{[i]}$ is homogenous).

Example. The polynomials x_0, $x_0^2 + x_0 x_1$ are homogenous but $x_0^2 + x_1$ is not.

We have a decomposition of $k[x_0, \dots, x_n]$ as an internal direct sum

$$k[x_0, \dots, x_n] = \bigoplus_{l \geqslant 0} k[x_0, \dots, x_n]_{[l]}$$

where $k[x_0, \dots, x_n]_{[l]}$ is the k-vector space of homogenous polynomials of degree l. In particular, we have $k[x_0, \dots, x_n]_{[0]} = k$. This decomposition into a direct sum makes $k[x_0, \dots, x_n]$ into a graded ring in the sense of section 11.2 of CA.

Example. We have $(x_0^2 + x_1)_{[2]} = x_0^2$, $(x_0^2 + x_1)_{[1]} = x_1$, $(x_0^2 + x_1)_{[0]} = 0$.

Note the following elementary fact. If $P(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is homogenous then $P(s \cdot x_0, \dots, s \cdot x_n) = s^{\deg(P)} P(x_0, \dots, x_n)$ for all $s \in k$.

We thus see that if $P(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ is a homogenous polynomial and $\bar{v} \in k^{n+1}$ is non zero, we have $P(\bar{v}) = 0$ iff $P(s \cdot \bar{v}) = 0$ for all $s \in k^*$. This gives rise to the following definition. Let $S \subseteq k[x_0, \dots, x_n]$ be a set of homogenous polynomials. We define

$$\mathrm{Z}(S) := \{[\bar{v}] \in \mathbb{P}^n(k) \mid \bar{v} \in k^{n+1} \backslash \{0\}, \forall P \in S : P(\bar{v}) = 0\}.$$

A *projective algebraic set* in $\mathbb{P}^n(k)$ is a subset of the form $\mathrm{Z}(S)$, where $S \subseteq k[x_0, \dots, x_n]$ is a set of homogenous polynomials.

For convenience, we shall extend the operator $\mathrm{Z}(\cdot)$ to non homogenous polynomials. For any set $S \subseteq k[x_0, \dots, x_n]$ (not necessarily consisting of homogenous polynomials), we set

$$\mathrm{Z}(S) := \{[\bar{v}] \mid \bar{v} \in k^{n+1} \backslash \{0\}, P_{[i]}(\bar{v}) = 0 \, \forall i \geqslant 0\}.$$

Just as in the affine case, we have $\mathrm{Z}(S) = \mathrm{Z}(S \cdot k[x_0, \dots, x_n])$ (why?). Hence the projective algebraic sets in $\mathbb{P}^n(k)$ are the sets of the type $\mathrm{Z}(I)$, where $I \subseteq k[x_0, \dots, x_n]$ is an ideal generated by homogenous elements.

We shall say that an ideal of $k[x_0, \dots x_n]$ is *homogenous* if it is generated by homogenous elements.

Lemma 7.1. *Let $I \subseteq k[x_0, \dots x_n]$ be an ideal. Then I is homogenous iff for all $P \in I$ and all $i \geqslant 0$, we have $P_{[i]} \in I$. If I is homogenous then its radical $\mathfrak{r}(I)$ is also homogenous.*

In other words, a homogenous ideal is a graded ideal in $k[x_0, \dots, x_n]$ (ie a graded $k[x_0, \dots, x_n]$-submodule of $k[x_0, \dots, x_n]$).

Proof. See exercises. □

Proposition 7.2. *Projective algebraic sets are closed in $\mathbb{P}^n(k)$. Furthermore, if $C \subseteq \mathbb{P}^n(k)$ is a closed subset and J is the ideal generated by the homogenous polynomials which vanish on C, then $\mathrm{Z}(J) = C$. In particular, the closed subsets of $\mathbb{P}^n(k)$ are precisely the projective algebraic sets.*

Proof. Let $S := \{P_l\}$ be a set of homogenous polynomials in $k[x_0, \dots, x_n]$. By construction, we have

$$u_i^{-1}(\mathrm{Z}(S)) = \mathrm{Z}(\{P_l(x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n)\})$$

so that $u_i^{-1}(\mathrm{Z}(S))$ is closed in k^n. By Proposition 6.1, the set $\mathrm{Z}(S) \cap U_i$ is thus closed in U_i (for the induced topology). Since the U_i cover $\mathbb{P}^n(k)$, we thus see that $\mathrm{Z}(S)$ is closed in $\mathbb{P}^n(k)$.

As to the second assertion, we clearly have $\mathrm{Z}(J) \supseteq C$. So we need to prove that $\mathrm{Z}(J) \subseteq C$. In other words, we have to prove that if $[\bar{v}] \notin C$, then there is a homogenous polynomial $H \in J$, such that $H([\bar{v}]) \neq 0$. Now let $j \in \{0, \dots, n\}$ and suppose that $[\bar{v}] \in U_j$. We then have $[\bar{v}] \notin C \cap U_j$. Since $u_j^{-1}(C)$ is the zero set of an ideal in $k[x_0, \dots \check{x}_j, \dots, x_n]$, there is a polynomial $P(x_0, \dots, \check{x}_j, \dots x_n) \in k[x_0, \dots \check{x}_j, \dots, x_n]$ such that $P(u_j^{-1}([\bar{v}])) \neq 0$ and such that $P \in \mathcal{I}(u_j^{-1}(C))$. Let

$$\beta_j(P) := x_j^{\deg(P_j)} P(\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j} \dots, \frac{x_n}{x_j}).$$

This is a homogenous polynomial (the "homogenisation" of P with respect of the variable x_j) such that

$$(\beta_j(P))(x_0,\dots,x_{j-1},1,x_j,\dots,x_n) = P_j.$$

In particular we have $\mathrm{Z}(\beta_j(P)) \supseteq C\cap U_j$ and $(\beta_j(P))([\bar{v}]) = P(u_j^{-1}([\bar{v}])) \neq 0$. Now let $Q_j = x^j\beta_j(P)$. Then Q_j is still homogenous and we have $Q_j([\bar{v}]) \neq 0$ and $\mathrm{Z}(Q_j) \supseteq C$ (because x_j vanishes on $\mathbb{P}^n(k)\backslash U_j$). Hence we may set $H = Q_j$. This completes the proof. $\square$

If $A \subseteq \mathbb{P}^n(k)$ is a subset, we shall write $\mathcal{I}(A) \subseteq k[x_0,\dots,x_n]$ for the ideal generated by the homogenous polynomials vanishing on A. This notation clashes with the notation in the affine case but the context should make it clear which definition of $\mathcal{I}(\cdot)$ we use.

Now we have the analogue of Proposition 2.3:

Proposition 7.3. *Let $C \subseteq \mathbb{P}^n(k)$ be a closed subset and let $J \subseteq k[x_0,\dots,x_n]$ be a homogenous radical ideal. Suppose that $\mathrm{Z}(J) \neq \emptyset$. Then $\mathcal{I}(C)$ is a (by definition homogenous) radical ideal and we have $\mathrm{Z}(\mathcal{I}(C)) = C$ and $\mathcal{I}(\mathrm{Z}(J)) = J$.*

Proof. We first show that $\mathcal{I}(C)$ is a radical ideal. To see this, let $H \subseteq \mathfrak{r}(\mathcal{I}(C))$ be the subset of $\mathfrak{r}(\mathcal{I}(C))$ consisting of the homogenous elements of $\mathfrak{r}(\mathcal{I}(C))$. By the definition of the nilradical of an ideal, all the elements of H vanish on C. On the other hand, $\mathfrak{r}(\mathcal{I}(C))$ is a homogenous ideal by Lemma 7.1 and so H generates $\mathfrak{r}(\mathcal{I}(C))$. Hence $\mathfrak{r}(\mathcal{I}(C)) \subseteq \mathcal{I}(C)$. Hence $\mathfrak{r}(\mathcal{I}(C)) = \mathcal{I}(C)$.

The equality $\mathrm{Z}(\mathcal{I}(C)) = C$ is contained in Proposition 7.2. For the second equality, note first that the inclusion $J \subseteq \mathcal{I}(\mathrm{Z}(J))$ follows from the definitions. We thus only have to prove that $J \supseteq \mathcal{I}(\mathrm{Z}(J))$. So let Q be a non zero homogenous polynomial vanishing on $\mathrm{Z}(J)$. We need to show that $Q \in J$. Note that $\deg(Q) > 0$. Indeed, if $\deg(Q) = 0$ then Q is a non zero constant polynomial and then $\mathrm{Z}(Q) = \emptyset$, which implies that $\mathrm{Z}(J) = \emptyset$. This is not possible by assumption. Also, note that J does not contain any constant polynomial, for otherwise $\mathrm{Z}(J) = \emptyset$. Now consider the map

$$q : k^{n+1}\backslash\{0\} \to \mathbb{P}^n(k)$$

given by the formula $q(\bar{v}) := [\bar{v}]$. Note that $q^{-1}(\mathrm{Z}(J))$ is by construction the set of zeroes of J in $k^{n+1}\backslash\{0\}$. Hence the set of zeroes of J in k^n is the set $q^{-1}(\mathrm{Z}(J))\cup\{0\}$ (since every non constant homogenous polynomial vanishes at the 0 vector). Now Q also vanishes on $q^{-1}(\mathrm{Z}(J)) \cup \{0\}$ and so by the strong Nullstellensatz we have $Q \in \mathfrak{r}(J) = J$. $\square$

Lemma 7.4. *Let $J \subseteq k[x_0,\dots,x_n]$ be a homogenous radical ideal. Then the subset $\mathrm{Z}(J)$ of $\mathbb{P}^n(k)$ is empty iff $J = k[x_0,\dots,x_n]$ or $J = k[x_0,\dots,x_n]_+$.*

Here $k[x_0,\dots,x_n]_+$ is the homogenous ideal of $k[x_0,\dots,x_n]$ generated by all the non constant homogenous polynomials.

Proof. We first prove the $\Leftarrow$ direction of the equivalence. So let $\bar{v} = \langle v_1,\dots,v_n\rangle \in k^{n+1}\backslash\{0\}$. Suppose that $v_{i_0} \neq 0$ for some $i_0 \in \{0,\dots,n\}$. The homogenous polynomial $x_{i_0} \in k[x_0,\dots,x_n]_+$ does not vanish at $[\bar{v}]$. Since $\bar{v} \in k^{n+1}\backslash\{0\}$ was arbitrary, we see that $\mathrm{Z}(J)$ is empty if $J = k[x_0,\dots,x_n]_+$ or $J = k[x_0,\dots,x_n]$.

We now prove the $\Rightarrow$ direction. So suppose that $\mathrm{Z}(J) = \emptyset$. To avoid notational confusion, write $Z_{\mathrm{aff}}(I)$ for the set of common zeroes in k^{n+1} of the elements of a (not necessarily homogenous) ideal $I \subseteq k[x_0,\dots,x_n]$. By using the map $q : k^{n+1}\backslash\{0\} \to \mathbb{P}^n(k)$ described in the proof of Proposition 7.3, we see that

$$Z_{\mathrm{aff}}(J) \cap (k^{n+1}\backslash\{0\}) = \emptyset.$$

Now suppose first that J does not contain any non zero constant polynomials. Then $0 \in Z_{\text{aff}}(J)$ (because J is generated by non constant homogenous polynomials) so that $Z_{\text{aff}}(J) = \{0\}$. Using the correspondence described after Proposition 2.3, we conclude that J is the radical ideal of $k[x_0, \ldots, x_n]$ associated with the point 0, which is $k[x_0, \ldots, x_n]_+$. If J contains a non zero constant polynomial then $J = k[x_0, \ldots, x_n]$ (because J contains a unit). So we conclude that if $\mathrm{Z}(J) = \emptyset$ then either $J = k[x_0, \ldots, x_n]_+$ or $J = k[x_0, \ldots, x_n]$. $\square$

We shall call the ideal $k[x_0, \ldots, x_n]_+$ the *irrelevant* ideal of $k[x_0, \ldots, x_n]$.

We conclude from Lemma 7.4 and Proposition 7.3 that there is a correspondence

$$\{\text{closed sets in } \mathbb{P}^n(k)\} \underset{Z}{\overset{\mathcal{I}}{\rightleftarrows}} \{\text{non irrelevant homogenous radical ideals in } R_n\}$$

where the maps $\mathrm{Z}(\cdot)$ and $\mathcal{I}(\cdot)$ are inverse to each other.

A *projective variety* is a variety isomorphic (as a variety) to a closed subvariety of $\mathbb{P}^n(k)$ (for some $n \geqslant 0$).

A *quasi-projective variety* is a variety isomorphic to an open subvariety of a projective variety.

8 Dimension

Let T be a topological space. Then T is said to be *noetherian* if for any descending sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$$

of closed subsets of T, there is an $i_0 \geqslant 0$ such that $C_{i_0} = C_{i_0+1} = \ldots$. In this situation, we say that the sequence stabilises at i_0. Note that any subset of a noetherian topological space is also noetherian (in the induced topology) (why?). Finally, note that a noetherian topological space is quasi-compact (ie any covering of the space has a finite subcovering). See exercises.

The topological space T is said to be *irreducible* if T is not empty and any open subset of T is dense in T.

Example. The Zariski topology on k^n is noetherian. Indeed any descending sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$$

of closed subsets of k^n corresponds uniquely to a sequence

$$\mathcal{I}(C_1) \subseteq \mathcal{I}(C_2) \subseteq \mathcal{I}(C_3) \subseteq \ldots$$

(see the first section) and such a sequence stabilises for some index because $k[x_1, \ldots, x_n]$ is a noetherian ring (by Hilbert's basis theorem). Consequently, the topology of any algebraic set is noetherian. A closed subspace Z of k^n is irreducible iff Z is irreducible as an algebraic set (why?).

Lemma 8.1. *Let T be a non empty noetherian topological space. Then there is a unique finite collection $\{T_i\}$ of irreducible closed subsets of T such that*

(1) $T = \cup_i T_i$

(2) $T_i \not\subseteq \cup_{j \neq i} T_j$ for all i.

Note that a consequence of the lemma is that the T_i are the irreducible closed subsets of T which are maximal for the relation of inclusion among all the irreducible closed subsets contained in T (why?).

Proof. See exercises. $\square$

The closed subsets T_i described in Lemma 8.1 are called the *irreducible components* of T. If T is an algebraic set, the decomposition of T into irreducible components coincides with the decomposition given by Lemma 2.6 (why?).

Lemma 8.2. *A variety is noetherian.*

Proof. Let V be a variety. Let

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

be a descending sequence of closed subsets of V. Let $\{U_i\}$ be a finite covering of V by open affine subvarieties. Since the U_i are noetherian (as topological spaces) by the remark above and since there are only finitely many U_i, there is an integer $l \geqslant 1$ such that $C_l \cap U_i = C_{l+1} \cap U_i = \dots$ for all i. Since the U_i cover V, this implies that $C_l = C_{l+1} = \dots$ $\square$

Now consider again a non empty topological space T. The *dimension* $\dim(T)$ of T is

$$\dim(T) := \sup\{t \,|\, \text{there are irreducible closed subsets } C_0, \dots, C_t \subseteq T \text{ such that } C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_t\}.$$

Note that $\dim(T)$ might be infinite. Dimension is not defined for the empty topological space (note that some authors define the dimension of the empty topological space to be -1).

Lemma 8.3. *Let $V \subseteq k^n$ be an algebraic set. Then* $\dim(V) = \dim(\mathcal{C}(V))$.

Here $\dim(\mathcal{C}(V))$ is the dimension of $\mathcal{C}(V)$ as a ring (see Def. 11.1 in CA). Recall that by definition we have

$$\dim(R) := \sup\{n \,|\, \exists\, \mathfrak{p}_0, \dots, \mathfrak{p}_n \in \mathrm{Spec}(R) : \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \dots \supsetneq \mathfrak{p}_n\}$$

for any ring R.

Proof. We have already seen that irreducible closed subsets of V correspond to prime ideals of $\mathcal{C}(V)$ (see Lemma 2.5). Hence the definition of $\dim(\mathcal{C}(V))$ corresponds with the definition of $\dim(V)$ under the correspondence between radical ideals of $\mathcal{C}(V)$ and closed subsets of V described at the beginning of section one. $\square$

Theorem 8.4. (1) *The dimension of k^n is n.*

(2) *The dimension of $\mathbb{P}^n(k)$ is n.*

Proof. (1) We saw in CA that $\dim(k[x_1, \dots, x_n]) = n$ (see Cor. 11.27 in CA). Hence $\dim(k^n) = n$ by Lemma 8.3. (2) Apply Q6 in exercise sheet 2 to the open covering of $\mathbb{P}^n(k)$ by its standard coordinate charts and use (1). $\square$

Definition 8.5. *Let T be a topological space. Let $C \subseteq T$ be a closed irreducible subspace. The codimension, or height of C is*

$$\mathrm{cod}(C, T) = \mathrm{ht}(C, T) := \sup\{t \,|\, \text{there are irreducible closed subsets } C_1, \dots, C_t \subseteq T \text{ such that } C \subsetneq C_1 \subsetneq \dots \subsetneq C_t\}$$

We shall sometimes write $\mathrm{cod}(C)$ and $\mathrm{ht}(C)$ instead of $\mathrm{cod}(C,T)$ and $\mathrm{ht}(C,T)$, respectively, when the ambient topological space T is clear from the context.

Note that from the definitions, we have

$$\dim(T) = \sup_{C \text{ closed irreducible subset of } T} \mathrm{ht}(C,T).$$

Suppose that $C, V \subseteq k^n$ are algebraic sets in k^n and that $C \subseteq V$. Suppose that C is irreducible. Then the height of C in V is the height of the prime ideal $\mathcal{I}(C) \,(\mathrm{mod}\, \mathcal{I}(V))$ of $\mathcal{C}(V)$ (in the sense of section 11 of CA). The proof is similar to the proof of Lemma 8.3 (we leave the details to the reader).

Proposition 8.6. *Let V be a variety. Let $C \subseteq V$ be an irreducible closed subset. Then $\dim(V)$ and $\mathrm{cod}(C,V)$ are finite.*

Proof. See Q6 (4) in Sheet 2. □

Finally, we also have the following difficult result of commutative algebra, which justifies the use of the word "codimension".

Theorem 8.7. *Let R be a finitely generated k-algebra. Suppose that R is an integral domain. Let $\mathfrak{p} \subseteq R$ be a prime ideal. Then we have*

$$\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$$

The proof of this theorem is given in the Appendix. The proof is in several steps and is structured as an exercise with model solution. We suggest the reader go through the steps by themselves first without looking at the model solution. Note that the proof of Theorem 8.7 is not examinable.

Corollary 8.8. *Let V be an irreducible variety. Let $C \subseteq V$ be an irreducible closed subset. Then*

$$\mathrm{cod}(C,V) + \dim(C) = \dim(V)$$

Note first that from the definitions, we have

$$\mathrm{cod}(C,V) + \dim(C) \leqslant \dim(V)$$

(why?). So we only have to to prove that $\mathrm{cod}(C,V) + \dim(C) \geqslant \dim(V)$.

Proof. Let $\{V_i\}$ be a finite open covering of V. We suppose that each V_i is isomorphic to an affine variety when viewed as an open subvariety of V. Note that since V is irreducible, each V_i is irreducible as well (why?). We use Q6 of Sheet 2 and we obtain

$$\sup_{i, C \cap V_i \neq \emptyset} \mathrm{cod}(C \cap V_i, V_i) = \mathrm{cod}(C,V)$$

and

$$\sup_{i, C \cap V_i \neq \emptyset} \dim(C \cap V_i) = \dim(C)$$

Let R_i be the coordinate ring associated with V_i (so that $V_i \simeq \mathrm{Spm}(R_i)$ as a set). Then R_i is integral by Q4 (3) of Sheet 2. Hence we may apply Theorem 8.7 and we compute

$$\mathrm{cod}(C \cap V_i, V_i) + \dim(C \cap V_i) = \dim(V_i)$$

if $C \cap V_i \neq \emptyset$. Applying $\sup_{i, C\cap V_i \neq \emptyset}(\cdot)$ to both sides of this equality and using Q6 of Sheet 2 again, we see that there is an index i_0 such that $C \cap V_{i_0} \neq \emptyset$ and such that

$$\mathrm{cod}(C \cap V_{i_0}, V_{i_0}) + \dim(C \cap V_{i_0}) = \dim(V)$$

and hence

$$\mathrm{cod}(C, V) + \dim(C) \geqslant \mathrm{cod}(C \cap V_{i_0}, V_{i_0}) + \dim(C \cap V_{i_0}) = \dim(V).$$

which is what we wanted to prove. □

Here is another fundamental result from the CA course, which is relevant to the theory of dimension.

Theorem 8.9. *Let $n \geqslant 0$ and let $V, W \subseteq k^n$ be algebraic sets. Suppose that $V \subseteq W$. Suppose that $I \subseteq k[x_1, \dots, x_n]$ is such that $\mathrm{Z}(I) = V$.*

Let $l \geqslant 1$ and suppose that the ideal $I \,(\mathrm{mod}\, \mathcal{I}(W)) \subseteq \mathcal{C}(W)$ is generated by l elements.

Then every irreducible component of V has codimension $\leqslant l$ in W.

Furthermore, if C is an irreducible component of V then there is an ideal $J \subseteq \mathcal{I}(C) \subseteq \mathcal{C}(W)$ which is generated by $\mathrm{cod}(C, W)$ elements and such that C is an irreducible component of $\mathrm{Z}(J) \subseteq W$ (in other words $\mathcal{I}(C)$ is a prime ideal, which is minimal among the prime ideals which contain J).

See Cor. 11.15 and Cor. 11.17 in CA for the proof. This is a consequence of Krull's principal ideal theorem.

9 Rational maps

Let V, W be varieties. Consider the set $H = H_{V,W}$ whose elements are morphisms $f : U \to W$, where U is a non empty open subvariety of V. Let $\sim = \sim_{V,W}$ be the relation on H, such that $f : U \to W$ and $g : O \to W$ are related by $\sim$ iff there is a open subvariety UO of $U \cap O$, which is dense in V and which is such that $f|_{UO} = g|_{UO}$. The relation $\sim$ is easily seen to be an equivalence relation. We shall write $\mathrm{Rat}(V, W)$ for the set of equivalences classes of H under the relation $\sim$. We call elements of $\mathrm{Rat}(V, W)$ *rational maps* from V to W. Beware that rational maps are not actual maps but equivalence classes of maps.

Suppose now until further notice that V is irreducible.

Note the following. Let $f : U \to W$ be a representative of a rational map from V to W. If f is dominant, then any other representative of the same rational map is dominant as well. Indeed, let $g : O \to W$ be another representative of the rational map defined by f. Then $f|_{U\cap O} = g|_{U\cap O}$. Suppose for contradiction that g is not dominant. Then $W \backslash g(O)$ contains a non empty open subset W_1. Since $f : U \to W$ is dominant, we know that $f^{-1}(W_1) \neq \emptyset$. Thus, since V is irreducible, we have $f^{-1}(W_1) \cap (U \cap O) = g^{-1}(W_1) \cap (U \cap O) \neq \emptyset$. In particular $g^{-1}(W \backslash g(O)) \neq \emptyset$, which is a contradiction. So g is also dominant.

We thus see from the discussion in the last paragraph that it makes sense to speak of a *dominant rational map* from V to W: it is a rational map all of whose representative are dominant (or equivalently, it is a rational map with one dominant representative). We shall write $\mathrm{Rat}_{\mathrm{dom}}(V, W)$ for the set of dominant rational maps from V to W.

We shall write $\kappa(V)$ as a shorthand for $\mathrm{Rat}(V, k)$. If $f : U \to k$ and $g : O \to k$ are two elements of $H_{V,k}$, one may define a new element $f + g : U \cap O \to k$ of $H_{V,k}$ by declaring that $(f + g)(u) = f(u) + g(u)$ for all $u \in U \cap O$ (note that $U \cap O$ is not empty because V is irreducible). Note that $f + g : U \cap O \to k$ is a morphism of varieties (use Proposition 4.5). Similarly, one may define an element $fg = f \cdot g : U \cap O \to k$ by declaring that $(f \cdot g)(u) = f(u) \cdot g(u)$ for all $u \in U \cap O$. Again, $f \cdot g : U \cap O \to k$ is a morphism (same reasoning as before). Finally, if $f : U \to k$ does not vanish on all of U, then we may define $f^{-1} : U \backslash \mathrm{Z}(f) \to k$ by the formula $f^{-1}(u) = 1/f(u)$. Here again, $f^{-1} : U \backslash \mathrm{Z}(f) \to k$ is a morphism (reason as before and use the remark after Proposition 4.5). It is again easily verified that these operations are compatible with $\sim_{V,k}$ and we thus obtain a structure of field on $\kappa(V)$. This field is called the *function field* of V. There is an obvious injection $k \hookrightarrow \kappa(V)$ which makes $\kappa(V)$ into a k-algebra. Note finally that for any $v \in V$, there is a natural injection $\mathcal{O}_{V,v} \hookrightarrow \kappa(V)$, which sends any representative of an equivalence class in $\mathcal{O}_{V,v}$ to its equivalence class in $\kappa(V)$. So $\kappa(V)$ naturally contains the local rings at all the points of V.

Now suppose that we are given a dominant morphism of irreducible varieties $a : V \to W$. Then we may define a map $H_{W,k} \to H_{V,k}$ by the recipe

$$(f : O \to k) \mapsto (f \circ a|_{f^{-1}(O)} : f^{-1}(O) \to k)$$

where O is a non empty open subvariety of W and $f : O \to k$ is an element of $H_{W,k}$. This definition makes sense because $f^{-1}(O) \neq \emptyset$ as f is dominant. One checks (we skip te details) that this map is compatible with the relations $\sim_{W,k}$ and $\sim_{V,k}$ and also with the operations $+$, $(\cdot)^{-1}$ and $\cdot$. One thus obtains a map of rings

$$a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V).$$

Note that since $\kappa(W)$ is a field, the map $a^{*,\mathrm{rat}}$ is injective. Also, if $a : V \to W$ is the inclusion of an open subvariety of V into W, it follows from the definitions that the map $a^{*,\mathrm{rat}}$ is a bijection (check!). Finally, the construction of $a^{*,\mathrm{rat}}$ is clearly compatible with compositions of dominant morphisms (ie if $b : W \to W_1$ is another dominant morphism of irreducible varieties, then $(b \circ a)^{*,\mathrm{rat}} = a^{*,\mathrm{rat}} \circ b^{*,\mathrm{rat}}$). We conclude from all this that the homomorphism $a^{*,\mathrm{rat}}$ only depends on the element of $\mathrm{Rat}(V, W)$ defined by a. In turn, any dominant representative $g : O \to W$ of an element of $\mathrm{Rat}(V, W)$ defines a map of k-algebras $g^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V) \simeq \kappa(O)$ and again it follows from the definitions that this map only depends on the class of g in $\mathrm{Rat}(V, W)$. So all in all, any dominant rational map $\rho \in \mathrm{Rat}_{\mathrm{dom}}(V, W)$ gives rise to an injection of fields $\rho^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V)$.

Lemma 9.1. *Let X be an irreducible affine variety. Let $V \subseteq k^n$ be an algebraic set giving rise to X. Then there is a canonical isomorphism of k-algebras $\kappa(X) \to \mathrm{Frac}(\mathcal{C}(V))$. This isomorphism is compatible with dominant regular maps between irreducible algebraic sets and the corresponding morphisms of varieties.*

Note that by Q4 of Sheet 2, the fact that V irreducible implies that the ring $\mathcal{C}(V)$ is an integral domain. So it makes sense to talk about the fraction field $\mathrm{Frac}(\mathcal{C}(V))$ of $\mathcal{C}(V)$.

Proof. Define a map

$$\tau : \mathcal{C}(V) \to \kappa(X)$$

by sending an element of $\mathcal{C}(V)$ to the equivalence class of the corresponding morphism $X \to k$. By construction, this is a map of k-algebras. Now suppose that $f : V \to k$ is a regular map and suppose that $\tau(f) = 0$.

Then by definition f vanishes on an open subset of V. However the vanishing set of f is equal to $\mathrm{Z}(f)$ and so is closed. Hence $\mathrm{Z}(f)$ contains the closure in V of an open subset of V and thus $\mathrm{Z}(f) = V$ (because V is irreducible). Hence f is the map with constant value 0 and thus $\tau(f) = 0$. We thus see that h is injective. Hence τ extends to a (necessarily injective) map of fields $\mathrm{Frac}(\mathcal{C}(V)) \to \kappa(X)$ by the universal property of localisation (see Lemma-Definition 5.1 in CA). We have to show that this last map is surjective. To see this, let O be an open subset of V and let $g : O \to k$ be a representative of an element of $\kappa(X)$. By Lemma 4.1, we may assume without restriction of generality that $O = V\backslash\mathrm{Z}(f)$, where $f \in \mathcal{C}(V)$. By Corollary 4.4, we know that $g = \frac{g_1}{g_2}|_O$, where $g_1, g_2 \in \mathcal{C}(V)$. Hence $\tau(g_1/g_2) = g$. Since g was arbitrary, we have shown that the map $\mathrm{Frac}(\mathcal{C}(V)) \to \kappa(X)$ is surjective, and thus an isomorphism.

The fact that this isomorphism is compatible with dominant regular maps between irreducible algebraic sets follows directly from the definition of τ. □

Proposition 9.2. *Let V be an irreducible variety. Then $\kappa(V)$ is finitely generated over k as a field and the dimension of V is equal to the transcendence degree of $\kappa(V)$ over k.*

Recall that the transcendence degree of $\kappa(V)$ over k is the largest integer $n \geqslant 0$ such that there exists an injection of k-algebras

$$k[x_1, \dots, x_n] \hookrightarrow \kappa(V)$$

See section 11.1 of CA for details.

Proof. (of Proposition 9.2) Let $\{V_i\}$ be a finite open covering of V and suppose that each V_i is an affine variety. By a remark at the beginning of this section, the function field of V_i is isomorphic to the function field of V as a k-algebra. On the other hand, we have $\dim(V) = \sup_i \dim(V_i))$ by Q6 of sheet 2. Hence it is sufficient to show that the transcendence degree of $\kappa(V_i)$ over k is equal to $\dim(V_i)$ for all i. So we may suppose without restriction of generality that V is affine. In that case, the statement is a consequence of Lemma 8.3, Lemma 9.1 and Cor. 11.28 in CA (which follows from the computation of the dimension of polynomial rings and the Noether normalisation lemma). □

Proposition 9.3. *Let $a : V \to W$ be a dominant morphism of irreducible subvarieties. Then $a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V)$ is an isomorphism iff there exist open subvarieties $V_0 \subseteq V$ and $W_0 \subseteq W$ such that $a(V_0) \subseteq W_0$ and such that the induced morphism $a|_{V_0} : V_0 \to W_0$ is an isomorphism.*

Proof. The $\Leftarrow$ direction of the equivalence is clear (why?) so we only have to establish the $\Rightarrow$ direction. Let $W_{00} \subseteq W$ be an open affine subvariety and let V_{00} be an open affine subvariety of $a^{-1}(W_0)$ (this exists by Proposition 5.1). We claim that the map $V_{00} \to W_{00}$ induced by a is also dominant. To prove this claim, suppose for contradiction that the map $V_{00} \to W_{00}$ is not dominant. Then there is a non empty subset O of W_{00} such that $O \subseteq W_{00}\backslash a(V_{00})$. Hence $a^{-1}(O) \cap V_{00} = \emptyset$. Now $a^{-1}(O) \neq \emptyset$ since a is dominant, so this contradicts the irreducibility of V. We have thus established the claim. Since the inclusions $V_{00} \to V$ and $W_{00} \to W$ induce isomorphisms of function fields, we may thus assume without restriction of generality that V and W are affine to begin with. In view of Lemma 9.1 and Q4 (3) of Sheet 2, it is thus sufficient to prove the following statement of commutative algebra.

Let $\phi : A \to B$ be a homomorphism of finitely generated integral k-algebras. Suppose that $\mathrm{Spm}(\phi)(\mathrm{Spm}(B))$ is dense in $\mathrm{Spm}(A)$ and suppose that the induced map $\mathrm{Frac}(\phi) : \mathrm{Frac}(A) \to \mathrm{Frac}(B)$ is an isomorphism. Then there is an element $f \in A$ such that the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is an isomorphism.

To prove this assertion, note that by Q5 of Sheet 1 we already know that under the given assumptions, ϕ must be injective. Note also that since we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Frac}(A) & \xrightarrow{\mathrm{Frac}(\phi)} & \mathrm{Frac}(B) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\phi} & B \end{array}$$

all whose maps are injective, the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is injective for any choice of $f \in A\backslash\{0\}$ (remember that A and B are integral domains). Thus we only have to show that there is $f \in A\backslash\{0\}$ such that the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is surjective. Now let $b_1, \dots, b_l$ be generators of B as a k-algebra. Let $a_1/c_1, \dots, a_l/c_l \in \mathrm{Frac}(A)$ such that

$$b_i/1 = \phi(a_i)/\phi(c_i) =: \mathrm{Frac}(\phi)(a_i/c_i)$$

for all $i \in \{1, \dots, l\}$. Let $f := \prod_i c_i$. Then $b_i/1 = \mathrm{Frac}(\phi)(a_i(\prod_{j \neq i} c_j)/f)$. Hence the image of

$$A[f^{-1}] \to B[\phi(f)^{-1}]$$

contains $b_i/1$ for all $i \in \{1, \dots, l\}$ and also contains $1/\phi(f) = \mathrm{Frac}(\phi)(1/f)$. Since $B[\phi(f)^{-1}]$ is generated as a k-algebra by $1/\phi(f)$ and by the elements $b_i/1$ (use Lemma 5.3 in CA), we see that $A[f^{-1}] \to B[\phi(f)^{-1}]$ is surjective. $\square$

If V and W are irreducible varieties, and $V_0 \subseteq V$ and $W_0 \subseteq W$ are open subvarieties such that $V_0 \simeq W_0$, we shall say that V and W are *birational*, or *birationally isomorphic*.

A *birational map* from V to W is a rational map from V to W which has a representative $f : O \to W$, such that $f(O)$ is open and such that the induced map $O \to f(O)$ is an isomorphism (where $f(O)$ is endowed with its structure of open subvariety of W). A *birational morphism* from V to W is a morphism $V \to W$ which induces a birational map.

Proposition 9.3 implies that a dominant rational map $\rho \in \mathrm{Rat}_{\mathrm{dom}}(V, W)$ is birational iff $a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V)$ is bijective.

Proposition 9.4. *Let V, W be irreducible varieties. Let $\kappa(W) \hookrightarrow \kappa(V)$ be a field extension compatible with the k-algebra structures. Then there is an open subvariety V_0 of V and a dominant morphism $a : V_0 \to W$ such that the extension $a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V_0)$ is isomorphic to $\kappa(W) \hookrightarrow \kappa(V)$ as a $\kappa(W)$-extension.*

A different wording of the conclusion of the proposition is that there is an isomorphism of rings between $\kappa(V)$ and $\kappa(V_0)$ compatible with the given $\kappa(W)$-algebra structures.

Proof. We may suppose without restriction of generality that V and W are affine varieties (why?). Let B (resp. A) be the coordinate ring of V (resp. W). Let $\iota : \mathrm{Frac}(A) \simeq \kappa(W) \hookrightarrow \kappa(V) \simeq \mathrm{Frac}(B)$ be the given field extension.

We claim that there is an $g \in B\backslash\{0\}$ such that $\iota(A) \subseteq B[g^{-1}] \subseteq \mathrm{Frac}(B)$ (where A is identified with its image in $\mathrm{Frac}(A)$). To prove this, let $a_1, \dots, a_l$ be generators of A as a k-algebra. For all $i \in \{1, \dots, l\}$ let $b_i, c_i \in B$ be such that $b_i/c_i = \iota(a_i/1)$. Let $g := \prod_i c_i$. We then have $\iota(a_i/1) \in B[g^{-1}]$ and thus $\iota(A) \subseteq B[g^{-1}]$, proving the claim.

Now let V_0 be the open affine subvariety associated with $B[g^{-1}]$. Let $\iota_0 : A \to B[g^{-1}]$ be the map induced by ι and the natural map from A to $\mathrm{Frac}(A)$. Since the map ι_0 is injective, it induces a dominant map $V_0 \to W$ by Q5 of Sheet 1. Hence V_0 and the map $V_0 \to W$ satisfy the requirements of the proposition. $\square$

Finally, note the following. Let V and W be irreducible varieties. Consider the map

$$\mathrm{Rat}_{\mathrm{dom}}(V,W) \to \text{homomorphisms of } k\text{-algebras } \kappa(W) \to \kappa(V) \quad (*)$$

which sends $a \in \mathrm{Rat}_{\mathrm{dom}}(V,W)$ to $a^{*,\mathrm{rat}} : \kappa(W) \to \kappa(V)$. *Proposition 9.4 implies that this map is surjective.* On the other hand we have

Lemma 9.5. *The map* $(*)$ *is injective.*

Proof. Let $a_1, a_2 \in \mathrm{Rat}_{\mathrm{dom}}(V,W)$ and suppose that $a_1^{*,\mathrm{rat}} = a_2^{*,\mathrm{rat}}$. We have to show that $a_1 = a_2$.

Now there is a by construction an open subset $O \subseteq V$ and morphisms $\alpha_1, \alpha_2 : O \to W$ which represent a_1 and a_2, respectively. Replacing W by one of its open affine subvarieties O' and replacing V by an open affine subvariety of $\alpha_1^{-1}(O')$, we may assume that both V and W are affine and that a_1 (resp. a_2) is represented by a morphism. Recycling notation, call $\alpha_1 : V \to W$ (resp. $\alpha_2 : V \to W$) a morphism representing a_1 (resp. a_2).

Now let B (resp. A) be the coordinate ring of V (resp. W).

Let $\iota : \mathrm{Frac}(A) \simeq \kappa(W) \hookrightarrow \kappa(V) \simeq \mathrm{Frac}(B)$ be the field extension given by $a_1^{*,\mathrm{rat}} = a_2^{*,\mathrm{rat}}$. We have by construction a commutative diagram

$$\begin{array}{ccc} \mathrm{Frac}(A) & \xrightarrow{\alpha_i^{*,\mathrm{rat}}} & \mathrm{Frac}(B) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\alpha_i^{*}} & B \end{array}$$

for $i \in \{1,2\}$. Since the vertical maps are injective and $a_1^{*,\mathrm{rat}} = a_2^{*,\mathrm{rat}}$, we thus have $\alpha_1^* = \alpha_2^*$. $\square$

In view of the last lemma and the comment preceding it, we thus see that *the map* $(*)$ *is bijective.* In particular, there is a one-to-one correspondence between dominant rational maps from V to W and $\kappa(W)$-algebra structures on the field $\kappa(V)$.

We shall from now on often write a^* for $a^{*,\mathrm{rat}}$ when V and W are irreducible varieties and $a \in \mathrm{Rat}(V,W)$. This is justified by the proof of Lemma 9.5.

10 Products

We wish to endow the cartesian product of two varieties with the structure of a variety. We shall do this for quasi-projective varieties.

Definition 10.1. *Let V and W be varieties. A product of V and W is a triple $(V \prod W, \pi_V, \pi_W)$, where $V \prod W$ is a variety and $\pi_V : V \prod W \to V$ and $\pi_W : V \prod W \to W$ are morphisms of varieties. This triple is required to have the following property* (PROD).

(PROD) *If X is a variety and $a : X \to V$ and $b : X \to W$ are morphisms of varieties, then there is a unique morphism of varieties $a \prod b : X \to V \prod W$ such that $\pi_V \circ (a \prod b) = a$ and $\pi_W \circ (a \prod b) = b$.*

Note that property (PROD) in Definition 10.1 characterises the triple $(V \prod W, \pi_V, \pi_W)$ uniquely up to unique isomorphism of triples (an isomorphism of triples is an isomorphism of the underlying varieties which is compatible with the morphisms in play). This is an example of categorical product. Note that if V and W are varieties, it is not clear *a priori* that they have a product. However, if the product of V and W exists, it is uniquely defined. Abusing language, we shall often say that $V \prod W$ is the product of V and W without writing the associated morphisms π_V and π_W. We first show

Theorem 10.2. *Let $m, n \geqslant 0$. The product $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$ exists.*

Before starting with the proof, we make a construction. We shall consider the projective space $\mathbb{P}^{mn+m+n}$. The space $\mathbb{P}^{mn+m+n}$ is by definition the set of lines through the origin in the vector space $k^{mn+m+n+1=(m+1)(n+1)}$ and we shall index its standard basis using double indices.

Let $b_1, \dots, b_{mn+m+n+1}$ be the standard basis of $k^{mn+m+n+1}$, indexed in the usual manner. If $i \in \{0, \dots, m\}$ and $j \in \{0, \dots, n\}$, we shall write b_{ij} for the element $b_{j(m+1)+i+1}$. With this convention, each b_l corresponds to precisely one b_{ij}. Since we shall exclusively work with double indices, this formula for b_{ij} is actually not important. One only needs to know that the b_{ij} form a basis of $k^{mn+m+n+1}$.

Let $\sigma : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \to \mathbb{P}^{mn+m+n}$ be the map given by the formula

$$\sigma(([X_0, \dots, X_m], [Y_0, \dots Y_n])) = [(X_i Y_j)_{ij}]$$

where $(\cdot)_{ij}$ means that we put $(\cdot)$ in the coordinate ij (corresponding to b_{ij}). We will write Z_{ij} for a variable quantity in the coordinate ij.

Lemma 10.3. *The map σ is injective and its image is closed in $\mathbb{P}^{mn+m+n}$.*

Proof. (of Lemma 10.3). For each $[Z_{ij}] \in \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$ let $i_0 j_0 = i_0([Z_{ij}]) j_0([Z_{ij}])$ be a pair of indices such that $Z_{i_0 j_0} \neq 0$. The map $\tau : \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k)) \to \mathbb{P}^m(k) \times \mathbb{P}^n(k)$ sending $[Z_{ij}]$ to

$$([Z_{0j_0}, Z_{1j_0}, \dots, Z_{mj_0}], [Z_{i_0 0}, Z_{i_0 1}, \dots Z_{i_0 n}])$$

has the property that $\tau \circ \sigma = \mathrm{Id}_{\mathbb{P}^m(k) \times \mathbb{P}^n(k)}$ (why?). Hence σ is injective. To show that $\sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$ is closed, consider the subvariety of $\mathbb{P}^{mn+m+n}$ described by the homogenous equations $Z_{ij} Z_{rs} - Z_{is} Z_{rj}$ (for all $i, r \in \{0, \dots, m\}$ and $j, s \in \{0, \dots, n\}$. We clearly have

$$X_i X_j X_r X_s = X_i X_s X_r X_j$$

and so $\mathrm{Z}((Z_{ij} Z_{rs} - Z_{is} Z_{rj})) \supseteq \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$. On the other hand, if we let $[Z_{ij}] \in \mathrm{Z}((Z_{ij} Z_{rs} - Z_{is} Z_{rj}))$ and $Z_{i_0 j_0} \neq 0$ (say) then

$$\sigma(([Z_{0j_0}, Z_{1j_0}, \dots, Z_{mj_0}], [Z_{i_0 0}, Z_{i_0 1}, \dots Z_{i_0 n}])) = [(Z_{ij_0} Z_{i_0 j})_{ij}] = [(Z_{ij} Z_{i_0 j_0})_{ij}] = [Z_{ij}]$$

so we also have $\mathrm{Z}((Z_{ij} Z_{rs} - Z_{is} Z_{rj})) \subseteq \sigma(\mathbb{P}^m(k) \times \mathbb{P}^n(k))$. □

We record one output of the proof of the last lemma: the image of the map σ is the zero set of the set of quadratic equations $Z_{ij} Z_{rs} = Z_{is} Z_{rj}$. The map σ is called the *Segre embedding*. Its image is called the *Segre variety* (which is a closed subvariety of $\mathbb{P}^{mn+m+n}$).

Proof. (of Theorem 10.2). Endow $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ with the variety structure inherited from the Segre variety via the Segre embedding. We will show that $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ has the properties listed in Definition 10.1. We first show that the projections $\pi_1 : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \to \mathbb{P}^m(k)$ and $\pi_2 : \mathbb{P}^m(k) \times \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ are morphisms of varieties. For any $i_0 \in \{0, \dots, m\}$ and any $j_0 \in \{0, \dots, n\}$, let $U_{i_0 j_0} \subseteq \mathbb{P}^{mn+m+n}$ be the open subset of the elements $[Z_{ij}]$ such that $Z_{i_0j_0} \neq 0$ (this is a standard coordinate chart of $\mathbb{P}^{mn+m+n}$). Let $\pi_{i_0j_0,1} : U_{i_0j_0} \to \mathbb{P}^m(k)$ be given by the formula

$$\pi_{i_0j_0,1}([Z_{ij}]) := [Z_{0j_0}, Z_{1j_0}, \ldots, Z_{mj_0}]$$

By Q2 of Sheet 2, this defines a morphism from $U_{i_0j_0}$ to $\mathbb{P}^m(k)$. Now suppose that

$$\sigma(([X_0, \ldots, X_m], [Y_0, \ldots Y_n])) = [(X_iY_j)_{ij}] \in U_{i_0j_0}$$

In other words, $X_{i_0}, Y_{j_0} \neq 0$. Then

$$\begin{aligned} & \pi_{i_0j_0,1}(\sigma(([X_0, \ldots, X_m], [Y_0, \ldots Y_n]))) = \pi_{i_0j_0,1}([(X_iY_j)_{ij}]) = [X_0Y_{j_0}, X_1Y_{j_0}, \ldots, X_mY_{j_0}] \\ = \; & [X_0, X_1, \ldots, X_m] = \pi_1(([X_0, \ldots, X_m], [Y_0, \ldots Y_n])) \end{aligned}$$

Hence π_1 is a morphism on the open subset $\sigma^{-1}(U_{i_0j_0})$ of $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$. Now if we vary the indices i_0 and j_0, the open subsets $\sigma^{-1}(U_{i_0j_0})$ cover all of $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ and hence π_1 is a morphism (by Definition 4.7 and the fact that functions on open subsets of varieties are regular iff there are regular locally [see Definition 4.6]). Similarly π_2 is a morphism.

Choosing $\pi_{\mathbb{P}^m(k)} := \pi_1$ and $\pi_{\mathbb{P}^n(k)} := \pi_2$, we shall now verify (PROD) in Definition 10.1. So let X be a variety and $a : X \to \mathbb{P}^m(k)$ and $b : X \to \mathbb{P}^n(k)$ be morphisms of varieties. We have to show that there is a unique morphism of varieties $c : X \to \mathbb{P}^m(k) \times \mathbb{P}^n(k)$ such that $\pi_1 \circ c = a$ and $\pi_2 \circ c = b$. Now note that the set $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ is the cartesian product of the sets $\mathbb{P}^m(k)$ and $\mathbb{P}^n(k)$. Hence, if the morphism c exists, it must be given by the formula $c(x) = (a(x), b(x))$ for all $x \in X$. Hence we only have to verify that c is a morphism of varieties. Since by the definition of a Topskf, a morphism is a morphism iff it is every locally a morphism, we may assume that X is affine and that $a(X) \subseteq U_{\mathbb{P}^m(k),i_0}$ and $b(X) \subseteq U_{\mathbb{P}^n(k),j_0}$ for some indices i_0 and j_0. Here $U_{\mathbb{P}^m(k),i_0}$ is the i_0-th standard coordinate chart of $\mathbb{P}^m(k)$ (resp. $U_{\mathbb{P}^n(k),j_0}$ is the j_0-th standard coordinate chart of $\mathbb{P}^n(k)$). So let us suppose that X is associated with an algebraic set $V \subseteq k^t$. The map a is then the restriction to V of a map $k^t \to U_{\mathbb{P}^m(k),i_0}$ of the form

$$\bar{v} \in k^t \mapsto [P_0(\bar{v}), \ldots, P_{i_0-1}(\bar{v}), 1, P_{i_0+1}(\bar{v}), \ldots P_m(\bar{v})]$$

where the P_h are polynomials in the entries $v_1, \ldots, v_t$ of the vector $\bar{v}$. Similarly, the map b is the restriction to V of a map $k^t \to U_{\mathbb{P}^n(k),j_0}$ of the form

$$\bar{v} \in k^t \mapsto [Q_0(\bar{v}), \ldots, Q_{j_0-1}(\bar{v}), 1, Q_{j_0+1}(\bar{v}), \ldots Q_n(\bar{v})]$$

where the P_l are polynomials in the entries $v_1, \ldots, v_t$ of the vector $\bar{v}$. We now compute

$$\sigma(c(\bar{v})) = [(P_i(\bar{v})Q_j(\bar{v}))_{ij}]$$

and so by Q2 of Sheet 2 again, we conclude that $\sigma \circ c$ is a morphism from V to $\mathbb{P}^{mn+m+n}$. Applying Lemma 5.3, we conclude that the morphism c is a morphism of varieties. $\square$

In the proof above, we have shown that $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$ can be realised as the Cartesian product $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ endowed with a certain variety structure. Furthermore, the projections $\pi_{\mathbb{P}^m(k)}$ and $\pi_{\mathbb{P}^m(k)}$ are then simply the ordinary projections on the two factors. We shall thus often write $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ instead of $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$.

We shall now use Theorem 10.2 to prove that any two quasi-projective varieties have a product.

We start with the

Lemma 10.4. *Let $C_1 \subseteq \mathbb{P}^m(k)$ and $C_2 \subseteq \mathbb{P}^n(k)$ be closed subsets. Let $V_1 \subseteq \mathbb{P}^m(k)$ and $V_2 \subseteq \mathbb{P}^n(k)$ be open subsets. Then the Cartesian product $C_1 \times C_2$ is closed in $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$ and the Cartesian product $V_1 \times V_2$ is open in $\mathbb{P}^m(k) \prod \mathbb{P}^n(k)$.*

Proof. Note that the second statement is a consequence of the first, because the complement of $V_1 \times V_2$ is $(\mathbb{P}^m(k) \backslash V_1) \times \mathbb{P}^n(k) \cup \mathbb{P}^m(k) \times (\mathbb{P}^n(k) \backslash V_2)$, which is closed according to the first statement. The proof of the first statement is similar to the proof of Lemma 10.3 and we leave it to the reader. □

Corollary 10.5. *Let V and W be two quasi-projective varieties. Then the product $V \prod W$ exists.*

Proof. By assumption, there are integers $m, n \geqslant 0$ and open subvarieties $O_1 \subseteq \mathbb{P}^m(k)$ and $O_2 \subseteq \mathbb{P}^m(k)$ such that V is isomorphic to a closed subvariety of O_1 and W is isomorphic to a closed subvariety of O_2. We may thus assume that V is a closed subvariety of O_1 and that W is a closed subvariety of O_2, where O_1 and O_2 are as above. Let $C_1 \subseteq \mathbb{P}^m(k)$ and $C_2 \subseteq \mathbb{P}^n(k)$ be closed subsets such that $C_1 \cap O_1 = V$ and $C_2 \cap O_2 = W$. We then have $V \times W = (C_1 \times C_2) \cap (O_1 \times O_2)$ and hence $V \times W$ is closed in the open set $O_1 \times O_2$ by Lemma 10.4. We endow the set $V \times W$ with the structure of variety which comes from its inclusion into $O_1 \times O_2$ as a closed subset. We now claim that $V \times W$ is a product of V and W. To see this, let X be a variety and let $a : X \to V$, $b : X \to W$ be two morphisms of varieties. Since the set $V \times W$ is the Cartesian product of V and W, we see as before that if the morphism $a \prod b$ exists, it must be given by the unique map $a \times b : X \to V \times W$ sending $x \in X$ to $(a(x), b(x))$. So we only have to verify that this map is a morphism. To see this, let $a' : X \to O_1$ be the map obtained by composing a with the inclusion $V \to O_1$ (resp. $b' : X \to O_1$ be the map obtained by composing b with the inclusion $W \to O_2$). Let $a'' : X \to \mathbb{P}^m(k)$ be the map obtained by composing a' with the inclusion $O_1 \to \mathbb{P}^m(k)$ (resp. $b'' : X \to \mathbb{P}^n(k)$ be the map obtained by composing b' with the inclusion $O_2 \to \mathbb{P}^n(k)$). We know that $a'' \times b''$ is a morphism by Theorem 10.2. Next, we know that $a' \times b'$ is a morphism because $(a' \times b')(X) \subseteq O_1 \times O_2$ and because $O_1 \times O_2$ is open in $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$ by Lemma 10.4. Finally, by Lemma 5.3, we know that $a \times b$ is a morphism, as $(a \times b)(X) \subseteq V \times W$, and $V \times W$ is closed in $O_1 \times O_2$ by the above reasoning. So we have verified that $a \times b$ is a morphism. This completes the proof. □

An outcome of the proof of Corollary 10.5 is the following. Let $m, n \geqslant 0$ and let $O_1 \subseteq \mathbb{P}^m(k)$ and $O_2 \subseteq \mathbb{P}^n(k)$ be open subvarieties. Suppose that V is a closed subvariety of O_1 and that W is a closed subvariety of O_2. Then $O_1 \times O_2$ is open in $\mathbb{P}^m(k) \times \mathbb{P}^n(k)$, the Cartesian product $V \times W$ is closed in $O_1 \times O_2$ and the product of V and W is the set $V \times W$ endowed with the variety structure it inherits from $O_1 \times O_2$ as a closed subvariety. The projections π_V and π_W are then the ordinary projections on the two factors.

Again, this justifies simply writing $V \times W$ instead of $V \prod W$.

Corollary 10.6. *Let V_1, V_2 be quasi-projective varieties. Let $C_1 \subseteq V_1$ and $C_2 \subseteq V_2$ be closed subsets. Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be open subsets. Then the set theoretic product $C_1 \times C_2$ (resp. the set theoretic product $U_1 \times U_2$) is closed (resp. open) in $V \times W = V \prod W$. If $C_1 \times C_2$ (resp. $U_1 \times U_2$) is endowed with its structure of closed (resp. open) subvariety of $V_1 \prod V_2$ and with the natural projection maps on the two factors, then $C_1 \times C_2$ (resp. $U_1 \times U_2$) is a product of C_1 and C_2 (resp. U_1 and U_2).*

Proof. Left to the reader. The proof is completely similar to the proof of Corollary 10.5. □

The next lemma is needed for the following proposition.

Lemma 10.7. *Let $I \subseteq k[x_1, \dots, x_n]$ (resp. $J \subseteq k[y_1, \dots, y_t]$) be an ideal. Let $\bar{I}$ (resp. $\bar{J}$) be the ideal generated by I (resp. J) in $k[x_1, \dots, x_n, y_1, \dots, y_t]$. If I and J are radical (resp. prime) then $\bar{I} + \bar{J}$ is radical (resp. prime).*

Proof. Suppose first that I and J are prime. Let $P, Q \in k[x_1, \dots, x_n, y_1, \dots, y_t]$. Suppose that $P \cdot Q \in \bar{I} + \bar{J}$. Suppose for contradiction that $P \notin \bar{I} + \bar{J}$ and $Q \notin \bar{I} + \bar{J}$. Note that we may without restriction of generality replace P (resp. Q) by $P + P_1$, where $P_1 \in \bar{I} + \bar{J}$ (resp. by $Q + Q_1$, where $Q_1 \in \bar{I} + \bar{J}$) without affecting the conclusion. Write

$$P = \sum_i A_i(x_1, \dots, x_n) B_i(y_1, \dots, y_t)$$

and

$$Q = \sum_j C_j(x_1, \dots, x_n) D_j(y_1, \dots, y_t)$$

where both sums are finite. We may assume the elements $B_i \pmod J$ (resp. the elements $D_j \pmod J$) are linearly independent over k. Indeed, if $B_{i_0} = R + \sum_{i \neq i_0} \lambda_i B_i$ where $R \in J$ and $\lambda_i \in k$, then we have

$$P = \sum_{i \neq i_0} A_i B_i + A_{i_0}(R + \sum_{i \neq i_0} \lambda_i B_i) = \sum_{i \neq i_0} (A_i + \lambda_i A_{i_0}) B_i + A_{i_0} R$$

where $A_{i_0} R \in \bar{J}$. By the preceding remark, we may thus replace P by $\sum_{i \neq i_0} (A_i + \lambda_i A_{i_0}) B_i$ and thus assume that $B_{i_0} = 0$. We can now repeat this process until all the elements $B_i \pmod J$ are linearly independent over k. The same construction applies to the elements $D_j \pmod J$.

Further, note that we may suppose that for some A_i, we have $A_i \notin I$, otherwise $P \in \bar{I}$ and there is nothing to prove. Similarly, we may suppose that for some C_j, we have $C_j \notin I$. So choose indices i_1, j_1 such that $A_{i_1}, C_{j_1} \notin I$.

Now let $\mathfrak{m}_0 \subseteq k[x_1, \dots, x_n]/I$ be a maximal ideal such that $A_{i_1} \pmod I \notin \mathfrak{m}$ and $C_{j_1} \pmod I \notin \mathfrak{m}$.

Such a maximal ideal exists. Indeed, suppose there is no such ideal. Then every maximal ideal of $k[x_1, \dots, n]/I$ contains $(A_{i_1} \pmod I) \cdot (C_{j_1} \pmod I))$ and hence $(A_{i_1} \pmod I) \cdot (C_{j_1} \pmod I))$ is contained in the Jacobson radical of $k[x_1, \dots, n]/I$. But $k[x_1, \dots, n]/I$ is Jacobson since it is finitely generated over k. Since $k[x_1, \dots, n]/I$ is a domain we thus conclude that $(A_{i_1} \pmod I) \cdot (C_{j_1} \pmod I)) = 0$. This implies that that either $A_{i_1} \pmod I) = 0$ or $C_{j_1} \pmod I) = 0$, which is a contradiction.

Now let $\mathfrak{m} \in \mathrm{Spm}(k[x_1, \dots, x_n])$ be the maximal ideal corresponding to $\mathfrak{m}_0$. By the weak Nullstellentsatz, we have an isomorphism of k-algebras $k[x_1, \dots, x_n]/\mathfrak{m} \simeq k$ and thus we obtain a map of k-algebras $\phi : k[x_1, \dots, x_n] \to k$ whose kernel is $\mathfrak{m}$. Let $\Phi : k[x_1, \dots, x_n, y_1, \dots, y_t] \to k[y_1, \dots, y_t]$ be the induced map. The map Φ sends any polynomial in the y variable on itself and any polynomial H in the x variable on $\phi(H) \in k$. In particular, we have $\Phi(\bar{J}) \subseteq J$ and $\Phi(A_{i_1}) = \phi(A_{i_1}) \neq 0$ and $\Phi(C_{j_1}) = \phi(C_{j_1}) \neq 0$. Note also that the kernel of Φ contains $\bar{\mathfrak{m}} := \mathfrak{m} \cdot k[x_1, \dots, x_n, y_1, \dots, y_t]$ and that $\bar{\mathfrak{m}} \supseteq \bar{I}$. Thus we have $\Phi(PQ) \in J$.

Now we compute

$$0 = \Phi(PQ) \pmod J = \Big(\sum_i \phi(A_i(x_1, \dots, x_n))(B_i(y_1, \dots, y_t) \pmod J) \Big) \Big(\sum_j \phi(C_j(x_1, \dots, x_n))(D_j(y_1, \dots, y_t) \pmod J) \Big).$$

Since the elements $B_i(y_1, \dots, y_t) \pmod J$ are linearly independent over k and $\phi(A_{i_0}) \neq 0$ we see that

$$\sum_i \phi(A_i(x_1, \dots, x_n)) B_i(y_1, \dots, y_t) \notin J.$$

Similarly

$$\sum_j \phi(C_j(x_1,\ldots,x_n))D_j(y_1,\ldots,y_t) \notin J.$$

This is a contradiction, since J is prime.

The proof that $\bar{I}+\bar{J}$ is radical if I and J are radical is completely similar and is left to the reader. $\square$

Proposition 10.8. *Let V and W be irreducible quasi-projective varieties. Then $V \times W = V \prod W$ is also irreducible.*

Proof. We first prove the result in the situation where V and W are affine. So suppose that $V \subseteq k^n$ and $W \subseteq k^t$ are algebraic sets in k^n and k^t, respectively. By Q3 of Sheet 3, we know that the subset $V \times W$ of $k^n \times k^t = k^{n+t}$ is an algebraic subset in k^{n+t} and is a product of V and W. So we have to show that $V \times W$ is irreducible, when endowed with the topology induced from k^{n+t}. Write $k[x_1,\ldots,x_n]$ for the coordinate ring of k^n and $k[y_1,\ldots,y_t]$ for the coordinate ring of k^t. Let

$$\bar{\mathcal{I}}(V) = \mathcal{I}(V)\cdot k[x_1,\ldots,x_n,y_1,\ldots,y_t]$$

and

$$\bar{\mathcal{I}}(W) = \mathcal{I}(W)\cdot k[x_1,\ldots,x_n,y_1,\ldots,y_t].$$

By construction we have $\mathrm{Z}(\bar{\mathcal{I}}(V)+\bar{\mathcal{I}}(W)) = V \times W$. Furthermore, by Lemma 10.7 the ideal $\bar{\mathcal{I}}(V)+\bar{\mathcal{I}}(W)$ is prime. Hence $\mathcal{I}(V\times W) = \bar{\mathcal{I}}(V)+\bar{\mathcal{I}}(W)$ and thus $V \times W$ is irreducible.

Now suppose that V and W are quasi-projective. Suppose for contradiction that $V\times W$ is not irreducible. Let $T_1,\ldots,T_l$ be the irreducible components of $V\times W$. By assumption, we have $l \geqslant 2$. Let $(v_1,w_1)\in T_1$ and $(v_2,w_2)\in T_2$. Let U_{v_1} be an open affine neighbourhood of v_1 in V and let U_{w_1} be an open affine neighbourhood of w_1 in W. Define U_{v_2} and U_{w_2} similarly. Then we have $(v_1,w_1)\in U_{v_1}\times U_{w_1}$ and $(v_2,w_2)\in U_{v_2}\times U_{w_2}$. Now from the first part and Lemma 10.6, we know that $U_{v_1}\times U_{w_1}$ and $U_{v_2}\times U_{w_2}$ are open irreducible subsets of $V\times W$. Hence $U_{v_1}\times U_{w_1}\subseteq T_1$ and $U_{v_2}\times U_{w_2}\subseteq T_2$ (why?). Also, we have $U_{v_1}\times U_{w_1}\cap U_{v_2}\times U_{w_2}=\emptyset$, for otherwise $T_1\backslash(T_1\cap T_2)$ is not dense in T_1. However, since V and W are irreducible there is a point $z_v\in U_{v_1}\cap U_{v_2}$ and a point $z_w\in U_{w_1}\cap U_{w_2}$. We have $(z_v,z_w)\in U_{v_1}\times U_{w_1}\cap U_{v_2}\times U_{w_2}$, which is a contradiction. So $V\times W$ is irreducible. $\square$

In the next proposition, we shall need Noether's normalisation lemma. This says the following: if R is a non zero finitely generated algebra over a field L, then there exists an injective map of L-algebras $L[x_1,\ldots,x_l]\hookrightarrow R$ for some $l\geqslant 0$, such that R is finite as a $L[x_1,\ldots,x_l]$-module. See Theor. 9.1 in CA.

Proposition 10.9. *Let V and W be irreducible quasi-projective varieties. Then*

$$\dim(V\times W) = \dim(V)+\dim(W).$$

Proof. First suppose that V and W are affine. So we may suppose that V (resp. W) is an algebraic set in k^n (resp. k^t). Let $I := \mathcal{I}(V)\subseteq k[x_1,\ldots,x_n]$ (resp. $J := \mathcal{I}(W)\subseteq k[y_1,\ldots,y_t]$). We saw in the proof of Proposition 10.8 that the product of V and W can be realised as the closed subset $V\times W$ of k^{n+t} and that the ideal of $V\times W$ in $k[x_1,\ldots,x_n,y_1,\ldots,y_t]$ is $\bar{I}+\bar{J}$, where

$$\bar{I} = \mathcal{I}(V)\cdot k[x_1,\ldots,x_n,y_1,\ldots,y_t]$$

and

$$\bar{J} = \mathcal{I}(W) \cdot k[x_1, \ldots, x_n, y_1, \ldots, y_t].$$

Now use Noether's normalisation lemma to obtain an injective map of k-algebras

$$\phi_V : k[X_1, \ldots, X_v] \hookrightarrow \mathcal{C}(V) = k[x_1, \ldots, x_n]/I$$

making $\mathcal{C}(V)$ into a finite $k[X_1, \ldots, X_v]$-module. This corresponding map of algebraic sets $\mathrm{Spm}(\phi_V) : V \to k^n$, is then surjective. This follows from Theorem 8.8 and Cor. 8.10 in CA. Incidentally, Proposition 8.12 in CA also implies that the map $\mathrm{Spm}(\phi_V)$ has finite fibres. Similarly let

$$\phi_W : k[Y_1, \ldots, Y_w] \hookrightarrow \mathcal{C}(W) = k[y_1, \ldots, y_t]/J.$$

be an injective map of k-algebras making $\mathcal{C}(W)$ into a finite $k[Y_1, \ldots, Y_w]$-module.

Now let

$$\phi_{VW} : k[X_1, \ldots, X_v, Y_1, \ldots, Y_w] \to k[x_1, \ldots, x_n, y_1, \ldots, y_t]/(\bar{I} + \bar{J})$$

be the map of k-algebras sending X_i to $\phi_V(X_i)$ and Y_j to $\phi_W(Y_j)$.

We claim that the map Φ_{VW} is injective. To see this, notice that by unrolling the definitions, we have that $\Phi_{VW} = (\mathrm{Spm}(\phi_V) \times \mathrm{Spm}(\phi_W))^*$, where

$$\mathrm{Spm}(\phi_V) \times \mathrm{Spm}(\phi_W) : V \times W \to k^n \times k^t = k^{n+t}$$

is the map given by the formula $(\mathrm{Spm}(\phi_V) \times \mathrm{Spm}(\phi_W))(a \times b) = \mathrm{Spm}(\phi_V)(a) \times \mathrm{Spm}(\phi_W)(b)$. In particular, the map $\mathrm{Spm}(\Phi_{VW})$ is surjective. Thus Φ_{VW} is injective by Q5 (1) of Sheet 1.

We also claim that Φ_{VW} makes $k[x_1, \ldots, x_n, y_1, \ldots, y_t]/(\bar{I}+\bar{J})$ into a finite $k[X_1, \ldots, X_v, Y_1, \ldots, Y_w]$-module. To see this, note that by definition, each x_i is integral over $k[X_1, \ldots, X_v]$ via ϕ_V (see section 8 in CA). This means that there are polynomials $P_{0,i}(X_1, \ldots, X_v), P_{1,i}(X_1, \ldots, X_v), \ldots, P_{\delta(i)-1,i}(X_1, \ldots, X_v)$ such that

$$x_i^{\delta_i} + \sum_{s=0}^{\delta(i)-1} \phi_V(P_{s,i}) x_i^s = 0$$

in $k[x_1, \ldots, x_n]/I$. In particular, we have

$$x_i^{\delta_i} + \sum_{s=0}^{\delta(i)-1} \phi_{VW}(P_{s,i}) x_i^s = 0$$

in $k[x_1, \ldots, x_n, y_1, \ldots, y_t]/(\bar{I} + \bar{J})$. So x_i is also integral over $k[X_1, \ldots, X_v, Y_1, \ldots Y_w]$ via ϕ_{VW}. The same reasoning applies to each y_j. Since the x_i and the y_j generate $k[x_1, \ldots, x_n, y_1, \ldots, y_t]/(\bar{I}+\bar{J})$ as a $k-$algebra and hence as a $k[X_1, \ldots, X_v, Y_1, \ldots, Y_w]$-algebra, we deduce that $k[x_1, \ldots, x_n, y_1, \ldots, y_t]/(\bar{I} + \bar{J})$ is finitely generated as a $k[X_1, \ldots, X_v, Y_1, \ldots, Y_w]$-module (about this see the discussion after Corollary 8.5 in CA).

Appealing to Lemma 11.29 in CA, we deduce that

$$\dim(k[x_1, \ldots, x_n, y_1, \ldots, y_t]/(\bar{I} + \bar{J})) = \dim(V \times W) = \dim(k[X_1, \ldots, X_v, Y_1, \ldots, Y_w]) = v + w$$

On the other hand, by Lemma 11.29 again, we have

$$v = \dim(\mathcal{C}(V)) = \dim(V)$$

and

$$w = \dim(\mathcal{C}(W)) = \dim(W).$$

Hence $\dim(V \times W) = \dim(V) + \dim(W)$.

Now we turn to the general case. Let V_1 (resp. W_1) be an open affine subvariety of V (resp. W). The set $V_1 \times W_1$ is open in $V \times W$ and is a product of V_1 and W_1 when considered as an open subvariety of $V \times W$ (by Lemma 10.6). Also, $V_1 \times W_1$ and $V \times W$ are irreducible by Proposition 10.8. Now we apply Proposition 9.2 and the above to obtain

$$\begin{aligned}
&\dim(V \times W) = \text{tr. deg. of } \kappa(V \times W) \text{ over } k = \text{tr. deg. of } \kappa(V_1 \times W_1) \text{ over } k \\
= \ &\dim(V_1 \times W_1) = \dim(V_1) + \dim(W_1) = (\text{tr. deg. of } \kappa(V_1) \text{ over } k) + (\text{tr. deg. of } \kappa(V_1) \text{ over } k) \\
= \ &(\text{tr. deg. of } \kappa(V) \text{ over } k) + (\text{tr. deg. of } \kappa(V) \text{ over } k) = \dim(V) + \dim(W).
\end{aligned}$$

□

We end with the following important remark. *One can show that for any varieties V, W the product $V \prod W$ exists.* The proof uses different methods. It proceeds roughly as follows. One covers V and W with open affine varieties V_i and W_j, respectively. It can be shown using commutative algebra that the products $V_i \prod W_j$ exist (see Q4 of Sheet 3). One then constructs the product $V \prod W$ by glueing the varieties $V_i \prod W_j$. The advantage of the above construction of the product of quasi-projective varieties is that it bypasses the need for such a glueing procedure, which is combinatorially cumbersome.

11 Intersections in affine and projective space

Proposition 11.1 (affine dimension theorem). *Let $n \geqslant 0$ and let $V, W \subseteq k^n$ be irreducible algebraic sets. Then every irreducible component of $V \cap W$ has dimension $\geqslant \dim(V) + \dim(W) - n$.*

Proof. Note that the Cartesian product $V \times W \subseteq k^{2n}$ is closed and is a product of V and W (see Q3 of Sheet 3). Let

$$\Delta := \{(a_1, \dots, a_n, a_1, \dots, a_n) \,|\, a_1, \dots, a_n \in k\}$$

be the diagonal of k^{2n}. Note that we have

$$\Delta = \mathrm{Z}(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$$

where we write $\mathcal{C}(k^{2n}) = k[x_1, \dots, x_n, y_1, \dots, y_n]$. We have a k-algebra map

$$\phi : k[x_1, \dots, x_n, y_1, \dots, y_n]/(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n) \to k[z_1, \dots, z_n]$$

such that $\phi(x_i) = \phi(y_i) = z_i$ for all $i \in \{1, \dots, n\}$. The map ϕ has an inverse given by the map sending z_i to $x_i \,(\mathrm{mod}\,(x_1 - y_1, x_2 - y_2, \dots, x_n - y_n))$. In particular $\mathrm{Spm}(\phi) : k^n \to \Delta$ is an isomorphism of algebraic sets. By construction, we have

$$\mathrm{Spm}(\phi)^{-1}(V \times W \cap \Delta) = V \cap W.$$

Thus we only have to prove that every irreducible component of $V \times W \cap \Delta$ has dimension $\geqslant \dim(V) + \dim(W) - n$. Now by construction we have

$$V \times W \cap \Delta = \mathrm{Z}(x_1 - y_1) \cap \mathrm{Z}(x_2 - y_2) \cap \dots \cap \mathrm{Z}(x_n - y_n) \cap V \times W.$$

Applying Theorem 8.9, we see that for any irreducible component C of $V \times W \cap \Delta$ we have

$$\operatorname{cod}(C, V \times W) \leqslant n$$

and by Corollary 8.8, Proposition 10.8 and Proposition 10.9, this translates as

$$\dim(V \times W) - \dim(C) = \dim(V) + \dim(W) - \dim(C) \leqslant n$$

which is equivalent to the conclusion of the proposition. $\square$

Proposition 11.2 (projective dimension theorem)**.** *Let $n \geqslant 0$ and let $V, W \subseteq \mathbb{P}^n(k)$ be closed irreducible subvarieties. Then every irreducible component of $V \cap W$ has dimension $\geqslant \dim(V) + \dim(W) - n$. Furthermore, we have $V \cap W \neq \emptyset$ if $\dim(V) + \dim(W) - n \geqslant 0$.*

Proof. We first prove the first assertion. Let C be an irreducible component of $V \cap W$. Let U_i be a standard coordinate chart of $\mathbb{P}^n(k)$ such that $C \cap U_i \neq \emptyset$.

We claim that $C \cap U_i$ is an irreducible component of $(V \cap W) \cap U_i$. To see this, note that since $C \cap U_i$ is irreducible (because $C \cap U_i$ is non empty and open in C), there is an irreducible component T of $(V \cap W) \cap U_i$, which contains $C \cap U_i$. Write $\bar{T}$ for the closure of T in $V \cap W$. Then $\bar{T}$ is also irreducible by Q4 (1) of Sheet 2 and hence $\bar{T} \subseteq C$. On the other hand, by construction, we also have $\bar{T} \supseteq C$ so that $C = \bar{T}$. Hence $T = \bar{T} \cap U_i = C \cap U_i$ so that $C \cap U_i$ is an irreducible component of $V \cap W$.

Now Proposition 11.1, we have

$$\dim(C \cap U_i) \geqslant \dim(V \cap U_i) + \dim(W \cap U_i) - n$$

and by Proposition 9.2, we have $\dim(V \cap U_i) = \dim(V)$, $\dim(W \cap U_i) = \dim(W)$ and $\dim(C \cap U_i) = \dim(C)$. This proves the first assertion.

For the second assertion, consider again the map $q : k^{n+1} \backslash \{0\} \to \mathbb{P}^n(k)$ such that $q(\bar{v}) = [\bar{v}]$ for all $\bar{v} \in k^{n+1} \backslash \{0\}$. Let $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{\dim(V)} = V$ be an ascending sequence of irreducible closed subsets of V, which is of maximal length. The closed subvarieties $q^{-1}(V_i)$ of $k^{n+1} \backslash \{0\}$ are all irreducible by Q5 (2) of Sheet 3. Write $\overline{q^{-1}(V_i)}$ for the closure of $q^{-1}(V_i)$ in k^{n+1}. The closed subsets $\overline{q^{-1}(V_i)}$ of k^{n+1} are then all irreducible by Q5 (2) of Sheet 3 and Q4 (1) of Sheet 2. We thus get an ascending sequence

$$\overline{q^{-1}(V_0)} \subsetneq \overline{q^{-1}(V_1)} \subsetneq \cdots \subsetneq \overline{q^{-1}(V_{\dim(V)})} = \overline{q^{-1}(V)}$$

of closed irreducible subsets of k^{n+1}. Now note that by maximality the variety V_0 is a point (otherwise one could extend the sequence $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{\dim(V)}$ one step further). We thus have

$$q^{-1}(V_0) = \{\lambda \bar{v}_0 \,|\, \lambda \in k\} \cap (k^{n+1} \backslash \{0\})$$

for some $\bar{v}_0 \in k^{n+1} \backslash \{0\}$.

We claim that the closure of $\{\lambda \bar{v}_0 \,|\, \lambda \in k\} \cap (k^{n+1} \backslash \{0\})$ in k^{n+1} is $\{\lambda \bar{v}_0 \,|\, \lambda \in k\}$. To see this, let $P(x_0, \ldots, x_n)$ be some (not assumed to be homogenous) polynomial such that $P(\lambda v_0) = 0$ for all $\lambda \in k \backslash \{0\}$. We then have

$$P(\lambda \bar{v}_0) = \sum_i P_{[i]}(\lambda \bar{v}_0) = \sum_i \lambda^i P_{[i]}(\bar{v}_0) = 0$$

for all $\lambda \neq 0$. Considering $\deg(P) + 1$ different values of λ we obtain a system of linear equations with the unique solution 0 (arising from a Vandermonde matrix). In other words, we have $P_{[i]}(v_0) = 0$ for all i and in particular $P_{[0]} = P(0) = 0$. Now note that for all $i \in \{1, \dots, \deg(P)\}$, the polynomial $P_{[i]}$ is a non constant homogenous polynomial and hence vanishes at 0. We conclude that P also vanishes at 0. In particular, the closure of $\{\lambda\bar{v}_0 \,|\, \lambda \in k\} \cap (k^{n+1}\backslash\{0\})$ in k^n contains 0 and is thus equal to $\{\lambda\bar{v}_0 \,|\, \lambda \in k\}$.

We thus obtain an ascending sequence of irreducible closet subsets

$$\{0\} \subsetneq \{\lambda\bar{v}_0 \,|\, \lambda \in k\} = \overline{q^{-1}(V_0)} \subsetneq \overline{q^{-1}(V_1)} \subsetneq \cdots \subsetneq \overline{q^{-1}(V_{\dim(V)})} = \overline{q^{-1}(V)}$$

and we thus see that $\overline{q^{-1}(V)}$ has dimension $\geqslant \dim(V) + 1$. Similarly, $\overline{q^{-1}(W)}$ is irreducible in k^{n+1} and has dimension $\geqslant \dim(W) + 1$. We conclude from Proposition 11.1 that every irreducible component of $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ has dimension larger or equal to

$$\dim(\overline{q^{-1}(V)}) + \dim(\overline{q^{-1}(W)}) - (n+1) \geqslant \dim(V) + \dim(W) + 2 - (n+1) = \dim(V) + \dim(W) - n + 1.$$

Hence, if $\dim(V) + \dim(W) - n \geqslant 0$ then every irreducible component of $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ has dimension $\geqslant 1$. On the other hand, both $\overline{q^{-1}(V)}$ and $\overline{q^{-1}(W)}$ contain the point 0, so $\overline{q^{-1}(V)} \cap \overline{q^{-1}(W)}$ is not empty. We conclude that $\overline{q^{-1}(V)} \cap \overline{q^{-1}(V)}$ contains points other than 0, or in other words that $q^{-1}(V) \cap q^{-1}(W) \neq \emptyset$. This implies that $V \cap W \neq \emptyset$. □

Corollary 11.3. *Let $n \geqslant 0$ and let $V \subseteq \mathbb{P}^n(k)$ be a closed irreducible subset. Let H be a closed irreducible subset such that* $\mathrm{cod}(H, \mathbb{P}^n(k)) = 1$. *If* $\dim(V) \geqslant 1$ *then* $H \cap C \neq \emptyset$.

Proof. Left to the reader. □

12 Separatedness and completeness

Separatedness is an algebraic analogue of the Hausdorff property in topology. Completeness is an algebraic analogue of the notion of compactness in topology.

If X is a quasi-projective variety. Write $\delta_X : X \to X \prod X$ for the map $\mathrm{Id}_X \prod \mathrm{Id}_X$. We shall write $\Delta_X \subseteq X \prod X$ for the image of δ_X. We call it the *diagonal* in $X \prod X$.

Definition 12.1. *Let X be a quasi-projective variety. We say that X is separated if the diagonal in $X \prod X$ is closed.*

Note that if Δ_X is closed in $X \prod X$ then δ_X induces an isomorphism between X and Δ_X, where Δ_X is seen as a closed subvariety of $X \prod X$. Indeed, the map δ_X induces a morphism $X \to \Delta_X$ by Lemma 5.3 and this map has an inverse, given by the projection on the first factor.

To understand this definition, note that if T is a topological space and $T \times T$ is endowed with the product topology, then T is Hausdorff iff the diagonal $\Delta_T \subseteq T \times T$ is closed. Indeed, let $a, b \in T$ and $a \neq b$. Then $(a, b) \notin \Delta_T$. If Δ_T is closed then there are open subsets $U, V \subseteq T$ such that $U \times V \cap \Delta_T = \emptyset$ and such that $(a, b) \in U \times V$. In particular, $a \in U$, $b \in V$ and $U \cap V = \emptyset$. So a and b have disjoint neighbourhoods. On the other hand, if a and b have disjoint neighbourhoods U and V, respectively, then $U \times V \cap \Delta_T = \emptyset$ and $(a, b) \in U \times V$. So $(T \times T)\backslash\Delta_T$ is open, ie Δ_T is closed.

Note that in the definition of separatedness given above, the variety $X \prod X$ does not in general carry the product topology induced by its natural identification with the Cartesian product $X \times X$ so that the algebraic definition differs significantly from the topological one (but serves the same purpose from a structural point of view).

Remark. As remarked at the end of section 10, any two varieties have a product. It is therefore possible to extend to definition of separatedness to all varieties. We shall see below that all quasi-projective are separated, but it is possible to give examples of non quasi-projective varieties, which are not separated.

Lemma 12.2. *Affine varieties are separated.*

Proof. We first prove that the varieties k^t are separated for $t \geqslant 0$. Recall that by Q4 of Sheet 3, $k^t \prod k^t \simeq k^{2t}$. Write $\mathcal{C}(k^{2t}) = k[x_1, \ldots, x_t, y_1, \ldots, y_t]$. Now note that

$$\Delta_{k^t} = \mathrm{Z}(x_1 - y_1, x_2 - y_2, \ldots, x_t - y_t).$$

Hence Δ_{k^t} is closed. The general case now follows from Lemma 12.4. □

Lemma 12.3. *Let X be a quasi-projective variety. Suppose that for any two points $a, b \in X$ there exists an open affine subvariety $U \subseteq X$ such that $a, b \in U$. Then X is separated.*

Proof. Let $(a, b) \in X \times X \backslash \Delta_X$ (ie $a, b \in X$ and $a \neq b$). Let $U_{a,b}$ be an open affine subvariety of X such that $a, b \in U_{a,b}$. Then $(a, b) \in U_{a,b} \times U_{a,b}$. Furthermore, $\Delta_{U_{a,b}} = \Delta_X \cap (U_{a,b} \times U_{a,b})$ and the Cartesian product $U_{a,b} \times U_{a,b}$ is a product of $U_{a,b}$ with itself as an open subvariety of $X \times X$ (by Corollary 10.6). Hence $\Delta_{U_{a,b}}$ is closed as a subset of $U_{a,b} \times U_{a,b}$ by Lemma 12.2. In particular, (a, b) is contained in an open subset of $X \times X$, which is disjoint from (a, b). Since $(a, b) \in X \times X \backslash \Delta_X$ was arbitrary, we conclude that $X \times X \backslash \Delta_X$ is open, ie Δ_X is closed. □

Lemma 12.4. *Let X be a separated quasi-projective variety. Let V be a closed (resp. open) subvariety of X. Then V is separated.*

Proof. Suppose that V is a closed subvariety of X. The Cartesian product $V \times V \subseteq X \times X$ is closed and represents the product of V with itself as a closed subvariety of $X \times X$ (by Corollary 10.6). On the other hand, we have $\Delta_V = \Delta_X \cap V \times V$ so Δ_V is closed in $V \times V$ since Δ_X is closed. In other words, V is separated. The proof in the situation where V is an open subvariety of X is similar. □

Proposition 12.5. *Any quasi-projective variety is separated.*

Proof. Suppose first that $X = \mathbb{P}^n(k)$ for some $n \geqslant 0$. Then X is separated by Lemma 12.3 and Q7 of Sheet 2. The general case follows from this and Lemma 12.4. □

Proposition-Definition 12.6 (The graph of a morphism)**.** *Let X and Y be quasi-projective varieties. Let $\gamma : X \to Y$ be a morphism. Let*

$$\Gamma_\gamma := \{(x, \gamma(x)) \,|\, x \in X\} \subseteq X \times Y$$

be the graph of γ. Then Γ_γ is closed in $X \times Y$.

Proof. Let $\widetilde{\gamma} : X \times Y \to Y \times Y$ be the morphism such that $\widetilde{\gamma}(x, y) := (\gamma(x), y)$ for all $(x, y) \in X \times Y$ (this is a morphism by the definition of products). We have

$$\Gamma_\gamma = \widetilde{\gamma}^{-1}(\Delta_Y)$$

and so Γ_γ is closed since Δ_Y is closed by 12.5. □

Definition 12.7. *Let X be a quasi-projective variety. We say that X is complete if for any quasi-projective variety B and any closed subset $C \subseteq X \times B$, the set $\pi_B(C)$ is closed.*

Here $\pi_B : X \times B \to B$ is the projection on the second factor.

Lemma 12.8. *Let X be a complete quasi-projective variety. Then any closed subvariety of X is also complete.*

Proof. Left to the reader. Unroll the definitions and use Corollary 10.6. $\square$

Theorem 12.9. *Projective varieties are complete.*

Proof. By Lemma 12.8, we only need to prove this for $X = \mathbb{P}^n(k)$.

So let B be a quasi-projective variety and let $\{B_i\}$ be an open affine covering of B. Let $C \subseteq \mathbb{P}^b(k) \times B$ be a closed subset. By Corollary 10.6, the Cartesian product $\mathbb{P}^b(k) \times B_i$ is open in $\mathbb{P}^b(k) \times B$ and if $\mathbb{P}^b(k) \times B_i$ is viewed as an open subvariety of $\mathbb{P}^b(k) \times B$ it is a product of $\mathbb{P}^n(k)$ and B_i. Now $\pi_B(C)$ is closed iff $\pi_B(C) \cap B_i$ is closed in B_i for all i and we have $\pi_B(C) \cap B_i = \pi_{B_i}(C \cap (\mathbb{P}^n(k) \times B_i))$. Hence we may suppose from the start that B is affine. In that case B is a closed subvariety of k^t for some $t \geqslant 0$. By Corollary 10.6 again, the subset $\mathbb{P}^n(k) \times B \subseteq \mathbb{P}^n(k) \times k^t$ is closed and is a product of $\mathbb{P}^n(k)$ and B if $\mathbb{P}^n(k) \times B$ is viewed as a closed subvariety of $\mathbb{P}^n(k) \times k^t$. Furthermore, $\pi_B(C)$ is closed in B iff it is closed in k^t. Some we might suppose that $B = k^t$.

Now let $i \in \{0, \ldots, n\}$ and let $U_i \subseteq \mathbb{P}^n(k)$ be the well-known coordinate chart. Recall that there is an isomorphism $u_i : k^n \to U_i$ given by the formula

$$u_i(\langle X_0, \ldots, \breve{X}_i, \ldots, X_n \rangle) = [X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n] \in \mathbb{P}^n(k).$$

By Q4 of Sheet 3, the variety $U_i \times k^t$ is affine and we have

$$\mathcal{C}(k^n \times k^t) = k[x_0, \ldots, \breve{x}_i, \ldots, x_n, y_1, \ldots, y_t]$$

where the x_j are the coordinates of k^n and the y_j are the coordinates of k^t.

Write

$$\phi_i : k[x_0, \ldots x_n, y_1, \ldots, y_t] \to k[x_0, \ldots, \breve{x}_i, \ldots, x_n, y_1, \ldots, y_t]$$

for the map of k-algebras such that $\phi(x_j) = x_j$ for all $j \neq i$, $\phi(x_i) = 1$ and $\phi(y_j) = y_j$ for all j.

Let $I_i := \mathcal{I}((u_i \times \mathrm{Id}_{k^t})^{-1}(C)) \subseteq k[x_0, \ldots, \breve{x}_i, \ldots, x_n, y_1, \ldots, y_t]$.

Note the following. Suppose that $H \in k[x_0, \ldots x_n, y_1, \ldots, y_t]$ and that H is homogenous in the x-variables. Then $H \in \phi_i^{-1}(I_i)$ iff $H(X_0, \ldots, X_n, Y_1, \ldots, Y_t) = 0$ for all $[X_0, \ldots, X_n] \times \langle Y_1, \ldots, Y_t \rangle \in C \cap (U_i \times k^t)$. This follows directly from the definitions.

In particular a polynomial $H \in k[x_0, \ldots x_n, y_1, \ldots, y_t]$ which is homogenous in the x-variables lies in $\cap_i \phi_i^{-1}(I_i)$ iff $H(X_0, \ldots, X_n, Y_1, \ldots, Y_t) = 0$ for all $[X_0, \ldots, X_n] \times \langle Y_1, \ldots, Y_t \rangle \in C$.

For any $N \geqslant 0$, write $S_N \subseteq k[x_0, \ldots x_n, y_1, \ldots, y_t]$ for the polynomials, which are homogenous in the x-variable and which are of degree N in the x-variable. This gives $k[x_0, \ldots x_n, y_1, \ldots, y_t]$ the structure of a graded ring with $S_0 = k[y_1, \ldots, y_t]$. In particular S_N is a $S_0 = k[y_1, \ldots, y_t]$-submodule of $k[x_0, \ldots x_n, y_1, \ldots, y_t]$. We also write $A_N = S_N \cap (\cap_i \phi_i^{-1}(I_i))$. It follows from the definitions that A_N is then a graded ideal in (= graded sub-$k[x_0, \dots x_n, y_1, \dots, y_t]$-module of) $k[x_0, \dots x_n, y_1, \dots, y_t]$. In particular, A_N is a $S_0 = k[y_1, \dots, y_t]$-submodule of S_N.

Now let $\bar{w} = \langle W_1, \dots W_t \rangle \in k^t$ and suppose that $\bar{w} \notin \pi_B(C)$. Let $\bar{m} = (y_1 - W_1, \dots, y_t - W_t) \subseteq k[y_1, \dots, y_t]$ be the maximal ideal associated with $\bar{w}$. Let $i \in \{0, \dots, n\}$. By assumption, we have

$$I_i + \mathfrak{m} \cdot k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t] = k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$$

(since the zero set of $\mathfrak{m} \cdot k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$ is $k^n \times \{w\}$ and by assumption $u_i^{-1}(C) = \mathrm{Z}(I_i)$, which does not meet $k^n \times \{w\}$). In particular, there is a polynomial $P_i \in I_i$ and polynomials $M_{il} \in \mathfrak{m}$ and $G_{il} \in k[x_0, \dots, \check{x}_i, \dots, x_n, y_1, \dots, y_t]$ such that

$$1 = P_i + \sum_l M_{il} \cdot G_{il}$$

Hence, for any $N \geqslant 0$ we have

$$\begin{aligned} & x_i^N = x_i^{N - \deg_x(P_i)} \big(x_i^{\deg_x(P_i)} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t) \big) \\ + & \sum_l M_{il}(y_1, \dots, y_t) \big[x_i^{N - \deg_x(G_{il})} \big(x_i^{\deg_x(G_{il})} G_{il}(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t) \big) \big] \end{aligned}$$

Now note that the polynomial $x_i^{\deg_x(P_i)} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)$ is by construction homogenous in the x-variable and of x-degree $\deg_x(P_i)$; the same polynomial also lies in $\phi_i^{-1}(I_i)$ since

$$\phi_i(x_i^{\deg_x(P_i)} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)) = P_i.$$

Furthermore, by definition, the polynomial

$$x_i^{\deg_x(P_i)+1} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)$$

vanishes when evaluated on $\langle X_0, \dots, X_n, Y_1, \dots, Y_t \rangle$ whenever $[X_0, \dots, X_n] \times \langle Y_1, \dots, Y_t \rangle \in C$ (remember that x_i vanishes on $(\mathbb{P}^n(k) \backslash U_i) \times k^t$). Hence

$$x_i^{\deg_x(P_i)+1} P_i(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t) \in A_{\deg_x(P_i)+1}$$

by the above discussion.

Similarly, the polynomial $x_i^{\deg_x(G_{il})} G_{il}(x_0/x_i, \dots, \check{x}_i, \dots, x_n/x_i, y_1, \dots, y_t)$ is also homogenous in the x-variable and is of x-degree $\deg_x(G_{il})$.

So if N is larger than $\deg_x(P_i) + 1$ and also larger than $\deg_x(G_{il})$ for all l, we have an equality

$$x_i^N = A_i + \sum_l M_{il} H_{il}$$

where $A_i \in A_N$ and $H_{il} \in S_N$. Since there is only a finite number of indices i, there is thus a natural number N_0 such that

$$x_i^N \in A_N + \mathfrak{m} S_N$$

for all $N \geqslant N_0$ and all $i \in \{0, \dots, n\}$.

Now note that if N_1 is sufficiently large, any monomial of degree $\geqslant N_1$ in the x_i becomes divisible by $x_j^{N_0}$ for some x_j. So if N_1 is sufficiently large then for all $N \geqslant N_1$ we have

$$S_N \subseteq (\oplus_{s \geqslant 0} S_s)(A_{N_0} + \mathfrak{m} S_{N_0})$$

Since $\oplus_{s\geqslant 0}A_s$ is a graded ideal, we then have

$$S_N \subseteq S_{N-N_0}(A_{N_0} + \mathfrak{m}S_{N_0}) \subseteq A_N + \mathfrak{m}S_N.$$

In particular, we have $(S_N/A_N) = \mathfrak{m}(S_N/A_N)$ where the quotient S_N/A_N is quotient of $k[y_1,\dots,y_t]$-modules.

We conclude from the generalised form of Nakayama's lemma (see Q4 in Sheet 1 of CA) that there is $Q \in 1+\mathfrak{m}$ such that $Q\cdot(S_N/A_N) = 0$. In particular $Q\cdot x_i^N \in A_N$ for all $i \in \{0,\dots,n\}$. In other words, for any i we have

$$X_i^N Q(X_0,\dots,X_n,Y_1,\dots,Y_t) = X_i^N Q(Y_1,\dots,Y_t) = 0$$

for all $[X_0,\dots,X_n]\times\langle Y_1,\dots,Y_t\rangle \in C$ (see the discussion above). In particular, whenever $Q(Y_1,\dots,Y_t)\neq 0$ the set $C\cap(U_i\times\{\langle Y_1,\dots,Y_t\rangle\})$ is empty. Since this holds for all $i\in\{0,\dots,n\}$, the set $C\cap(\mathbb{P}^n(k)\times\{\langle Y_1,\dots,Y_t\rangle\})$ is empty whenever $Q(Y_1,\dots,Y_t)\neq 0$. Said differently, if $\langle Y_1,\dots,Y_t\rangle \in k^t\backslash \mathrm{Z}(Q)$ then $\langle Y_1,\dots,Y_t\rangle \notin \pi_B(C)$. Finally, we have $Q(\bar{w})\neq 0$ since $Q\in 1+\mathfrak{m}$, so $k^t\backslash \mathrm{Z}(Q)$ is a neighbourhood of $\bar{w}$. Since $\bar{w}\in k^t\backslash \pi_B(C)$ was arbitrary, we conclude that $k^t\backslash\pi_B(C)$ is open, ie $\pi_B(C)$ is closed. □

Remark. Let $n,t\geqslant 0$ and consider the variety $\mathbb{P}^n(k)\times k^t$.

Suppose given polynomials $H_1,\dots,H_l \in k[x_0,\dots,x_n,y_1,\dots,y_t]$. Suppose that the H_j are homogenous in the variable x. Let

$$C := \{[X_0,\dots,X_n]\times\langle Y_1,\dots,Y_t\rangle \in \mathbb{P}^n(k)\times k^t \,|\, \forall j\in\{1,\dots,l\} : H_j(X_0,\dots,X_n,Y_1,\dots,Y_t) = 0\}.$$

It can be shown that C is a closed subset of $\mathbb{P}^n(k)\times k^t$ (prove this!). By Theorem 12.9, the set

$$\pi_{k^t}(C) := \{\langle Y_1,\dots,Y_t\rangle \in k^t \,|\, \exists [X_0,\dots,X_n]\in\mathbb{P}^n(k) : \forall j\in\{1,\dots,l\} : H_j(X_0,\dots,X_n,Y_1,\dots,Y_t) = 0\}$$

is then closed. In other words, there are polynomials $Q_1,\dots,Q_a \in k[y_1,\dots,y_t]$ such that $Q(Y_1,\dots,Y_t) = 0$ iff there is $X_0,\dots,X_n \in k^{n+1}\backslash\{0\}$ such that

$$H_1(X_0,\dots,X_n,Y_1,\dots,Y_t) = H_2(X_0,\dots,X_n,Y_1,\dots,Y_t) = \cdots = H_l(X_0,\dots,X_n,Y_1,\dots,Y_t) = 0.$$

Writing $R = k[y_1,\dots,y_t]$, one can rephrase this result in terms of commutative algebra:

Theorem. Let $H_1,\dots,H_l \in R[x_0,\dots,x_n]$ be homogenous polynomials. Then there are elements $Q_1,\dots,Q_a\in R$ with the following property. If $\phi: R\to L$ is a ring homomorphism from R to an algebraically closed field L, then the equations

$$\phi(H_1)(X_0,\dots,X_n) = \phi(H_2)(X_0,\dots,X_n) = \cdots = \phi(H_l)(X_0,\dots,X_n) = 0$$

have a non vanishing solution in L iff $Q_1,\dots,Q_a \in \ker(\phi)$. The radical of the ideal $(Q_1,\dots,Q_a)$ depends only on the polynomials $H_1,\dots,H_l$.

This theorem is called the *main theorem of elimination theory.* If $t=1$, the unique monic generator of $\mathfrak{r}((Q_1,\dots,Q_a))$ is called the *resultant* of the polynomials $H_1,\dots,H_l$.

One can show that the theorem holds for any noetherian commutative ring R. Theorem 12.9 proves this when $R = k[y_1,\dots,y_t]$ (and more generally if R is any finitely generated reduced k-algebra). The scheme-theoretic generalisation of Theorem 12.9 implies the general form of the main theorem of elimination theory.

Corollary 12.10 (of Theorem 12.9)**.** *Let X,Y be quasi-projective varieties and suppose that X is complete. Let $\phi: X\to Y$ is a morphism. Then $\phi(X)$ is closed.*

Proof. The image of $\phi(X)$ is the projection of the graph $\Gamma_\phi \subseteq X \times Y$ by the projection to Y. Hence Proposition-Definition 12.6 implies the result. $\square$

Proposition 12.11. *A complete quasi-projective variety is projective.*

Proof. Let X be a quasi-projective complete variety. By definition, we may suppose that there is an open subvariety U of $\mathbb{P}^n(k)$ such that X is a closed subvariety of U. By Corollary 12.10, X is closed in $\mathbb{P}^n(k)$. Hence, from the definition of subvarieties, X is a closed subvariety of $\mathbb{P}^n(k)$. Hence X is projective. $\square$

Lemma 12.12. *Let X be an affine complete variety. Then X consists of a finite number of points.*

Proof. By Q2 of Sheet 3, $\mathcal{C}(X)$ is a finite dimensional k vector space. In particular, $\mathcal{C}(X)$ is finite over k. We deduce from Prop. 8.12 in CA that $\mathcal{C}(X)$ has only finitely maximal ideals. Hence X has only finitely many points by the discussion before Lemma 2.8. Alternatively, we can reason as follows. We see from Lemma 12.8 that all the irreducible components of X are complete. Let V be an irreducible component of X. Then $\mathcal{C}(V)$ is an integral domain by Lemma 2.5 and it is finite over k by Q2 of Sheet 3. Hence $\mathcal{C}(V)$ is a field by the (elementary) Lemma 8.9 in CA. Hence V is a point. Since X has only finitely many irreducible components, we conclude again that X only has a finite number of points. $\square$

13 Smoothness

A variety is smooth if it has "no kinks". For a curve C in the plane given by one equation $f(x, y) = 0$, this can analysed by looking at its gradient $\mathrm{grad}(f) = \langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f \rangle$. The curve will be smooth if $\mathrm{grad}(f)$ does not vanishes for any point of C. The general definition has a similar flavour.

Definition 13.1. *Let $V \subseteq k^n$ be an algebraic set. Suppose that $\mathcal{I}(V) = (P_1, \ldots, P_t) \subseteq k[x_1, \ldots, x_n]$. Let $\bar{v} \in V$. We say that V is nonsingular at $\bar{v}$ if the matrix $[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij}$ has rank $n - \mathrm{cod}(\{v\}, V)$. If V is irreducible then $\mathrm{cod}(\{v\}, V) = \dim(V)$ so that in that case V is nonsingular at $\bar{v}$ iff the matrix $[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij}$ has rank $n - \dim(V)$.*

Note that when C is a curve in the plane, we recover the definition given above. To make sense of this definition, we need to show that it does not depend on the polynomials P_i. In fact, we will show that the definition only depends on the coordinate ring $\mathcal{C}(V)$.

On the way to this result, we first make another definition.

Definition 13.2. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k_0 := R/\mathfrak{m}$. We say that R is a regular local ring if $\dim(R) = \dim_{k_0} \mathfrak{m}/\mathfrak{m}^2$.*

A few comments are in order. Note that with the notation of the last definition, we have $\dim(R) = \mathrm{ht}(\mathfrak{m})$ (this follows from the definition of dimension and the fact that R is local). On the other hand, by Nakayama's lemma (see Cor. 3.6 in CA), the ideal $\mathfrak{m}$ can be generated by $\dim_{k_0} \mathfrak{m}/\mathfrak{m}^2$ elements. Hence by a corollary of Krull's theorem (see CA Cor. 11.15), we have

$$\dim(R) = \mathrm{ht}(\mathfrak{m}) \leqslant \dim_{k_0} \mathfrak{m}/\mathfrak{m}^2.$$

The local ring R is regular iff this last inequality is an equality.

Proposition 13.3. *Let $V \subseteq k^n$ be an algebraic set. Then V is nonsingular at $\bar{v} \in V$ (for some and hence any choice of generators P_i of $\mathcal{I}(V)$) iff the local ring $\mathcal{O}_{V,v} \simeq \mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})}$ is regular.*

For the proof, we shall need the

Lemma 13.4. *Let R be a ring and let $\mathfrak{m} \subseteq R$ be a maximal ideal. Let $\phi : R \to R_{\mathfrak{m}}$ be the natural map of rings. Let $n \geqslant 0$. Then the unique maximal ideal $\underline{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ is the ideal of $R_{\mathfrak{m}}$ generated by $\phi(\mathfrak{m})$. Furthermore, we have $\phi^{-1}(\underline{\mathfrak{m}}^n) = \mathfrak{m}^n$ and the map of R-modules induced by ϕ*

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \to \underline{\mathfrak{m}}^n/\underline{\mathfrak{m}}^{n+1}$$

is an isomorphism.

Note that the lemma is obviously false if $\mathfrak{m}$ is not maximal (look eg at the case $n = 0$).

Proof. (of Lemma 13.4) The first assertion is contained in Lemma 5.6 in CA (standard properties of localisations - you can also prove this directly). This also implies that $\underline{\mathfrak{m}}^n$ is the ideal generated by $\phi(\mathfrak{m}^n)$. In particular, any element of $\underline{\mathfrak{m}}^n$ can be written in the form r/u, where $r \in \mathfrak{m}^n$ and $u \in R\backslash\mathfrak{m}$ (prove this directly or refer to Lemma 5.6 in CA).

We now prove that $\phi^{-1}(\underline{\mathfrak{m}}^n) = \mathfrak{m}^n$. To see this, note that by the definition of localisation, the ideal $\phi^{-1}(\underline{\mathfrak{m}}^n)$ is the set of elements $r \in R$, such that for some $t \in \mathfrak{m}^n$ and for some $u, v \in R\backslash\mathfrak{m}$, we have $v(ur - t) = 0$ (use the definition of localisation). Hence if $r \in R$ and there is $u \in \mathrm{R}\backslash\mathfrak{m}$ such that $ur \in \mathfrak{m}$ then $r \in \phi^{-1}(\underline{\mathfrak{m}}^n)$. On the other hand, if $r \in R$ and for some $t \in \mathfrak{m}^n$ and $u, v \in R\backslash\mathfrak{m}$ we have $vur = vt$, then $(vu)r \in \mathfrak{m}$. Thus

$$\phi^{-1}(\underline{\mathfrak{m}}^n) = \{r \in R \,|\, \exists u \in R\backslash\mathfrak{m} : ur \in \mathfrak{m}^n\}.$$

Now suppose that $r \in R$ and that $u \in R\backslash\mathfrak{m}$ is such that $ur \in \mathfrak{m}^n$. Recall that $\mathfrak{m}^n$ is $\mathfrak{m}$-primary (see Lemma 6.4 in CA). Since $u \notin \mathfrak{m}^n$, we deduce that either $r \in \mathfrak{m}^n$ or both $u \,(\mathrm{mod}\, \mathfrak{m}^n)$ and $r \,(\mathrm{mod}\, \mathfrak{m}^n)$ are nilpotent in $R/\mathfrak{m}^n$. The second possibility cannot occur because all the powers of u lie in $R\backslash\mathfrak{m}$ (since $\mathfrak{m}$ is prime). Hence we must have $r \in \mathfrak{m}^n$. In other words we have $\phi^{-1}(\underline{\mathfrak{m}}^n) = \mathfrak{m}^n$.

We now show that the natural map $\mathfrak{m}^n/\mathfrak{m}^{n+1} \to \underline{\mathfrak{m}}^n/\underline{\mathfrak{m}}^{n+1}$ is an isomorphism. Since $\phi^{-1}(\underline{\mathfrak{m}}^{n+1}) = \mathfrak{m}^{n+1}$, we see that the map is injective. To prove surjectivity, let $r/u \in \underline{\mathfrak{m}}^n$, where $r \in \mathfrak{m}^n$ and $u \in R\backslash\mathfrak{m}$. Let $v \in R\backslash\mathfrak{m}$ be such that $uv = 1 \,(\mathrm{mod}\, \mathfrak{m})$ (such a v exists because $R/\mathfrak{m}$ is a field). Then there is an $a \in \mathfrak{m}$ such that

$$rv/1 = ruv/u = (r + ra)/u = r/u + (a/1)(r/u) = r/u \,(\mathrm{mod}\, \underline{\mathfrak{m}}^{n+1})$$

and thus $r/u \,(\mathrm{mod}\, \underline{\mathfrak{m}}^{n+1})$ is in the image of $\mathfrak{m}^n$ in $\underline{\mathfrak{m}}^n/\underline{\mathfrak{m}}^{n+1}$. □

Proof. (of Proposition 13.3) Let $\bar{v} = \langle v_1, \dots, v_n\rangle \in V \subseteq k^n$. Suppose that $\mathcal{I}(V) = (P_1, \dots, P_t)$. Write

$$\mathfrak{m} := \mathcal{I}(\{\bar{v}\}) = (x_1 - v_1, \dots, x_n - v_n)$$

be the maximal ideal of $k[x_1, \dots, x_n]$ associated with $\bar{v}$. Let $\mathfrak{n} = \mathfrak{m} \,(\mathrm{mod}\, \mathcal{I}(V)) \subseteq \mathcal{C}(V)$ be the maximal ideal of $\mathcal{C}(V)$ associated with $\bar{v}$. Define a map of k-vector space $\phi : \mathfrak{m} \to k^n$ by the formula

$$\phi(Q) = \langle(\frac{\partial}{\partial x_1}Q)(\bar{v}), \dots, (\frac{\partial}{\partial x_n}Q)(\bar{v})\rangle.$$

Since $\mathfrak{m}^2$ is generated by the elements $(x_i - v_i)(x_j - v_j)$, we see that $\phi(\mathfrak{m}^2) = 0$ (apply the Leibniz rule). We thus obtain a k-linear map $\mathfrak{m}/\mathfrak{m}^2 \to k^n$. This map is surjective because $\phi(x_i - v_i)$ is the i-the element of the standard basis of k^n. On the other hand, $\mathfrak{m}/\mathfrak{m}^2$ is generated by n elements as a $R/\mathfrak{m} = k$-vector space and so is of dimension $\leqslant n$. Hence the map $\mathfrak{m}/\mathfrak{m}^2 \to k^n$ is an isomorphism of k-vector spaces. Now the image $(\mathcal{I}(V)+\mathfrak{m}^2)/\mathfrak{m}^2$ of $\mathcal{I}(V) \subseteq \mathfrak{m}$ in $\mathfrak{m}/\mathfrak{m}^2$ is generated by $P_1 \pmod{\mathfrak{m}^2}, \dots, P_t \pmod{\mathfrak{m}^2}$ has a $R/\mathfrak{m} = k$-vector space. Hence

$$\dim_k((\mathcal{I}(V)+\mathfrak{m}^2)/\mathfrak{m}^2) = \dim_k(\phi(\mathcal{I}(V))) = \mathrm{rk}\begin{pmatrix} (\frac{\partial}{\partial x_1}P_1)(\bar{v}) & \dots & (\frac{\partial}{\partial x_n}P_1)(\bar{v}) \\ (\frac{\partial}{\partial x_1}P_2)(\bar{v}) & \dots & (\frac{\partial}{\partial x_n}P_2)(\bar{v}) \\ \vdots & \vdots & \vdots \\ (\frac{\partial}{\partial x_1}P_t)(\bar{v}) & \dots & (\frac{\partial}{\partial x_n}P_t)(\bar{v}) \end{pmatrix} =: \mathrm{rk}[(\frac{\partial}{\partial x_j}P_i)(\bar{v})]_{ij}.$$

On the other hand, we have by construction a complex of $R/\mathfrak{m} = k$-vector spaces

$$0 \to (\mathcal{I}(V)+\mathfrak{m}^2)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0$$

We claim that this complex is exact. The second arrow from the left is injective by definition and likewise it follows from the definitions that the third arrow from the left is surjective. So we only have to show that the complex is exact at $\mathfrak{m}/\mathfrak{m}^2$. To see this, suppose that $P \in \mathfrak{m}$ and that $P \pmod{\mathcal{I}(V)} \in \mathfrak{n}^2$. Since $\mathfrak{n}^2 = (\mathfrak{m}^2 + \mathcal{I}(V))/\mathcal{I}(V)$, there is $Q \in \mathfrak{m}^2 + \mathcal{I}(V)$ such that $P \pmod{\mathcal{I}(V)} = Q \pmod{\mathcal{I}(V)}$. We then have $(P - Q) \pmod{\mathcal{I}(V)} = 0$, or in other words $P - Q \in \mathcal{I}(V)$. Hence P is the sum of an element of $\mathcal{I}(V)$ and an element of $\mathfrak{m}^2$. This shows that the complex is exact at $\mathfrak{m}/\mathfrak{m}^2$ and is thus an exact complex.

We conclude that

$$\mathrm{rk}[(\frac{\partial}{\partial x_j}P_i)(\bar{v})]_{ij} + \dim_k(\mathfrak{n}/\mathfrak{n}^2) = n. \tag{1}$$

Now we have $\mathrm{cod}(V, \{\bar{v}\}) = \mathrm{ht}(\mathfrak{n}) = \dim(\mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})})$ (follows from the definition of dimension - see Lemma 11.2 in CA). Using Lemma 13.4, we see that the local ring $\mathcal{C}(V)_{\mathcal{I}(\{\bar{v}\})}$ is regular iff

$$\mathrm{rk}[(\frac{\partial}{\partial x_j}P_i)(\bar{v})]_{ij} = n - \mathrm{cod}(V, \{\bar{v}\}).$$

This proves the first assertion.

For the second assertion, note that if V is irreducible, we have $\mathrm{cod}(V, \{\bar{v}\}) = \dim(V)$ by Theorem 8.7 (note that a point has dimension 0). □

Remark. (1) Keep the notation of the proof of Proposition 13.3. From the remark preceding the proposition, we have $\dim_k(\mathfrak{n}/\mathfrak{n}^2) \geqslant \mathrm{cod}(V, \{\bar{v}\})$ and so we always have

$$\mathrm{rk}[(\frac{\partial}{\partial x_j}P_i)(\bar{v})]_{ij} = n - \dim_k(\mathfrak{n}/\mathfrak{n}^2) \leqslant n - \mathrm{cod}(V, \{\bar{v}\})$$

even if V is singular at $\bar{v}$.

(2) Note that equation (1) gives an effective way to compute $\dim_k(\mathfrak{n}/\mathfrak{n}^2)$.

We also record the following lemma, which will be useful in calculations.

Lemma 13.5. *Keep the assumptions and notation of Proposition 13.3. Let $Q_1, \dots Q_s \in \mathcal{I}(V)$. Suppose that $[(\frac{\partial}{\partial x_j}Q_i)(\bar{v})]_{ij}$ has rank $n - \mathrm{cod}(V, \{v\})$. Then V is nonsingular at $\bar{v}$.*

This lemma will allow us to check nonsingularity in situations where it is difficult to find generators of $\mathcal{I}(V)$.

Proof. We use the notation of the proof of Proposition 13.3. Let $J \subseteq \mathcal{I}(V)$ be the ideal generated by $Q_1, \ldots, Q_s$. It was shown in the proof of Proposition 13.3 that

$$\mathrm{rk}[(\frac{\partial}{\partial x_j} Q_i)(\bar{v})]_{ij} = \dim_k(\phi(J))$$

and in particular that $\mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} = \dim_k(\phi(\mathcal{I}(V)))$. On the other hand, we have $\dim_k(\phi(\mathcal{I}(V))) \geqslant \dim_k(\phi(J))$ since $J \subseteq \mathcal{I}(V)$. Hence by the remark preceding the lemma, we have

$$\mathrm{rk}[(\frac{\partial}{\partial x_j} Q_i)(\bar{v})]_{ij} \leqslant \mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} \leqslant n - \mathrm{cod}(V, \{\bar{v}\}).$$

The assumptions of the lemma now imply that the two last inequalities are equalities, hence the conclusion. □

Let now X be any variety. We shall write $\mathrm{Sing}(X)$ for the set of points $x \in X$ such that the local ring $\mathcal{O}_{X,x}$ is a regular local ring. This clearly specialises to Definition 13.1 when X is an affine variety.

A variety X is *nonsingular* or *smooth* if $\mathrm{Sing}(X) = \emptyset$.

Proposition 13.6. *Let X be a non empty irreducible variety. Then the set* $\mathrm{Sing}(X)$ *is closed and* $\mathrm{Sing}(X) \neq X$.

For the proof, we shall need the following definition and the subsequent proposition. Let R be a UFD with fraction field K. If

$$Q(x) = x^m + r_{m-1}x^{m-1} + \cdots + r_0 \in R[x],$$

we define the *content* $\mathrm{cont}(Q)$ to be the gcd of the coefficients of Q (note that the gcd is only well-defined up to multiplication by a unit of R). If $Q(x) \in K[x]$, we define the content of $Q(x)$ to be $\mathrm{cont}(d \cdot Q)/d$, where $d \in R$ is such that $d \cdot Q(x) \in R[x]$. One can show that this last definition does not depend on the choice of d. Moreover, one can show that $\mathrm{cont}(Q_1 \cdot Q_2) = \mathrm{cont}(Q_1) \cdot \mathrm{cont}(Q_2)$ for any two $Q_1, Q_2 \in K[x]$. Note that if $Q(x) \in K[x]$ and $\mathrm{cont}(Q)$ is a unit, then $Q(x) \in R[x]$ (why?). The all-important result concerning the content function is the

Lemma (generalisation of Gauss's lemma). The irreducible elements of $R[x]$ are the irreducible elements of R and the polynomials $P(x) \in R[x]$, whose content is a unit and which are irreducible (and hence non constant) in $K[x]$.

The proofs of all these statement are similar to the ones considered in the Rings and Modules course in the situation where $R = \mathbb{Z}$ (but they are not examinable). See IV, §2 in S. Lang's book *Algebra* (Springer) for more details.

Proposition 13.7. *Let X be a non empty irreducible variety. Then X is birational to an algebraic set $V \subseteq k^n$ such that $\mathcal{I}(V) \subseteq k[x_1, \ldots, x_n]$ is prime and principal.*

Proof. (of Proposition 13.7) We shall only prove this in the situation where $\mathrm{char}(k) = 0$ (but the result holds without this assumption). So suppose that $\mathrm{char}(k) = 0$. Restricting to an open affine subset of X, we may assume wlog that X is an irreducible affine variety. Let $K := \mathrm{Frac}(\mathcal{C}(X))$ be the function field of X.

Since the k-algebra $\mathcal{C}(X)$ is finitely generated over k, the field K is finitely generated as a field over k. Let $b_1, \ldots, b_t \in K$ be a transcendence basis for K over k. By definition, this means that the b_i are algebraically independent over k (ie the map of k-algebras $k[y_1, \ldots, y_t] \to K$ sending y_i to b_i is injective) and that the field extension $K|k(b_1, \ldots, b_t)$ is algebraic. A transcendence basis always exists. See Prop. 11.3 in CA for this. Since $\mathrm{char}(k) = 0$, the extension $K|k(b_1, \ldots, b_t)$ is a separable extension. This extension is also a finite extension because K is finitely generated as a field over $k(b_1, \ldots, b_t)$ (since K is finitely generated as a field over k). Hence the extension $K|k(b_1, \ldots, b_t)$ is a simple extension by the "primitive element theorem" (see the course on Galois theory) and so there is an element $b \in K$, such that $k = k(b_1, \ldots, b_t)(b)$ and an irreducible polynomial $Q(x) \in k(b_1, \ldots, b_t)[x]$ such that $Q(b) = 0$.

Now note that every element of $k(b_1, \ldots, b_t)$ can be written as quotient c/d, where $c, d \in k[b_1, \ldots, b_t]$ (here $k[b_1, \ldots, b_t]$ is the k-subalgebra of K generated by the b_i). Write

$$Q(x) = x^m + \frac{c_{m-1}}{d_{m-1}}x^{m-1} + \cdots + \frac{c_1}{d_1}x + \frac{c_0}{d_0}$$

where $c_i, d_i \in k[b_1, \ldots, b_t]$. Let $d = \prod_i d_i$. Consider the polynomial $dQ \in k[b_1, \ldots, b_t][x]$ and let

$$P := dQ/\mathrm{cont}(dQ) \in k[b_1, \ldots, b_t][x],$$

where (abusing language) $\mathrm{cont}(dQ) \in k[b_1, \ldots, b_t]$ is an arbitrary representative of the content of dQ. By construction, the polynomial $P(x)$ is irreducible in $k(b_1, \ldots, b_t)[x]$ (since it is a constant multiple of a minimal polynomial) and its content is a unit. By the generalised Gauss lemma (see the discussion above), $P(x)$ is thus irreducible in $k[b_1, \ldots, b_t][x]$.

Now let

$$\phi : k[b_1, \ldots, b_t][x] \to K$$

be the homomorphism of k-algebras sending the b_i to themselves and x to b.

The kernel $\ker(\phi)$ is then a prime ideal (since the image of ϕ is a domain) and by construction we have $P(x) \in \ker(\phi)$. Now the ideal $(P) \subseteq k[b_1, \ldots, b_t][x]$ is also prime, since P is irreducible. Hence $\mathrm{cod}((P), k[b_1, \ldots, b_t][x]) = 1$ by Krull's principal ideal theorem (see Th. 11.13 in CA). On the other hand, the fraction field of

$$\mathrm{Im}(\phi) = k[b_1, \ldots, b_t, b] \simeq k[b_1, \ldots, b_t][x]/\ker(\phi)$$

is the field K and K has transcendence degree t by assumption. Thus

$$\dim(k[b_1, \ldots, b_t][x]/\ker(\phi)) = t$$

by Corollary 11.28 in CA. Using Theorem 8.7, we deduce that

$$\mathrm{cod}(\ker(\phi), k[b_1, \ldots, b_t][x]) = \dim(k[b_1, \ldots, b_t][x]) - t = t + 1 - t = 1.$$

Hence we must have $\ker(\phi) = (P)$, for otherwise we would have $\mathrm{cod}(\ker(\phi), k[b_1, \ldots, b_t][x]) \geqslant 2$.

So we conclude that $k[b_1, \ldots, b_t][x]/(P) \simeq k[b_1, \ldots, b_t, b]$.

Now the b_i are algebraically independent and thus the k-algebra $k[b_1, \ldots, b_t][x]$ can be viewed as the coordinate ring of k^{t+1}. The ring $k[b_1, \ldots, b_t][x]/(P)$ is thus isomorphic to the coordinate ring of an irreducible algebraic set V in k^{t+1}, whose (prime) radical ideal is generated by a single irreducible polynomial. Since the function field of V is isomorphic to K as a K-algebra, it satisfies the conclusion of the proposition (by Proposition 9.3). □

Proof. (of Proposition 13.6) We first show that $\mathrm{Sing}(X)$ is closed. Let $\{U_i\}$ be an open affine covering of X. By Proposition 13.3, a point $x \in U_i$ is nonsingular in X iff it is nonsingular in U_i, ie we have

$\mathrm{Sing}(X) \cap U_i = \mathrm{Sing}(U_i)$. On the other hand, the set $\mathrm{Sing}(X)$ is closed iff $\mathrm{Sing}(X) \cap U_i$ is closed for all i (why?). Hence we may assume that X is isomorphic to an algebraic set $V \subseteq k^n$ for some n.

Let $P_1, \dots, P_t$ be generators of $\mathcal{I}(V) \subseteq k[x_1, \dots, x_n]$. From the remark following the proof of Proposition 13.3, we have

$$\mathrm{Sing}(V) = \{\bar{v} \in V \,|\, \mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} < n - \dim(V)\}.$$

Now recall that

$$\mathrm{rk}[(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} = \max\{h \in \mathbb{N} \,|\, \text{there exists a } h \times h\text{-submatrix } M \text{ in } [(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij} \text{ such that } \det(M) \neq 0\}$$

and hence

$$\mathrm{Sing}(V) = \{\bar{v} \in V \,|\, \det(M) = 0 \text{ for all the } (n - \dim(V)) \times (n - \dim(V))\text{-submatrices } M \text{ in } [(\frac{\partial}{\partial x_j} P_i)(\bar{v})]_{ij}\}$$

and hence $\mathrm{Sing}(V)$ is the zero set of a set of polynomials and is thus closed.

We now prove that $\mathrm{Sing}(X) \neq X$. Again, we only show this when $\mathrm{char}(k) = 0$ (but the statement holds without that assumption). We may replace wlog X by any of its open subsets and so thanks to Proposition 13.7 we may suppose that X is an algebraic set $V \subseteq k^n$ such that $\mathcal{I}(V) = (P)$, where $P \in k[x_1, \dots, x_n]$ is an irreducible polynomial. In this situation, we have to show that

$$\mathrm{Sing}(V) = \{\bar{v} \in V \,|\, (\frac{\partial}{\partial x_1} P)(\bar{v}) = (\frac{\partial}{\partial x_2} P)(\bar{v}) = \dots = (\frac{\partial}{\partial x_n} P)(\bar{v}) = 0\} \neq V.$$

Suppose for contradiction that $\mathrm{Sing}(V) = V$. Then $\mathfrak{r}((\frac{\partial}{\partial x_i} P)) \subseteq \mathcal{I}(V) = (P)$ for all i. In particular we have an inclusion of ideals $(\frac{\partial}{\partial x_i} P) \subseteq (P)$ for all i, since (P) is a prime ideal. In other words, $P | \frac{\partial}{\partial x_i} P$ for all i. Now let i_0 be such that P_{i_0} has a monomial divisible by x_{i_0}. This exists since P is irreducible and in particular not constant. In that case $\frac{\partial}{\partial x_{i_0}} P \neq 0$ (note that we use the fact that $\mathrm{char}(k) = 0$ here) and $\deg_{x_{i_0}}(\frac{\partial}{\partial x_{i_0}} P) < \deg_{x_{i_0}}(P)$. In particular, $\frac{\partial}{\partial x_{i_0}} P$ is not divisible by P. This is a contradiction, so $\mathrm{Sing}(V) \neq V$. □

14 Appendix. Proof of Theorem 8.7.

The material in this appendix is not examinable.

Let R be a ring and let $\mathfrak{p}$ be a prime ideal of R. From the definition of dimension and height, we have the inequality

$$\dim(R/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) \leqslant \dim(R).$$

This inequality is not an equality in general. However, it is an equality if R is a finitely generated domain over a field.

If R is a finite-dimensional noetherian ring and $n \geqslant 0$, we shall say that R satisfies $\mathrm{DE}(n)$ if

$$\dim(R/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) = \dim(R)$$

for all prime ideals $\mathfrak{p}$ of R such that $\mathrm{ht}(\mathfrak{p}) = n$.

Theorem A (= Theorem 8.7). *Let R be a finitely generated domain over k. Then R satisfies* $\mathrm{DE}(n)$ *for all* $n \geqslant 0$.

The assumption that k is algebraically closed will not be used in the proof, so the result holds without that assumption.

Proceed as follows.

(1) Show that to prove Theorem A, it is sufficient to show that any finitely generated domain over a field satisfies DE(1). [Hint: induction on n.]

Solution. Suppose that any finitely generated domain over k satisfies DE(1). Let R be a finitely generated domain over k. We prove that

$$\dim(R/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) = \dim(R)$$

by induction on $\mathrm{ht}(\mathfrak{p})$. The case $\mathrm{ht}(\mathfrak{p}) = 0$ is clear, since in that case $\mathfrak{p} = (0)$ (since R is a domain) so we may suppose that $\mathrm{ht}(\mathfrak{p}) > 1$. Let $\mathfrak{p}$ be a prime ideal and let

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_\delta$$

be a descending chain of prime ideals of length $\delta := \mathrm{ht}(\mathfrak{p})$. We suppose that any finitely generated domain over k satisfies $\mathrm{DE}(0), \mathrm{DE}(1), \ldots, \mathrm{DE}(\delta - 1)$. Since $\mathfrak{p}_0$ is minimal over $\mathfrak{p}_1$, we have $\mathrm{ht}(\mathfrak{p}/\mathfrak{p}_1) = 1$. We thus have

$$\mathrm{ht}(\mathfrak{p}/\mathfrak{p}_1) + \dim(R/\mathfrak{p}) = \dim(R/\mathfrak{p}_1)$$

Also, we have $\mathrm{ht}(\mathfrak{p}_1) = \delta - 1 \geqslant 1$ by construction. So we have

$$\dim(R/\mathfrak{p}_1) + \mathrm{ht}(\mathfrak{p}_1) = \dim(R)$$

We conclude that

$$1 + \dim(R/\mathfrak{p}) = \dim(R) - (\delta - 1)$$

or in other words that

$$\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$$

as required.

(2) Show that to prove that any finitely generated domain over a field satisfies DE(1), it is sufficient to prove that any polynomial ring $k[x_1, \ldots, x_d]$ ($d \geqslant 0$) satisfies DE(1). [Hint: apply Noether's normalisation lemma and the Going-up theorem.]

Solution. Suppose that any polynomial ring over k satisfies DE(1). Let R be a finitely generated domain over k. By Noether's normalisation lemma, there is an injection $k[x_1, \ldots, x_d] \hookrightarrow R$ ($d \geqslant 0$) making R into a finite $k[x_1, \ldots, x_d]$-algebra. Let $P := k[x_1, \ldots, x_d]$. Let $\mathfrak{p}$ be a prime ideal of height one in R. By the Going-up theorem and Q1 of Sheet 3 in CA, the height of $\mathfrak{p}$ and the height of $\mathfrak{p} \cap P$ is the same. Hence

$$1 + \dim(P/\mathfrak{p} \cap P) = \dim(P).$$

By Lemma 11.28 in CA, we have $\dim(P) = \dim(R)$. Also $R/\mathfrak{p}$ is an integral extension of $P/\mathfrak{p} \cap P$ and thus $\dim(R/\mathfrak{p}) = \dim(P/\mathfrak{p} \cap P)$ by the same lemma. We conclude that $1 + \dim(R/\mathfrak{p}) = \dim(R)$, as required.

(3) Prove that the height of a maximal ideal in $k[x_1, \ldots, x_d]$ is d for any $d \geqslant 1$. [Hint: use the method of Sheet 3, Q6 in CA.]

Solution. Let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_d] =: P$. In Sheet 3, Q6 of CA it is shown that $\mathfrak{m}$ is generated by polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \ldots, P_d(x_1, \ldots, x_d)$. In the course of the proof (this can be seen in the solution, which is available), it is also shown that the ideals

$$\mathfrak{a}_i := (P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_i(x_1, \dots, x_i))$$

are maximal in $k[x_1, \dots, x_i]$ for all $i = 1, \dots, d$. It was shown at the beginning of section 11.4 of CA that we have $\mathfrak{a}_i P = \mathfrak{a}_i[x_{i+1}, \dots, x_d]$. This implies that $\mathfrak{a}_i P \neq \mathfrak{a}_{i+1} P$ for otherwise

$$\mathfrak{a}_i[x_{i+1}] = \mathfrak{a}_i P \cap k[x_1, \dots, x_{i+1}] = \mathfrak{a}_{i+1} P \cap k[x_1, \dots, x_{i+1}] = \mathfrak{a}_{i+1}.$$

This is not possible because $\mathfrak{a}_{i+1}$ is maximal in $k[x_1, \dots, x_{i+1}]$ and $\mathfrak{a}_i[x_{i+1}]$ is not, because

$$k[x_1, \dots, x_{i+1}]/\mathfrak{a}_i[x_{i+1}] = (k[x_1, \dots, x_i]/\mathfrak{a}_i)[x_{i+1}],$$

which is a domain but not a field.

Hence we obtain a chain of prime ideals

$$\mathfrak{m} \supsetneq \mathfrak{a}_{d-1} P \supsetneq \mathfrak{a}_{d-2} P \supsetneq \cdots \supsetneq \mathfrak{a}_1 P \supsetneq (0)$$

and thus $\mathrm{ht}(\mathfrak{m}) \geqslant d$. Since we also have $\mathrm{ht}(\mathfrak{m}) \leqslant d = \dim(P)$ (by Corollary 11.26 in CA), we thus have $\mathrm{ht}(\mathfrak{m}) = d$.

(4) Prove that the height of a maximal ideal in a finitely generated domain R over k is $\dim(R)$. [Hint: apply Noether's normalisation lemma and (3). When k is algebraically closed, this is done in [1], Theorem 11.25.]

Solution. Let $\mathfrak{m}$ be a maximal ideal of R. By Noether's normalisation lemma, there is an injection $k[x_1, \dots, x_d] \hookrightarrow R$ ($d \geqslant 0$) making R into a finite $k[x_1, \dots, x_d]$-algebra. Let $P := k[x_1, \dots, x_d]$. Let $\mathfrak{p}$ be a prime ideal of P, which lies over R (this exists by Th. 8.8). By the Going-down theorem (Q1 of Sheet 4 in CA) and Q1 of Sheet 3, the height of $\mathfrak{p}$ and the height of $\mathfrak{p} \cap P$ is the same and by Corollary 8.20 of CA, $\mathfrak{p}$ is also a maximal ideal of P. By Lemma 11.28 in CA, we have $\dim(P) = \dim(R)$ and thus the height of $\mathfrak{p}$ is $\dim(R)$ by (3).

(5) Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$. Let $f \in \mathfrak{m}$. Prove that $\dim(R/(f)) \geqslant \dim(R) - 1$. [Hint: this is a consequence of Krull's principal ideal theorem; you may follow the proof given in [3], Th. 2.5.15, p. 72.]

(6) Show that if R is noetherian and a UFD, then any prime ideal of height 1 is principal. [Hint: this is a classical statement, which can be deduced from Krull's principal ideal theorem; you may follow the proof given in Cor. 10.6, [2], p. 236.]

Solution. The argument is given in the proof of Cor. 10.6, [2], p. 236.

(7) Deduce that any polynomial ring $k[x_1, \dots, x_d]$ ($d \geqslant 0$) satisfies DE(1). [Hint: localise at a maximal ideal and apply (4), (5), (6).]

Solution. Let $P := k[x_1, \dots, x_d]$ and let $\mathfrak{p} \in \mathrm{Spec}(P)$ be a prime ideal of height 1. Let $\mathfrak{m}$ be a maximal ideal of P, which contains $\mathfrak{p}$. The equality

$$\dim(P) = 1 + \dim(P/\mathfrak{p})$$

is equivalent to the equality

$$\dim(P_{\mathfrak{m}}) = 1 + \dim((P/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}})$$

by (4). Note also that we have $(P/\mathfrak{p})_{\mathfrak{m}/\mathfrak{p}} = P_{\mathfrak{m}}/\mathfrak{p}P_{\mathfrak{m}}$ by Lemma 5.5. in CA.

On the other hand, by (6) and the fact that P is a UFD, we have $\mathfrak{p} = (f)$ for some $f \in \mathfrak{p}$ and so $P_\mathfrak{m}/\mathfrak{p}P_\mathfrak{m} = P_\mathfrak{m}/(f)$. By (5) and (4), we thus have

$$\dim(P_\mathfrak{m}/\mathfrak{p}P_\mathfrak{m}) \geqslant \dim(P_\mathfrak{m}) - 1 = \dim(P) - 1.$$

Hence

$$1 + \dim(P/\mathfrak{p}) \geqslant \dim(P)$$

Since we have $1+\dim(P/\mathfrak{p}) \leqslant \dim(P)$ by the definition of dimension and height, we thus have $1+\dim(P/\mathfrak{p}) = \dim(P)$, as required.

(8) Prove the theorem.

Solution: (7), (2), (1).

References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Student economy edition, Addison-Wesley Series in Mathematics, Westview Press, Boulder, CO, 2016.

[2] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[3] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné; Oxford Science Publications.

Exercise sheet 1. Week 4. Chapters 1-4.

Q1. (1) Describe the Zariski topology of k.

(2) Show that the Zariski topology of k^2 is not the product topology of $k \times k = k^2$.

Q2. Let $V \subseteq k^n$ be an algebraic set. Show that V is the disjoint union of two non empty algebraic sets in k^n iff there are two non-zero finitely generated reduced k-algebras T_1 and T_2 and an isomorphism of k-algebras $T_1 \oplus T_2 \simeq \mathcal{C}(V)$.

Q3. Let $V \subseteq k^3$ be the set

$$V := \{(t, t^2, t^3) \,|\, t \in k\}.$$

Show that V is an algebraic set and that it is isomorphic to k as an algebraic set. Provide generators for $\mathcal{I}(V)$.

Q4. (1) Let $V \subseteq k^2$ be the set of solutions of the equation $y = x^2$. Show that V is isomorphic to k as an algebraic set.

(2) Let $V \subseteq k^2$ be the set of solutions of the equation $xy = 1$. Show that V is not isomorphic to k as an algebraic set.

(3) [difficult] (optional) Let $P(x, y) \in k[x, y]$ be an irreducible quadratic polynomial and let $V \subseteq k^2$ be the set of zeroes of $P(x, y)$. Show that V is isomorphic to one of the algebraic sets defined in (1) and (2).

Q5. Let $V \subseteq k^n$ and $W \subseteq k^t$ be two algebraic sets. Let $\psi : V \to W$ be a regular map.

(1) Show that $\psi(V)$ is dense in W iff the map of rings $\psi^* : \mathcal{C}(W) \to \mathcal{C}(V)$ is injective.

(2) Show that ψ^* is surjective iff $\psi(V)$ is closed and the induced map $V \to \psi(V)$ is an isomorphism of algebraic sets.

Q6. Let $V \subseteq k^3$ be the algebraic set described by the ideal $(x^2 - yz, xz - x)$. Show that V has three irreducible components. Find generators for the radical (actually prime) ideals associated with these components.

Q7. Let $V \subseteq k^n$ and $W \subseteq k^t$ be algebraic subsets. Let $V_0 \subseteq V$ and $W_0 \subseteq W$ be open subsets. View V_0 and W_0 as open subvarieties of V and W respectively. For $i \in \{1, \dots, t\}$ let $\pi_i : k^t \to k$ be the projection on the i-coordinate. Let $\psi : V_0 \to W_0$ be a map. Show that ψ is a morphism of varieties iff $\pi_i \circ \psi$ is a regular function on V_0 for all $i \in \{1, \dots, t\}$.

Q8. (optional) Show that the open subvariety $k^2 \backslash \{0\}$ of k^2 is not affine.

Exercise sheet 2. Week 6. Chapters 1-8.

Q1. Let $i \in \{0, \dots, n\}$ and let $u_i : k^n \to \mathbb{P}^n(k)$ be the standard map (with image the coordinate chart U_i). Let $C \subseteq k^n$ be a closed subvariety of k^n (ie an algebraic set in k^n). For any $P \in k[x_0, \dots, x_{i-1}, \check{x}_i, x_{i+1}, \dots, x_n]$ let

$$\beta_i(P) := x_i^{\deg(P)} P(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{\check{x}_i}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}) \in k[x_0, \dots, x_n].$$

(1) Let $\bar{C}$ be the closure of $u_i(C)$ in $\mathbb{P}^n(k)$. Show that $(\beta_i(\mathcal{I}(C))) = \mathcal{I}(\bar{C})$ (where $(\beta_i(\mathcal{I}(C)))$ is the ideal of $k[x_0, \dots, x_n]$ generated by all the elements of $\beta_i(\mathcal{I}(Z))$).

(2) Suppose that $\mathcal{I}(C) = (J)$ (ie $\mathcal{I}(C)$ is a principal ideal with generator J). Show that $(\beta_i(J)) = \mathcal{I}(\bar{C})$.

(3) Suppose that $n = 3$ and that C is the variety considered in Q3 of Sheet 1. Describe the closure of $u_0(C)$ in $\mathbb{P}^3(k)$. Find homogenous polynomials $(H_1, \dots, H_h)$ such that $\mathrm{Z}(H_1, \dots, H_h)$ is the closure of $u_0(C)$ in $\mathbb{P}^3(k)$.

Q2. Let V (resp. W) be a closed subvariety of $\mathbb{P}^n(k)$ (resp. $\mathbb{P}^t(k)$). Let $V_0 \subseteq V$ (resp. $W_0 \subseteq W$) be an open subset of V (resp. and open subset of W). View V_0 (resp. W_0) as an open subvariety of V (resp. W). Let $Q_0, \dots, Q_t \in k[x_0, \dots, x_n]$ be homogenous polynomials of the same degree. Suppose that $V_0 \cap \mathrm{Z}((Q_0, \dots, Q_t)) = \emptyset$. Let $f : V_0 \to \mathbb{P}^t(k)$ be the map given by the formula $f(\bar{v}) := [Q_0(\bar{v}), \dots, Q_t(\bar{v})]$. Suppose finally that $f(V_0) \subseteq W_0$. Show that the induced map $V_0 \to W_0$ is a morphism of varieties.

Q3. Prove Lemma 7.1.

Q4. Let T be a topological space.

(1) Let $S \subseteq T$ be a subset. Suppose that S is irreducible. Show that the closure of S in T is also irreducible.

(2) Suppose that T is noetherian. Show that T is Hausdorff iff T is finite and discrete.

(3) Let V be a variety. Show that V is irreducible iff the ring $\mathcal{O}_V(U)$ is an integral domain for all open subsets $U \subseteq V$.

(4) Suppose T is noetherian. Show that T is quasi-compact.

Q5. Prove Lemma 8.1.

Q6. Let T be a topological space. Let $\{V_i\}$ be an open covering of T. Let $C \subseteq T$ be an irreducible closed subset (hence non empty).

(1) Show that $C \cap V_i$ is irreducible if $C \cap V_i \neq \emptyset$ and that $\sup_{i, C \cap V_i \neq \emptyset} \mathrm{cod}(C \cap V_i, V_i) = \mathrm{cod}(C, T)$ and $\sup_i \dim(V_i) = \dim(T)$.

(2) Prove Proposition 8.6. Give an example of a noetherian topological space of infinite dimension.

Q7. (1) Show that any element of $\mathrm{GL}_{n+1}(k)$ (= group of $(n+1) \times (n+1)$-matrices with entries in k and with non zero determinant) defines an automorphism of $\mathbb{P}^n(k)$.

(2) Show that if V is a projective variety, then for any two points $v_1, v_2 \in V$, there is an open affine subvariety $V_0 \subseteq V$ such that $v_1, v_2 \in V_0$.

Q8. (optional) (1) Let $P(x_0, \dots, x_n)$ be a homogenous polynomial. Show that all the irreducible factors of P are also homogenous.

(2) Let $D \subseteq \mathbb{P}^n(k)$ be a closed subvariety. Suppose that D is irreducible and that $\mathrm{cod}(D, \mathbb{P}^n(k)) = 1$. Show that there is a homogenous irreducible polynomial $P \in k[x_0, \dots, x_n]$ such that $D = \mathrm{Z}(P)$.

Exercise sheet 3. Week 8. Chapters 1-12.

Q1. Let $V_0 = \mathrm{Z}(x_0x_3 - x_1^2) \subseteq \mathbb{P}^3(k)$ and $V_1 = \mathrm{Z}(x_1x_3 - x_2^2) \subseteq \mathbb{P}^3(k)$. Let $C := V_0 \cap V_1 \subseteq \mathbb{P}^3(k)$. Let $U := \mathbb{P}^3 \backslash \mathrm{Z}(x_0, x_1, x_2)$ and endow U with its structure of open subvariety of $\mathbb{P}^3(k)$. Let $g : U \to \mathbb{P}^2(k)$ be the morphism such that $g([X_0, X_1, X_2, X_3]) = [X_0, X_1, X_2]$ for all $[X_0, X_1, X_2, X_3] \in U$ (see Q2 of Sheet 2).

(1) Show that the morphism $g|_{C\cap U} : C \cap U \to \mathbb{P}^2(k)$ extends to a morphism $f : C \to \mathbb{P}^2(k)$.

(2) Show that $f(C)$ is closed and that $f(C) = \mathrm{Z}(z_0z_2^2 - z_1^3)$.

(3) Show that the induced map $f : C \to f(C)$ is an isomorphism.

Q2. (1) Let $f : X \to Y$ be a surjective morphism of quasi-projective varieties. Suppose that X is complete. Show that Y is also complete.

(2) Show that a noetherian topological space only has finitely many connected components.

(2) Let $(V, \mathcal{O}_V)$ be a projective variety. Show that the k-vector space $\mathcal{O}_V(V)$ is finite-dimensional.

Q3. Let V and W be quasi-projective varieties. Suppose that V is irreducible. Let $\mathrm{Mor}(V, W)$ be the set of morphisms from V to W and let $\rho : \mathrm{Mor}(V, W) \to \mathrm{Rat}(V, W)$ be the natural map (ie ρ sends a morphism to the rational map it represents). Show that ρ is injective.

Q4. (1) Show that for any $m, n \geqslant 0$, $k^m \prod k^n \simeq k^{n+m}$.

(2) Let $V \subseteq k^m$ and $W \subseteq k^n$ be algebraic sets. Show that $V \times W \subseteq k^{n+m}$ is an algebraic set and describe $\mathcal{I}(V \times W)$. Show that the affine variety associated with the algebraic set $V \times W \subseteq k^{n+m}$ is a product of the affines varieties associated with V and W.

Q5. Let $a : X \dashrightarrow Y$ be a rational map between two varieties. Show that there is a unique representative $f : O \subseteq X$ of a (where $O \subseteq X$ is an open subvariety of X) such that if $f : U \to Y$ is a representative of a then $U \subseteq O$. The open set O is called the *open set of definition* of a.

Q6. Let $n \geqslant 0$ and let $q : k^{n+1}\backslash\{0\} \to \mathbb{P}^n(k)$ be the map such that $q(\bar{v}) = [\bar{v}]$ for all $\bar{v} \in k^{n+1}\backslash\{0\}$. Let $V \subseteq \mathbb{P}^n(k)$ be a closed subset. Endow $k^{n+1}\backslash\{0\}$ with the structure of variety it inherits from k^{n+1} as an open subset.

(1) Show that q is a morphism of varieties.

(2) Show that $\mathcal{I}(V)$ is prime iff V is irreducible.

(3) Show that $q^{-1}(V)$ is irreducible iff V is irreducible.

Q7. (1) Let $U \subseteq \mathbb{P}^1(k)$ be an open subset (for the Zariski topology). Let $f : U \to \mathbb{P}^1(k)$ be a morphism of varieties. Show that there exists a morphism of varieties $g : \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ such $g|_U = f$.

(2) Show that every automorphism of $\mathbb{P}^1(k)$ is of the form described in Q7 of Sheet 2.

(3) Show that k is not isomorphic to any of its proper open subvarieties (an open subvariety is proper if it is not equal to k).

Q8. Show that k^2 is not homeomorphic to $\mathbb{P}^2(k)$.

Exercise sheet 4. W1 of Hilary Term. All lectures.

Q1. Suppose in this exercise that $\text{char}(k) = 0$. Find the singularities of the following curves C in k^2. For each singular point $P \in C$ compute the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k-vector space. Here $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{C,P}$.

(1) $\mathrm{Z}(x^6 + y^6 - xy)$

(2) $\mathrm{Z}(y^2 + x^4 + y^4 - x^3)$

You may assume that the polynomials $x^6 + y^6 - xy$ and $y^2 + x^4 + y^4 - x^3$ are irreducible.

Q2. (blowing up the origin in affine space) Let $n \geqslant 1$. Let $x_1, \dots, x_n$ be variables for k^n and let $y_1, \dots, y_n$ be homogenous variables for $\mathbb{P}^{n-1}(k)$. Note that contrary to what is customary, the index of the homogenous variables runs between 1 and n here (not 0 and $n-1$).

(1) Let Z be the subset of $k^n \times \mathbb{P}^{n-1}(k)$ defined by the equations $\{x_i y_j - x_j y_i = 0\}_{i,j \in \{1,\dots,n\}}$ (note that this makes sense because the polynomials are homogenous in the y-variables). Show that Z is a closed subvariety of $k^n \times \mathbb{P}^{n-1}(k)$. The variety Z is called the *blow-up* of k^n at the origin of k^n. Let $\phi : Z \to k^n$ the map obtained by restricting the projection $k^n \times \mathbb{P}^{n-1}(k) \to k^n$ to Z.

(2) Show that $\phi^{-1}(\{0\})$ is canonically isomorphic to $\mathbb{P}^{n-1}(k)$. Show that the points of $\phi^{-1}(0)$ are in one-to-one correspondence with the lines going through the origin of k^n.

(3) Show that the restriction of ϕ to the open subvariety $\phi^{-1}(k^n \backslash \{0\})$ of Z induces an isomorphism $\phi^{-1}(k^n \backslash \{0\}) \simeq k^n \backslash \{0\}$.

Q3. (blowing up a point of an affine variety) Let $X \subseteq k^n$ be a closed subvariety (ie an algebraic set). Let $\bar{v} := \langle v_1, \dots, v_n \rangle \in X$ and suppose that $\{\bar{v}\}$ is not an irreducible component of X. Let $\tau_{\bar{v}} : k^n \to k^n$ be the map such that $\tau_{\bar{v}}(\langle w_1, \dots, w_n \rangle) = \langle w_1 + v_1, \dots, w_n + v_n \rangle$ for all $\bar{w} = \langle w_1, \dots, w_n \rangle \in k^n$ (note that this is an automorphism of the variety k^n). Let $Y := \tau_{-\bar{v}}(X)$. Note that by construction we have $0 \in Y$. Let $\phi : Z \to k^n$ be the morphism defined in Q2.

We define the *blow-up* $\mathrm{Bl}(X, \bar{v})$ of X at $\bar{v}$ to be the closure of $\phi^{-1}(Y \backslash \{0\})$ in Z.

(1) Show that $\phi(\mathrm{Bl}(X, \bar{v})) = Y$.

Let $b : \mathrm{Bl}(X, \bar{v}) \to X$ be the morphism $\tau_{\bar{v}} \circ \phi|_{\mathrm{Bl}(X,\bar{v})}$.

(2) Suppose that X is irreducible. Show that $\mathrm{Bl}(X, \bar{v})$ is an irreducible component of $\phi^{-1}(Y) \subseteq k^n \times \mathbb{P}^{n-1}(k)$. Show that b is a birational morphism. If $X \neq k^n$, show that the irreducible components of $\phi^{-1}(Y)$ are $\mathrm{Bl}(X, \bar{v})$ and $\{0\} \times \mathbb{P}^{n-1}(k)$.

The closed set $b^{-1}(\{v\}) = \mathrm{Bl}(X, \bar{v}) \cap (\{0\} \times \mathbb{P}^{n-1}(k))$ is called the *exceptional divisor* of $\mathrm{Bl}(X, \bar{v})$.

Q4. Let C be the plane curve considered in (1) of Q1. Consider the blow-up B of C at each of its singular points in turn. How many irreducible components does the exceptional divisor of B have? Is B nonsingular?

Q5. Let C be the curve $y^2 = x^3$ in k^2. Let $b : \mathrm{Bl}(C, 0) \to C$ of C be the blow-up of C at the origin.

(1) Show that $\mathrm{Bl}(C, 0) \simeq k$.

(2) Show that the map b is a homeomorphism but is not an isomorphism.

Q6. Let $V \subseteq k^2$ be the algebraic set defined by the equation $x_1 x_2 = 0$. Show that $\mathrm{Bl}(V, 0)$ has two disjoint irreducible components and that each of these components is isomorphic to k.

Q7. (1) Let $f : X \to Y$ be a dominant morphism of varieties. Suppose that Y is irreducible. Show that $\dim(X) \geqslant \dim(Y)$.

(2) Let $f : X \to Y$ be a dominant morphism of irreducible varieties. Suppose that the field extension $\kappa(X)|\kappa(Y)$ is algebraic. Show that there are affine open subvarieties $U \subseteq X$ and $W \subseteq Y$ such that $f(U) = W$ and such that the map of rings $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ is injective and finite.

(3) Let $f : X \to Y$ be a dominant morphism of irreducible quasi-projective varieties. Show that there is a $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$. [*Hint. Reduce to the situation where Y is affine and apply Noether's normalisation lemma to show that you may assume wlog that $Y = k^n$ for some n. Now use the existence of transcendence bases and (2) to show that there is an open subvariety $U \subseteq X$ and an open subvariety W of $k^{\dim(X)-\dim(Y)} \times k^n$ such that $f|_U$ factors as a finite and surjective morphism $U \to W$, followed by the projection to k^n. Now deduce the result from (1) and a computation of the dimension of the fibres of the projection $k^{\dim(X)-\dim(Y)} \times k^n \to k^n$.*]

(4) Deduce that in the situation of (3), the set of $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$ is dense in Y.

Q8. (1) Show that all the morphisms from $\mathbb{P}^2(k)$ to $\mathbb{P}^1(k)$ are constant. *[Hint: Use Q7 and the projective dimension theorem.]*

(2) Deduce from (1) that for any $n \geqslant 2$ the morphisms from $\mathbb{P}^n(k)$ to $\mathbb{P}^1(k)$ are constant. *[Hint: Use (1) and Q7 of Sheet 2.]*